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# HUB-INTEGRITY OF LINE GRAPHS

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Abstract. The hub-integrity of a graph $G = (V(G), E(G))$ is denoted as $HI(G)$ and defined by $HI(G) = \min\{|S| + m(G - S), S \textit{ is a hub set of } G\}$, where $m(G - S)$ is the order of a maximum component of $G - S$. This paper includes results on the hub-integrity of line graphs of some graphs.

## 1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph without loops or multiple edges. For any graph $G$, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. For graph theoretic terminology, we refer to [4].

The complement $\overline{G}$ of a graph $G$ has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$ [4].

Networks appear in many different applications and settings. The most common networks are telecommunication networks, computer networks, the internet, road and rail networks and other logistic networks. In all applications, vulnerability and reliability are crucial and important features. Network designers often build a network configuration around specific processing, performance and cost requirements. But there is little consideration given to the stability of the networks communication structure when under the pressure of link or node loses. This lack of consideration makes the networks have low survivability. Therefore network design process must identify the critical points of failure and be able to modify the design to eliminate them [6].

The stability of a communication network composed of processing nodes and communication links is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. In an analysis of the stability of a communication network to disruption, two questions that come to mind are: (i) How many vertices can still communicate? (ii) How difficult is it to reconnect the graph? The concept of integrity was introduced as a measure of graph stability by Barefoot, Entringer and Swart [1]. Formally, the integrity is $I(G) = \min_{S\subset V}\{|S| + m(G - S)\}$, where $m(G - S)$ denotes the order

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of a largest component of $G - S$. The integrity is a measure which deals with the first question stated above, namely how many vertices can still communicate? If the set $S$ achieves the integrity, then it is called an $I$-set of $G$. That is, if $|S| + m(G - S) = I(G)$ for any set $S$, then $S$ is called an $I$-set.

Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An $H$-path between $x$ and $y$ is a path where all intermediate vertices are from $H$. (This includes the degenerate cases where the path consists of the single edge $xy$ or a single vertex $x$ if $x = y$, call such an $H$-path trivial). A set $H \subseteq V(G)$ is a hub set of $G$ if it has the property that, for any $x, y \in V(G) - H$, there is an $H$-path in $G$ between $x$ and $y$. The smallest size of a hub set in $G$ is called a hub number of $G$, and is denoted by $h(G)$ [10]. A set $S \subseteq V(G)$ is called a dominating set of $G$ if each vertex of $V - S$ is adjacent to at least one vertex of $S$. The domination number of a graph $G$, denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in $G$.

Sultan et al. [7] have introduced the concept of hub-integrity of a graph as a new measure of vulnerability which is defined as follows.

**Definition 1.1** [7] The hub-integrity of a graph $G$ denoted by $HI(G)$ is defined by, $HI(G) = \min\{|S| + m(G - S), S \textit{ is a hub set of } G\}$, where $m(G - S)$ is the order of a maximum component of $G - S$. For more details on the hub-integrity see [8, 9].

**Definition 1.2** [7]A subset S of $V(G)$ is said to be a $HI$-set, if $HI(G) = |S| + m(G - S)$.

The integrity of middle graphs is discussed by Mamut and Vumar [5], while integrity of total graphs is discussed by Dndar and Ayta [2]. In the present work we investigate hub-integrity of line graphs. We need the following to prove main results.

**Proposition 1.3** [7] For any complete graph $K_p$, $HI(K_p) = p$.

## 2. Some properties of hub-integrity of Line Graphs

**Definition 2.1** [4] The line graph $L(G)$ of $G$ has the edges of $G$ as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in $G$.

**Theorem 2.2** If $G$ is a simple graph such that $\overline{G} \cong L(G)$, then $HI(G) = HI(L(G)) = HI(\overline{G})$ if and only if $G = C_5$ or $G$ is the graph shown on the Figure 1 below.

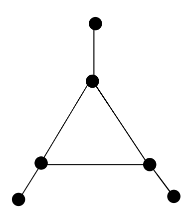


Figure 1.

**Remark 2.3** If $HI(L(G_1)) = HI(L(G_2))$, then it is not necessary that $HI(G_1) = HI(G_2)$, for example, the graphs $G_1$ and $G_2$ in Figure 2,

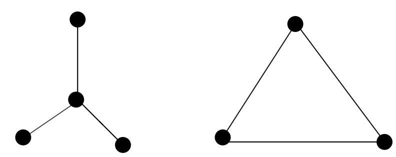



Figure 2: $G_1, G_2$

we have $HI(G_1) = 2$, while $HI(G_2) = 3$. Also $L(G_1) = L(G_2) \cong C_3$, hence $HI(L(G_1)) = HI(L(G_2)) = 3$.

**Proposition 2.4** If $G$ is regular graph of degree 2, then $HI(G) = HI(L(G))$.

**Proof.** $G$ is regular of degree 2, hence $G \cong C_p$, then the proof is completed.

**Lemma 2.5** Let $G$ be a connected graph and let $\alpha(G) = 1$, where $\alpha(G)$ is the vertex cover number. Then $HI(L(G)) = p$.

**Proof.** Suppose $\alpha(G) = 1$, then $G \cong K_{1,p}$. Then $L(G) = K_p$, so proof follows from Proposition 1.3.

**Observation 2.6** If $T$ is a tree with $p$ vertices and $\alpha(T) = 2$, then $HI(L(T)) \leq p - 2$.

**Lemma 2.7** Let $G$ be a connected graph with $\Delta(G) \leq 2$. Then $HI(L(G)) = |E(L(G))|$ if and only if $G = P_p$ or $G = C_3$.

**Proof.** Suppose that $G$ is a connected graph with $\Delta(G) \leq 2$. Then $G$ is path or cycle. But if $G$ is cycle. If $G = C_p, p \geq 4$, then we have $HI(L(C_p)) \leq p - 1 \neq |E(L(C_p))|$, thus $G$ is $C_3$ or path. The converse is obvious. □

**Proposition 2.8** If $HI(L(G)) = |E(G)|$, then $G \cong K_{1,p}$ or $G \cong C_3$.

**Theorem 2.9** For any subset $D$ of vertices in a graph $L(G)$, $HI(L(G) - D) \geq HI(L(G)) - |D|$.

**Proof.** Let $S$ be a $HI$- set of $L(G)$, let $D \subseteq V(L(G))$ and $S^\star$ be a $HI$-set of $L(G) - D$ such that $S^{\star\star} = S^\star \cup D$. Then $|S^{\star\star}| = |S^\star| + |D|$ and $L(G) - S^{\star\star} = L(G) - (S^\star \cup D) = (L(G) - D) - S^\star$. Therefore

$$\begin{aligned} HI(L(G)) &= |S| + m(L(G) - S) \\ &\leq |S^{\star\star}| + m(L(G) - S^{\star\star}) \\ &= |S^\star| + |D| + m[(L(G) - D) - S^\star] \\ &= HI(L(G) - D) + |D|. \end{aligned}$$

**Proposition 2.10** If $G$ is a non-trivial graph, then for all $v \in V(L(G))$, $HI(L(G) - v) \geq HI(L(G)) - 1$. The bound is sharp for $G = K_{1,p}$.

**Proposition 2.11** Let $G$ be a graph, then for all $e \in E(L(G))$, $HI(L(G) - e) \geq HI(L(G)) - 1$. The bound is sharp for $G = K_{1,p}$.

**Proposition 2.12** For any graph $G$,

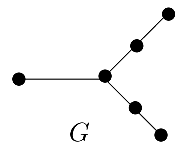
(1) $1 \leq HI(L(G)) \leq p$. The lower bound attains for $P_2$ and the upper bound attains for star graph $K_{1,p}$.
(2) For any graph $H \leq G$, $HI(L(H)) \leq HI(L(G))$.

**Lemma 2.13** $HI(L(K_{1,p})) + HI(\overline{L(K_{1,p})}) = 2p$.

**Proof.** Since $L(K_{1,p}) \cong K_p$, it follows from Proposition 1.3 that $HI(K_p) = p$, and $\overline{L(K_{1,p})} \cong \overline{K_p}$, so $HI(\overline{K_p}) = p$, hence the result. □

**Proposition 2.14** For any graph $G$, $p \leq HI(L(G)) + HI(\overline{L(G)}) \leq 3p - 1$.

The lower bound attains for a graph $G$ below, and the upper bound attains for wheel graph $W_{1,3}$.



$G$

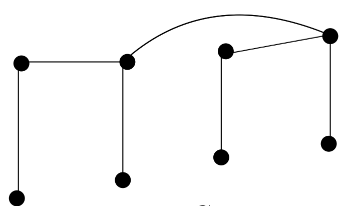
**Remark 2.15** The hub-integrity of graph $G$ and hub-integrity of line graph are not comparable. For this situation consider the graphs in the following cases:

- In the star $K_{1,p}$, $HI(L(K_{1,p})) > HI(K_{1,p})$.
- In the cycle $C_p$, $HI(L(C_p)) = HI(C_p)$.
- In the path $P_p$, $p > 3$, $HI(L(P_p)) < HI(P_p)$.

**Proposition 2.16** For any path $P_p$,

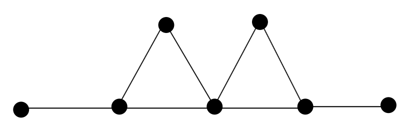
$$HI(L(P_p)) + HI(\overline{L(P_p)}) = \begin{cases} p+1, \text{ if } p = 3, 4 ; \\ 2p-4, p \geq 5. \end{cases}$$

**Remark 2.17** If $G$ is a connected graph, and $|E(L(G))| < |E(G)|$, then $HI(L(G)) < HI(G)$. We note that $|E(L(G))| < |E(G)|$ obtained only in a path graph, hence the result.
But the converse is not true, for example, the graphs shown in Figure 3, and Figure 4.



$G$

Figure 3



$L(G)$

Figure 4

$HI(G) = 5$ and $HI(L(G)) = 4$, while $|E(L(G))| > |E(G)|$.

**Proposition 2.18** Let $S$ be a $HI$-set of $L(G)$. Then $m(L(G)-S) \leq HI(L(G)-S)$.

**Proof.** Let $S^\star$ be a $HI$-set of $L(G) - S$.

$$\begin{aligned} |S| + m(L(G) - S) &= HI(L(G)) \\ &\leq m(L(G) - (S \cup S^\star)) + |S \cup S^\star| \\ &= |S| + |S^\star| + m((L(G) - S) - S^\star) \\ &= |S| + HI(L(G) - S). \end{aligned}$$

Hence $m(L(G) - S) \leq HI(L(G) - S)$. $\square$

**Observation 2.19** If $G$ is connected graph, then $HI(L(G)) \leq |V(L(G))|$.
If $G$ is disconnected, then $HI(L(G))$ may be greater than $m(L(G))$. For example if $G = P_3 \cup K_3$, then $HI(L(G)) = 5 > 3 = m(L(G))$. Also may be $HI(L(G)) \leq m(L(G))$, for example, let $G = K_2 \cup C_7$, then $HI(L(G)) = 6$ and $m(L(G)) = 7$.

**Proposition 2.20** If a connected graph $G$ is isomorphic to its line graph, then $HI(G) = HI(L(G))$. But the converse is not true, for example the graph $G$ is given in the following Figure 5.

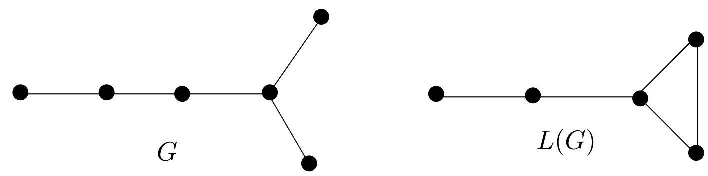



Figure 5

$HI(G) = 4 = HI(L(G))$, but $G$ and $L(G)$ are not isomorphic.

**Theorem 2.21** Let $G \cong K_p - e$, $e \in E(G)$. Then $HI(\overline{G}) = p$.

**Proof.** If $G \cong K_p - e$, then $\overline{G} \cong K_2 \cup (p-2)K_1$. By definition of hub-integrity of disconnected graph, we have

$$\begin{aligned} HI(\overline{G}) &= p - 2 + HI(K_2) \\ &= p - 2 + 2 = p. \end{aligned}$$

**Corollary 2.22** Let $G \cong K_p - e$, $e \in E(G)$. Then $HI(\overline{G}) = HI(G) + 1$. Therefore, $HI(\overline{G}) - HI(G) = 1$.

**Theorem 2.23** Let $G \cong K_p - e$, $e \in E(G)$. Then $HI(L(\overline{G})) = 1$.

**Proof.** Since $G \cong K_p - e$, then $\overline{G} \cong K_2 \cup (p-2)K_1$, and $L(\overline{G}) \cong K_1$. Thus $HI(L(\overline{G})) = 1$. □

In general, if $G \cong K_p - F$, where $F$ is a set of independent edges in $G$, then $\overline{G} \cong |F|K_2 \cup (p - 2|F|)K_1$. So we can conclude the following theorem.

**Theorem 2.24** Let $G \cong K_p - F$, where $F$ is a set of maximum number of independent edges in $G$, then $HI(\overline{G}) = p$.

**Theorem 2.25** If $G \cong K_p - F$, where $F$ is a set of maximum number of independent edges in $G$, then $HI(L(\overline{G})) = |F|$.

**Proof.** Since $G \cong K_p - F$, then $\overline{G} \cong |F|K_2 \cup (p - 2|F|)K_1$. Therefore $L(\overline{G}) = |F|K_1$. By definition of hub-integrity of disconnected graph, we have,

$$\begin{aligned} HI(L(\overline{G})) &= \underbrace{1 + 1 + 1 + \ldots + 1}_{|F|-1 \ times} + HI(K_1) \\ &= \underbrace{1 + 1 + 1 + \ldots + 1}_{|F|-1 \ times} + 1 \\ &= |F|. \end{aligned}$$

**Lemma 2.26** If $G$ is a graph, then $\gamma(L(G)) \leq h(L(G)) + 1$.

**Proof.** Let $S$ be a hub set in $L(G)$, and suppose that the set of vertices $Q$ which are not adjacent to anything in $S$ (nor in $S$ themselves). The only $S$-paths between vertices in $Q$ must therefore be trivial, since there are S-paths between all pairs of vertices in $V - S$, it follows that $G[Q]$ is complete. Then, $S \cup \{q\}$ must be a dominating set for any $q \in Q$. □

**Theorem 2.27** For any graph $G$, $\gamma(L(G)) \leq HI(L(G))$.

**Proof.** By the definition of $HI(G)$, $h(L(G)) + 1 \leq HI(L(G))$ and by Lemma 2.26, $\gamma(L(G)) \leq h(L(G)) + 1 \leq HI(L(G))$. Therefore $\gamma(L(G)) \leq HI(L(G))$. □

## 3. Hub-integrity of line graph of some graphs

**Theorem 3.1** Let $G$ be a star $K_{1,p}$, and $G'$ be a graph obtained from $G$ by dividing each edge of $G$ exactly once. Then $HI(L(G')) = p + 1$.

**Proof.** Since $L(G')$ contains a complete graph $K_p$ as its subgraph, if we choose the set $S$ as all vertices of $K_p$ of $L(G')$, then there exist $p$ components each containing only one vertex. So $HI(L(G')) = p + 1$. □

**Definition 3.2** [11] The $p$-sunlet graph is the graph on $2p$ vertices obtained by attaching $p$ pendant edges to a cycle graph $C_p$ and is denoted by $L_p$.

**Theorem 3.3** Let $L_p$ be $p$-sunlet graph, then $HI(L(L_p)) = p + 1$.

**Proof.** Let $V(L_p) = \{v_1, v_2, ..., v_p, u_1, u_2, ..., u_p\}$ and $E(L_p) = \{e_1, e_2, ..., e_p, e'_1, e'_2, ..., e'_p\}$. By the definition of line graph, $V(L(L_p)) = \{e_i, 1 \leq i \leq p\} \cup \{e'_i, 1 \leq i \leq p\}$ as shown in Figure 6. Consider $S = \{e_1, e_2, ..., e_p\}$ a hub set of $L(L_p)$ and $|S| = p$. Then $m(L(L_p) - S) = 1$, therefore

$$HI(L(L_p)) \leq |S| + m(L(L_p) - S) = p + 1. \tag{1}$$

We will show that the number $|S| + m(L(L_p) - S)$ is minimum. If $S_1$ is any hub set differently $S$ and $m(L(L_p) - S_1) \geq 1$, then trivially $|S_1| + m(L(L_p) - S_1) > p + 1$, hence for any hub set $S_1$,

$$|S_1| + m(L(L_p) - S_1) > p + 1. \tag{2}$$

From (1) and (2), $HI(L(L_p)) = p + 1$. □

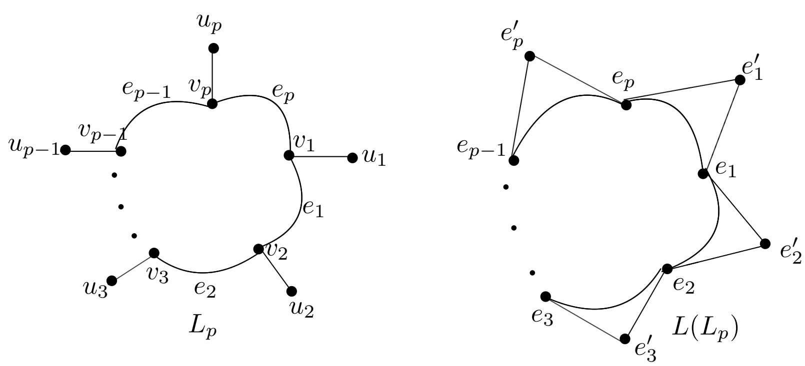



Figure 6.

**Corollary 3.4** $HI(L_p) = HI(L(L_p))$.

**Proposition 3.5**

(1) For any path $P_p$ with $p \geq 3$,

$$HI(L(P_p)) = \begin{cases} 2, & \text{if } p = 3 ; \\ p - 2, & \text{if } p \geq 4. \end{cases}$$

(2) For any cycle $C_p$, $p \geq 4$,

$$HI(L(C_p)) = \begin{cases} p - 1, & \text{if } p = 4, 5 ; \\ p - 2, & \text{if } p \geq 6. \end{cases}$$

(3) For the star $K_{1,p}$, $HI(L(K_{1,p})) = p$.

**Definition 3.6** [3] A double star $S_{n,m}$ is a tree with exactly two vertices that are not pendant vertices, with one adjacent to $n$ pendant vertices and the other to $m$ pendant vertices.

**Lemma 3.7** $h(L(S_{n,m}) = 1$.

**Proof.** The graph $L(S_{n,m})$ consists of two complete graphs of orders $n, m$ respectively, and the vertex $e$ that is adjacent to all vertices in $L(S_{n,m})$. The number of vertices of $L(S_{n,m})$ is $n + m - 1$. The graphs $S_{n,m}$ and $L(S_{n,m})$ are shown in Figure 7. Consider $S = \{e\}$ a hub set of $L(S_{n,m})$. Since $e$ is adjacent to all vertices in $L(S_{n,m})$, if we remove it from the graph $L(S_{n,m})$, there is no $S$-path between the vertices that are not adjacent. So $S$ is a minimum hub set. Therefore $h(L(S_{n,m}) = 1$. □

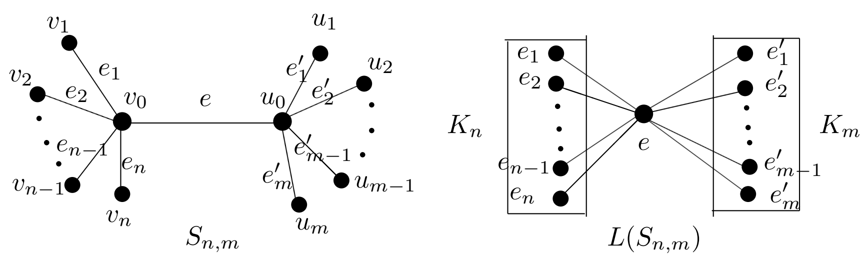



Figure 7: $S_{n,m}$ and $L(S_{n,m})$

**Theorem 3.8**

$$HI(L(S_{n,m}) = \begin{cases} n+1, \text{ if } n = m \text{ ;} \\ 1 + max\{n, m\}, \text{ if } n \neq m. \end{cases}$$

**Proof.** The number of the vertices of $L(S_{n,m})$ is $n + m - 1$. We have the two cases :

**Case 1 :** $n = m$. From Lemma 3, $h(L(S_{n,n})) = 1$ and $H = \{e\}$, a hub set of $L(S_{n,n})$. Then $m(L(S_{n,n}) - H) = n$. Therefore,

$$HI(L(S_{n,n})) \leq h(L(S_{n,}) + m(L(S_{n,n}) - H) = n + 1. \tag{3}$$

If $S$ is any hub set other than $H$ and $m(L(S_{n,n}) - S) = 1$, then $|S| \geq 2n - 1$, so

$$HI(L(S_{n,n})) \geq 2n. \tag{4}$$

If $S_1$ is any hub set other than $H$ and $m(L(S_{n,m}) - S_1) = 2$, then $|S_1| \geq 2n - 3$, so

$$HI(L(S_{n,n})) \geq 2n - 1. \tag{5}$$

If $S_2$ is any hub set other than $H$ and $m(L(S_{n,n}) - S_2) = n - 1$, then $|S_2| = 3$, so

$$HI(L(S_{n,n})) = n + 2. \tag{6}$$

Then from 3, 4, 5 and 6, we have $HI(L(S_{n,n})) = n + 1$.

**Case 2 :** $n \neq m$. From Lemma (3), $h(L(S_{n,m}) = 1$ and $H = \{e\}$, a hub set of $L(S_{n,m})$. Then $m(L(S_{n,m}) - H) = max\ \{n, m\}$. Therefore,

$$HI(L(S_{n,m})) \leq h(L(S_{n,m}) + m(L(S_{n,m}) - H) = 1 + max\ \{n, m\}. \tag{7}$$

If $S$ is any hub set other than $H$ and $m(L(S_{n,m}) - S) = 1$, then $|S| \geq max\ \{n, m\} + 1$, so

$$HI(L(S_{n,m})) \geq max\ \{n, m\} + 2. \tag{8}$$

If $S_1$ is any hub set other than $H$ and $m(L(S_{n,m}) - S_1) = max\ \{n, m\}$, then $|S_1| \geq 1$, so

$$HI(L(S_{n,m})) \geq max\ \{n, m\} + 1. \tag{9}$$

Therefore, from (7), (8) and (9), $HI(L(S_{n,m}) = max\ \{n, m\} + 1$. □

**Definition 3.9** [4] The (Cartesian)product $G \times H$ of graphs $G$ and $H$ has $V(G) \times V(H)$ as its vertex set and $(u_1, u_2)$ is adjacent to $(v_1, v_2)$ if either $u_1 = v_1$ and $u_2$ is adjacent to $v_2$ or $u_2 = v_2$ and $u_1$ is adjacent to $v_1$.

**Lemma 3.10** $h(L(K_2 \times P_p)) = 2p - 3, p \geq 3$.

**Proof.** The number of vertices of line graph $L(K_2 \times P_p)$ is $3p-2$.
Let $V(L(K_2 \times P_p)) = \{w_1, w_2, ..., w_{3p-2}\}$. Two vertices $w_p$ and $w_{2p}$ in $L(K_2 \times P_p)$ are adjacent to two vertices, and the vertices $w_1, w_{p-1}, w_{p+1}, w_{2p-1}$ are adjacent to three vertices, while, the remaining vertices are adjacent to 4 vertices. The graph $L(K_2 \times P_6)$ is shown in Figure 7, to understand more of the encoding and arrange of vertices.

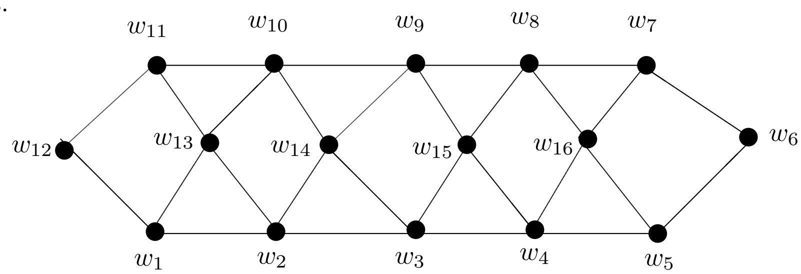



Figure 7. Graph $L(K_2 \times P_6)$

We have the two cases :
**Case 1 :** $p$ is even. Consider $S = \{w_1, w_3, ..., w_{p-1}\} \cup \{w_{p+2}, w_{p+4}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\}$, a hub set of $L(K_2 \times P_p)$ and $|S| = 2p-3$. We claim that $S$ is minimum hub set. If $w_1$ or $w_{p-1}$ is removed from set $S$, then there does not exist $S$-path between $w_p$ with $w_{2p}$. Hence $S$ is minimum hub set, therefore, $h(L(K_2 \times P_p)) = 2p-3$.
**Case 2 :** $p$ is odd. Consider $S = \{w_1, w_3, ..., w_{p-2}\} \cup \{w_{p+1}, w_{p+3}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\}$, a hub set of $L(K_2 \times P_p)$ and $|S| = 2p-3$. Proof is similar to Case 1. Hence $h(L(K_2 \times P_p)) = 2p-3$. □
**Theorem 3.11** $HI(L(K_2 \times P_p)) = 2p-1, p \geq 3$.
**Proof.** The number of vertices of line graph $L(K_2 \times P_p)$ is $3p-2$.
Let $V(L(K_2 \times P_p)) = \{w_1, w_2, ..., w_{3p-2}\}$.
From Lemma 3.10 , $h(L(K_2 \times P_p)) = 2p-3$, and $H = \{w_1, w_3, ..., w_{p-1}\} \cup \{w_{p+2}, w_{p+4}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\}$, if $p$ is even, and $H = \{w_1, w_3, ..., w_{p-2}\} \cup \{w_{p+1}, w_{p+3}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\}$, if $p$ is odd, then $m(L(K_2 \times P_p) - H) = 2$. Therefore,

$$HI(L(K_2 \times P_p)) \leq h(L(K_2 \times P_p) + m(L(K_2 \times P_p) - H) = 2p-1. \tag{10}$$

If $S$ is any hub set other than $H$ and $m(L(K_2 \times P_p) - S) = 1$, then $|S| \geq 2p-1$, so

$$HI(L(K_2 \times P_p)) \geq 2p. \tag{11}$$

Now, if $S_1$ is any hub set other than $H$ and $m(L(K_2 \times P_p) - S_1) \geq 2$, then trivially

$$|S_1| + m(L(K_2 \times P_p) - S_1) \geq 2p-1. \tag{12}$$

We consider $S_2 = \{w_1, w_2, ..., w_{p-1}\} \cup \{w_{p+1}, w_{p+2}, ..., w_{2p-1}\}$, and $m(L(K_2 \times P_p) - S_2) = 1$. Therefore,

$$HI(L(K_2 \times P_p)) \leq |S_2| + m(L(K_2 \times P_p) - S_2) = 2p-1. \tag{13}$$

Therefore, from (10), (11), (12) and (13), we have $HI(L(K_2 \times P_p)) = 2p-1$. □
**Corollary 3.12** For $p \geq 4$, $HI(L(K_2 \times P_p)) \geq 2HI(L(P_p)) + 3$.
**Theorem 3.13** $HI(L(K_2 \times C_p)) = \begin{cases} 2p+2, & \text{if } p \text{ is even ;} \\ 2p+1, & \text{if } p \text{ is odd.} \end{cases}$

**Proof.** The number of vertices of line graph $L(K_2 \times C_p)$ is $3p-2$.
Let $V(L(K_2 \times C_p)) = \{w_1, w_2, ..., w_{3p}\}$. The graph $L(K_2 \times C_6)$ is shown in Figure 8, to understand more of the encoding and arrange of vertices.

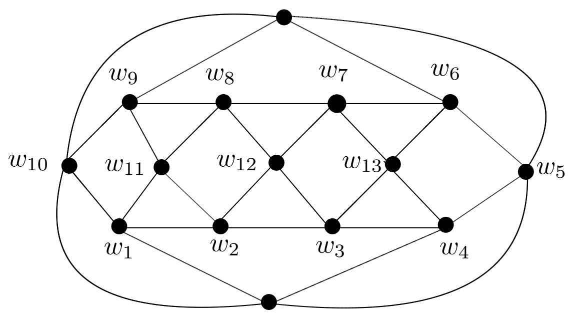



Figure 8. Graph $L(K_2 \times C_5)$

We have the two cases:
**Case 1:** $p$ is even. Consider $S = \{w_1, w_3, ..., w_{p-1}\} \cup \{w_{p+2}, w_{p+4}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\} \cup \{w_p, w_{2p}\}$, a hub set of $L(K_2 \times C_p)$ , $|S| = 2p-1$, then $m(L(K_2 \times C_p) - S) = 3$.
Therefore,

$$HI(L(K_2 \times C_p)) \leq |S| + m(L(K_2 \times C_p) - S) = 2p + 2. \tag{14}$$

If $S_1$ is any hub set other than $S$ with $m(L(K_2 \times C_p) - S_1) = 2$, then $|S_1| \geq 2p$.
This implies that

$$|S_1| + m(L(K_2 \times C_p) - S_1) \geq |S| + m(L(K_2 \times C_p) - S). \tag{15}$$

If $S_2$ is any hub set other than $S$ with $m(L(K_2 \times C_p) - S_2) = 1$, then $|S_2| \geq 2p+1$.
This implies that

$$|S_2| + m(L(K_2 \times C_p) - S_2) \geq |S| + m(L(K_2 \times C_p) - S). \tag{16}$$

Then from (14), (15) and (16), we have $HI(L(K_2 \times C_p)) = 2p + 2$.
**Case 2:** $p$ is odd. Consider $S = \{w_1, w_3, ..., w_{p-2}\} \cup \{w_{p+1}, w_{p+3}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\} \cup \{w_p, w_{2p}\}$, a hub set of $L(K_2 \times C_p)$ , $|S| = 2p-1$, then $m(L(K_2 \times C_p) - S) = 2$.
Therefore,

$$HI(L(K_2 \times C_p)) \leq |S| + m(L(K_2 \times C_p) - S) = 2p + 1. \tag{17}$$

If $S_1$ is any hub set other than $S$ with $m(L(K_2 \times C_p) - S_1) = 1$, then $|S_1| \geq 2p+1$.
This implies that,

$$|S_1| + m(L(K_2 \times C_p) - S_1) \geq 2p + 2. \tag{18}$$

If $m(L(K_2 \times C_p) - S_1) \geq 2$, then trivially

$$|S_1| + m(L(K_2 \times C_p) - S_1) \geq 2p + 1. \tag{19}$$

From (17), (18) and (19), we have $HI(L(K_2 \times C_p)) = 2p + 1$. □
**Lemma 3.14** $h(L(K_2 \times K_{1,p}) = p + 1$.
**Proof.** The graph $L(K_2 \times K_{1,p})$ consists of two complete graph $K_p$ each with $p$ vertices and $\{v, v_1, v_2, ..., v_p\}$ vertices as shown in Figure 9. Thus, the number of the vertices of $L(K_2 \times K_{1,p})$ is $3p + 1$.
A vertex $v$ in $L(K_2 \times K_{1,p})$ is adjacent to all vertices in both $K_p$ and the vertices

$\{v_1, v_2, ..., v_p\}$ are adjacent to one vertex of both complete $K_p$.

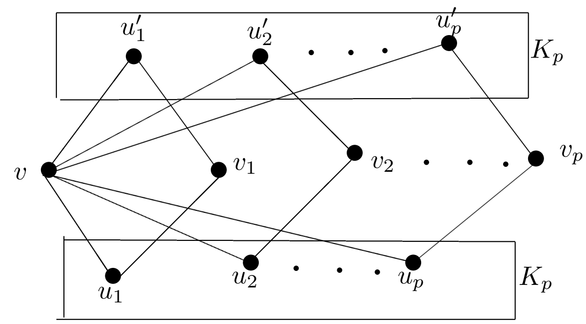



Figure 9. Graph $L(K_2 \times K_{1,p})$

Consider $H = \{v, v_1, v_2, ..., v_p\}$, a hub set of graph $L(K_2 \times K_{1,p})$.
We claim that $H$ is a minimum hub set. If the vertex $v$ is removed from set $H$, then there does not exist $H$-path between all the vertices of both complete graph $K_p$. Thus $h(L(K_2 \times K_{1,p}) = p + 1$. □

**Theorem 3.15** $HI(L(K_2 \times K_{1,p}) = 2p + 1$.

**Proof.** The graph $L(K_2 \times K_{1,p})$ consists of two complete graph $K_p$ each with $p$ vertices and $\{v, v_1, v_2, ..., v_p\}$ vertices. Thus, the number of the vertices of $L(K_2 \times K_{1,p})$ is $3p + 1$.
From Lemma 3.15, $h(K_2 \times K_{1,p}) = p + 1$ and $H = \{v, v_1, v_2, ..., v_p\}$, a hub set of graph $L(K_2 \times K_{1,p})$. Then $m(L(K_2 \times K_{1,p}) - H) = p$. This implies that

$$HI(L(K_2 \times K_{1,p}) \leq h(L(K_2 \times K_{1,p}) + m(L(K_2 \times K_{1,p}) - H) = 2p + 1. \quad (20)$$

If $S_1$ is any hub set other than $H$ with $m(L(K_2 \times K_{1,p}) - S_1) = p - 1$, then $|S_1| \geq 2p + 1$. This implies that

$$|S_1| + m(L(K_2 \times K_{1,p}) - S_1) \geq 3p. \quad (21)$$

If $S_2$ is any hub set other than $H$ and $S_1$ with $m(L(K_2 \times K_{1,p}) - S_2) = 1$, then $|S_2| \geq 2p + 1$. This implies that

$$|S_2| + m(L(K_2 \times K_{1,p}) - S_2) \geq 2p + 2. \quad (22)$$

If $m(L(K_2 \times K_{1,p}) - S_1) > p$, then

$$|S_1| + m(L(K_2 \times K_{1,p}) - S_1) \geq 2p + 1. \quad (23)$$

From (20), (21), (22) and (23), $HI(L(K_2 \times K_{1,p}) = 2p + 1$. □

**Corollary 3.16** $HI(L(K_2 \times K_{1,p})) \geq 2HI(L(K_{1,p})) + 1$.

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