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A NOTE ON GENERAL COINCIDENCE THEORY FOR SET–VALUED MAPS IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT. In this paper we present some coincidence results for maps which are not necessarily compact.

1. Introduction

Coincidence theory for compact maps (or alternatively maps whose domain is compact) has received some attention in the literature; we refer the reader to [[2], [3], [4], [5], [6], [7], [8], [9]] and the references therein. A typical example is the following result due to Granas and Liu [5]: Let X, Y be convex subsets of topological vector spaces and A, $B : X \to Y$ be set valued maps satisfying:
(i). A is upper semicontinuous and has nonempty compact acyclic values,
(ii). B has nonempty convex values and open fibres.
If A is compact then there exists $x_0 \in X$ with $A(x_0) \cap B(x_0) \neq \emptyset$. A special case of this result is when A compact is replaced by X compact.

In this paper we present results in the noncompact situation. With appropriate assumptions we use results in the literature (when the domain space is compact or alternatively one of the maps is compact) to establish our theory. We present four results in Section 2. Other approaches obtaining coincidence results in the noncompact case will be presented by the author is a series of future papers.

2. Coincidence Theory.

By a space we mean a Hausdorff topological space. We consider the classes $\mathbf{A}$ and $\mathbf{B}$. Let E be a space and X a subset of E.

Definition 2.1. We say $F \in M(X, E)$ if $F : X \to 2^E$ and $F \in \mathbf{A}(X, E)$; here 2^E denotes the family of nonempty subsets of E.

Definition 2.2. We say $G \in MB(X, E)$ if $G : X \to 2^E$ and $G \in \mathbf{B}(X, E)$.

Remark 2.3. Examples of the classes $\mathbf{A}$ and $\mathbf{B}$ can be found for example in [[7], [8], [9]].

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Theorem 2.4. *Let Ω be a closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $F : \Omega \to 2^{\Omega}$, $\Phi : \Omega \to 2^E$ and assume the following conditions hold:*

$$A \subseteq \Omega, \;\; A = \overline{co}\,(\{x_0\} \cup F(A)) \;\; \textit{implies} \;\; A \;\; \textit{is compact} \tag{2.1}$$

and

$$\begin{cases} \textit{for any nonempty convex compact subset } W \textit{ of } \Omega \\ \textit{with } F(W) \subseteq W, \; F \in M(W,W), \, \Phi \in MB(W,E) \;\; \textit{and} \\ (W,F,\Phi) \;\; \textit{has the coincidence property} \\ \textit{(i.e. there exists } \; x \in W \;\; \textit{with} \;\; F(x) \cap \Phi(x) \neq \emptyset \textit{).} \end{cases} \tag{2.2}$$

Then there exists $x \in \Omega$ with $F(x) \cap \Phi(x) \neq \emptyset$.

PROOF: Consider $\mathbf{S}$ the family of all closed, convex subsets D of Ω with $x_0 \in D$ and $F(x) \subseteq D$ for all $x \in D$. Note $\mathbf{S} \neq \emptyset$ since $\Omega \in \mathbf{S}$. Let

$$D_0 = \cap_{D \in \mathbf{S}} D.$$

Notice D_0 is nonempty, closed and convex and $F : D_0 \to 2^{D_0}$ (note if $x \in D_0$ then $F(x) \subseteq D$ for all $D \in \mathbf{S}$). Let

$$D_1 = \overline{co}\,(\{x_0\} \cup F(D_0)). \tag{2.3}$$

We now show $D_1 = D_0$. Now $F : D_0 \to 2^{D_0}$ together with D_0 closed and convex implies $D_1 \subseteq D_0$. Also $F(D_1) \subseteq F(D_0) \subseteq D_1$ (see (2.3)). Thus D_1 is closed and convex with $F(D_1) \subseteq D_1$. As a result $D_1 \in \mathbf{S}$, so $D_0 \subseteq D_1$. Consequently

$$D_0 = \overline{co}\,(\{x_0\} \cup F(D_0)). \tag{2.4}$$

Now (2.1) guarantees that D_0 is compact and (2.4) implies $F(D_0) \subseteq D_0$. Now (2.2) guarantees that there exists $x \in D_0$ with $F(x) \cap \Phi(x) \neq \emptyset$. $\square$

Note in Theorem 2.4 we could replace (2.2) with

$$\begin{cases} \text{for any } \; W \subseteq \Omega, \;\; W = \overline{co}\,(\{x_0\} \cup F(W)) \;\; \text{(so } W \text{ is convex and compact)} \\ \text{we have that } \; F \in M(W,W), \, \Phi \in MB(W,E) \;\; \text{and} \;\; (W,F,\Phi) \;\; \text{has the} \\ \text{coincidence property (i.e. there exists } \; x \in W \;\; \text{with} \;\; F(x) \cap \Phi(x) \neq \emptyset \text{).} \end{cases}$$

Theorem 2.5. *Let Ω be a closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$ and suppose $F : \Omega \to 2^{\Omega}$ and $\Phi : \Omega \to 2^E$. Let*

$$D_0 = \overline{co}\,(\{x_0\} \cup F(\Omega)), \;\; D_{n+1} = \overline{co}\,(\{x_0\} \cup F(D_n)) \;\; \textit{for} \;\; n \in \{0,1,2,...\},$$

and $D = \cap_{n=0}^{\infty} D_n$. Assume

$$D \;\; \textit{is compact} \tag{2.5}$$

and suppose (2.2) *holds. Then there exists $x \in \Omega$ with $F(x) \cap \Phi(x) \neq \emptyset$.*

Remark 2.6. In Theorem 2.5 note D is closed and convex. Also note (see below) that $\overline{co}\,(\{x_0\} \cup F(D)) \subseteq D$ but it is not clear whether D satisfies the inclusion $D \subseteq \overline{co}\,(\{x_0\} \cup F(D))$. To show $\overline{co}\,(\{x_0\} \cup F(D)) \subseteq D$ first note $F(D) \subseteq F(\Omega) \subseteq D_0$. Also note $D \subseteq D_0$ implies $F(D) \subseteq F(D_0) \subseteq D_1$, $D \subseteq D_1$ implies $F(D) \subseteq F(D_1) \subseteq D_2$ and by induction $F(D) \subseteq D_k$ for $k \in \{0,1,2,...\}$. Consequently $F(D) \subseteq \cap_{n=0}^{\infty} D_n = D$ so $\overline{co}\,(\{x_0\} \cup F(D)) \subseteq D$.

Remark 2.7. Examples where (2.5) holds can be found in the literature. For example if F is countably condensing then D is compact (see [[11], Theorem 2.2]).

PROOF: Note from Remark 2.6 that $F(D) \subseteq D$. Now apply (2.2). $\square$

Theorem 2.8. *Let Ω be a closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $F : \Omega \to 2^{\Omega}$ is a closed map (i.e. has closed graph) and $\Phi : \Omega \to 2^E$. Also assume the following conditions hold:*

$$A \subseteq \Omega, \;\; A = co\left(\{x_0\} \cup F(A)\right) \;\; \textit{implies} \;\; \overline{A} \;\; \textit{is compact} \tag{2.6}$$

and

$$\begin{cases} \textit{for any } \; A \subseteq \Omega, \;\; A = co\left(\{x_0\} \cup F(A)\right) \;\; (\textit{so } \; \overline{A} \; \textit{ is convex and compact}) \\ \textit{we have that } \; \Phi \in MB(\overline{A}, E) \; \textit{and the map } \; G \; \textit{ given by} \\ G(x) = F(x) \cap \overline{A}, \;\; x \in \overline{A}, \;\; \textit{is in } \; M(\overline{A}, \overline{A}) \; \textit{and} \\ (\overline{A}, G, \Phi) \;\; \textit{has the coincidence property.} \end{cases} \tag{2.7}$$

Then there exists $x \in \Omega$ with $F(x) \cap \Phi(x) \neq \emptyset$.

Remark 2.9. Notice G is well defined i.e. $G(x) \neq \emptyset$ for $x \in \overline{A}$. To see this it is enough to show $\overline{A} \subseteq F^{-1}(\overline{A})$. If $x \in \overline{A}$ then $x_\alpha \to x$ for some net $\{x_\alpha\}$ in A. Take any $y_\alpha \in F(x_\alpha)$. Since $F(A) \subseteq A$ (note $A = co\left(\{x_0\} \cup F(A)\right)$) we have $y_\alpha \in A \subseteq \overline{A}$. The compactness of $\overline{A}$ guarantees that we may assume without loss of generality that $y_\alpha \to y$ for some $y \in \overline{A}$. Since $(x_\alpha, y_\alpha) \in graph\, F$ and $graph\, F$ is closed, we have $(x, y) \in graph\, F$. Thus $y \in F(x) \cap \overline{A}$ i.e. $x \in F^{-1}(\overline{A})$.

PROOF: Let

$$D_0 = \{x_0\}, \quad D_n = co\left(\{x_0\} \cup F(D_{n-1})\right) \quad \text{for} \quad n \in N = \{1, 2, ...\}$$

and $D = \cup_{n=0}^{\infty} D_n$. Now for $n \in N$ notice D_n is convex and by induction we see that

$$D_0 \subseteq D_1 \subseteq \; .... \; \subseteq D_{n-1} \subseteq D_n \; .... \; \subseteq \Omega.$$

Consequently D is convex. It is also immediate that

$$D = \bigcup_{n=1}^{\infty} co\;\left(\{x_0\} \cup F(D_{n-1})\right) = co\left(\{x_0\} \cup F(D)\right);$$

for each $n \in \{1, 2, ...\}$; for one side note $D_{n-1} \subseteq D$ so $F(D_{n-1}) \subseteq F(D)$ whereas for the other side note $F(D_{n-1}) \subseteq D_n \subseteq D$ so $F(D) = F(\cup_{n=0}^{\infty} D_n) = \cup_{n=0}^{\infty} F(D_n) \subseteq D$. Now (2.6) guarantees that $\overline{D}$ is compact. Let $G(x) = F(x) \cap \overline{D}$ for $x \in \overline{D}$. Now (2.7) guarantees that there exists a $x \in \overline{D}$ with $G(x) \cap \Phi(x) \neq \emptyset$. □

Remark 2.10. (i). If we assume $F(\overline{Q}) \subseteq \overline{F(Q)}$ for any $Q \subseteq \Omega$ (or alternatively any $Q \subseteq \Omega$ with $Q = co\left(\{x_0\} \cup F(Q)\right)$) then the map G given in (2.7) is $G(x) = F(x)$, $x \in \overline{A}$. To see this notice $A = co\left(\{x_0\} \cup F(A)\right)$ so $F(A) \subseteq A$ and as a result $F(\overline{A}) \subseteq \overline{F(A)} \subseteq \overline{A}$

(ii). There are classes M where $F \in M(\Omega, \Omega)$ guarantees that $G \in M(\overline{A}, \overline{A})$; see for example [10].

Next we present a Mönch type coincidence result [[1], [10]].

Theorem 2.11. *Let Ω be a closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $F : \Omega \to 2^{\Omega}$ is a closed map which maps compact sets into relatively compact sets and $\Phi : \Omega \to 2^E$. Also assume the following conditions hold:*

$$\begin{cases} A \subseteq \Omega, \;\; A = co\left(\{x_0\} \cup F(A)\right) \;\; \textit{with} \;\; \overline{A} = \overline{C} \\ \textit{and} \;\; C \subseteq A \;\; \textit{countable, implies} \;\; \overline{A} \;\; \textit{is compact} \end{cases} \tag{2.8}$$

$$\begin{cases} \text{for any relatively compact subset } A \text{ of } \Omega \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A} \end{cases} \tag{2.9}$$

$$\text{if } A \text{ is a compact subset of } \Omega \text{ then } \overline{co}\,(A) \text{ is compact} \tag{2.10}$$

and

$$\begin{cases} \text{for any } A \subseteq \Omega,\ A = co\,(\{x_0\} \cup F(A)) \text{ with } \overline{A} = \overline{C} \\ \text{and } C \subseteq A \text{ countable (so } \overline{A} \text{ is convex and compact),} \\ \text{we have that } \Phi \in MB(\overline{A}, E) \text{ and the map } G \text{ given by} \\ G(x) = F(x) \cap \overline{A},\ x \in \overline{A}, \text{ is in } M(\overline{A}, \overline{A}) \text{ and} \\ (\overline{A}, G, \Phi) \text{ has the coincidence property.} \end{cases} \tag{2.11}$$

Then there exists $x \in \Omega$ with $F(x) \cap \Phi(x) \neq \emptyset$.

PROOF: Let

$$D_0 = \{x_0\}, \quad D_n = co\,(\{x_0\} \cup F(D_{n-1})) \quad \text{for} \quad n \in N \quad \text{and} \quad D = \cup_{n=0}^{\infty} D_n.$$

As in Theorem 2.8 we have that D is convex and $D = co\,(\{x_0\} \cup F(D))$. We now show D_n is relatively compact for $n \in \{0, 1, ...\}$. Suppose D_k is relatively compact for some $k \in N$. Then $F(\overline{D_k})$ is relatively compact and this together with (2.10) implies $\overline{co}\,(\{x_0\} \cup \overline{F(\overline{D_k})})$ is compact. Consequently D_{k+1} is relatively compact.

For each $n \in \{0, 1, ...\}$, (2.9) guarantees that there exists $C_n \subseteq D_n$ with C_n countable and $\overline{C_n} = \overline{D_n}$. Let $C = \cup_{n=0}^{\infty} C_n$. Now since

$$\bigcup_{n=0}^{\infty} D_n \subseteq \bigcup_{n=0}^{\infty} \overline{D_n} \subseteq \overline{\bigcup_{n=0}^{\infty} D_n}$$

we have

$$\overline{\bigcup_{n=0}^{\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{\infty} D_n} = \overline{D} \quad \text{and} \quad \overline{\bigcup_{n=0}^{\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{\infty} \overline{C_n}} = \overline{\bigcup_{n=0}^{\infty} C_n} = \overline{C}.$$

Thus $\overline{C} = \overline{D}$, and so (2.8) implies $\overline{D}$ is compact. Let $G(x) = F(x) \cap \overline{D}$ for $x \in \overline{D}$. Now (2.11) guarantees that there exists a $x \in \overline{D}$ with $G(x) \cap \Phi(x) \neq \emptyset$. □

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