Electronic Journal of Mathematical Analysis and Applications
Vol. 7(1) Jan. 2019, pp. 156-166.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

PERIODIC SOLUTIONS FOR A NONLINEAR ITERATIVE FUNCTIONAL DIFFERENTIAL EQUATION

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ABSTRACT. The main bulk of the present work is to study the existence and uniqueness of periodic solutions of the following iterative functional differential equation

$$\frac{d}{dt}x(t) = -a(t)\,x(t) + f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) + \frac{d}{dt}g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right).$$

By virtue of Schauder's fixed point theorem and under suitable conditions, we establish the existence of periodic solutions and by using the principle of contraction mapping, the uniqueness is proved. This paper, generalized some results obtained in [13].

1. Introduction

The first order differential equations result from the modeling of several phenomena such as population growth, carbon dating, ecological and medical models... The study of this kind of equations began circa 1671 with the pioneering work of Newton "The method of Fluxion and Infinite Series" and since then there has been much activity in this field [1, 7, 8].

In recent years, there exists a vast literature devoted to the existence of solutions by using the theory of fixed points (see [1]–[10], [12], [13]) and many authors was interested in iterative differential equations of the form

$$\frac{d}{dt}x(t) = f\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), ..., x^{[n]}(t)\right),$$

where $x^{[0]}(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t)), ..., x^{[n]}(t) = x^{[n-1]}(x(t))$. For instance, we mention only the work of Eder [4] on the equation

$$\frac{d}{dt}x(t) = x^{[2]}(t),$$

2010 *Mathematics Subject Classification.* Primary 34K13, 34A34; Secondary 34K30, 34L30.
Key words and phrases. Periodic solutions, iterative functional differential equations, fixed point theorem, Green's function.
Submitted Oct. 7, 2017. Revised May 14, 2018.

and the works of Si, Wang [9] and Hou, Jia [13] on the existence of solutions of the iterative equation

$$\frac{d}{dt}x(t) = c_1(t)x^{[1]}(t) + c_2(t)x^{[2]}(t) + ... + c_n(t)x^{[n]}(t).$$

Motivated by these works, we will establish sufficient conditions for the existence and uniqueness of periodic solution for the following equation

$$\begin{aligned}\frac{d}{dt}x(t) &= -a(t)x(t) + f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \\ &\quad + \frac{d}{dt}g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right). \qquad (1)\end{aligned}$$

Our approach is based on the application of Schauder's fixed point theorem and the principle of contraction mapping for proving the results.

The article is structured as follows. In the next section, we recall some notations and results which will be used in the next section. Also, we convert the considered equation into integral equation before applying Schauder's theorem in the next section. In the last section, we will establish the existence and uniqueness of periodic solutions of (1) via Schauder's fixed point theorem and the principle of contraction mapping.

2. Preliminaries

For $T > 0$ and L, $M \geq 0$, let

$$P_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}),\ x(t+T) = x(t)\},$$

equipped with the norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|$$

and

$$P_T(L, M) = \{x \in P_T,\ \ \|x\| \leq L,\ \ |x(t_2) - x(t_1)| \leq M|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbb{R}\},$$

then $(P_T, \|.\|)$ is a Banach space and $P_T(L, M)$ is a closed convex and bounded subset of P_T.

We assume that

$$a(t+T) = a(t),\ \int_0^T a(t)\,dt > 0. \qquad (2)$$

The functions $f(t, x_1, x_2, ..., x_n)$ and $g(t, x_1, x_2, ..., x_n)$ are supposed periodic in t with period T and globally Lipschitz in $x_1, ..., x_n$, i.e,

$$\begin{aligned} f(t+T, x_1, ..., x_n) &= f(t, x_1, ..., x_n), \\ g(t+T, x_1, ..., x_n) &= g(t, x_1, ..., x_n). \qquad (3)\end{aligned}$$

and there exist n positive constants $k_1, k_2, ..., k_n$ and n positive constants $c_1, c_2, ..., c_n$ such that

$$|f(t, x_1, ..., x_n) - f(t, y_1, ..., y_n)| \leq \sum_{i=1}^{n} k_i \|x_i - y_i\| \qquad (4)$$

and

$$|g(t, x_1, ..., x_n) - g(t, y_1, ..., y_n)| \leq \sum_{i=1}^{n} c_i \|x_i - y_i\|. \qquad (5)$$

The function $g(t, x_1, x_2, ..., x_n)$ is also supposed globally Lipschitz in t, i.e, there exists $K > 0$ such that

$$|g(t_2, x_1, ..., x_n) - g(t_1, x_1, ..., x_n)| \leq K |t_2 - t_1|. \tag{6}$$

Lemma 2.1. *Suppose (2) and (3) hold. If $x \in P_T(L, M)$, then x is a solution of (1) if and only if*

$$\begin{aligned} x(t) = \int_t^{t+T} & \Big[f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \\ & -a(s)\, g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \Big] G(t,s)\, ds \\ & + g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \end{aligned}$$

where

$$G(t,s) = \frac{\exp\left(\int_t^s a(u)\, du\right)}{\exp\left(\int_0^T a(u)\, du\right) - 1}.$$

Proof. First, we start by writing (1) in the following form

$$\begin{aligned} & \frac{d}{dt}x(t) + a(t)\, x(t) - \frac{d}{dt} g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \\ & = f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \frac{d}{dt}\left\{ \left[x(t) - g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right)\right] \exp\left(\int_0^t a(u)\, du\right)\right\} \\ & = \Big[f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \\ & -a(t)\, g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right)\Big] \exp\left(\int_0^t a(u)\, du\right). \end{aligned}$$

The integration from t to $t + T$ gives

$$\begin{aligned} & \int_t^{t+T} \frac{d}{ds}\left\{ \left[x(s) - g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right] \exp\left(\int_0^s a(u)\, du\right)\right\} ds \\ & = \int_t^{t+T} \Big[f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \\ & -a(s)\, g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\Big] \exp\left(\int_s^t a(s)\, ds\right) ds. \end{aligned}$$

Since

$$\begin{aligned} & \int_t^{t+T} \frac{d}{ds}\left\{ \left[x(s) + g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right] \exp\left(\int_0^s a(u)\, du\right)\right\} ds \\ & = \left\{ x(t) + g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right)\right\} \\ & \times \left(\left[\exp\left(\int_0^t a(u)\, du\right)\right] \left[\exp\left(\int_t^{t+T} a(u)\, du\right) - 1\right]\right), \end{aligned}$$

then

$$x(t)=\int_t^{t+T}\Big[f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)$$
$$-a\left(s\right)g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\Big]\frac{\exp\left(-\int_s^t a\left(u\right)du\right)}{\left[\exp\left(\int_t^{t+T}a\left(u\right)du\right)-1\right]}ds$$
$$+g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right).$$

Finally,

$$x(t)=\int_t^{t+T}\Big[f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)$$
$$-a\left(s\right)g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\Big]G\left(t,s\right)ds$$
$$+g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right).$$

□

Lemma 2.2. *Green function G satisfies the following properties*

$$G\left(t+T,s+T\right)=G\left(t,s\right),$$

and

$$\alpha=\frac{\exp\left(-\int_0^T a\left(u\right)du\right)}{\exp\left(\int_0^T a\left(u\right)du\right)-1}\leq G\left(t,s\right)\leq\frac{\exp\left(\int_0^T a\left(u\right)du\right)}{\exp\left(\int_0^T a\left(u\right)du\right)-1}=\beta.$$

Lemma 2.3 ([13])**.** *For any $\varphi,\psi\in P_T\left(L,M\right)$,*

$$\left\|\varphi^{[m]}-\psi^{[m]}\right\|\leq\sum_{j=0}^{m-1}M^j\left\|\varphi-\psi\right\|.$$

Theorem 2.4 (Schauder [11])**.** *Let $\mathbb{M}$ be a closed convex and bounded subset of a Banach space $(X,\|.\|)$ and let $H:\mathbb{M}\longrightarrow\mathbb{M}$ be a continuous and compact operator, then H has a fixed point.*

3. Main results

3.1. **Periodic solutions of equation (1).** In this section we will use the Schauder's fixed point theorem 2.4 to prove the existence of periodic solutions of the equation (1). For this , we define an operator $H:P_T\left(L,M\right)\longrightarrow P_T$ by

$$\left(H\varphi\right)(t)=\int_t^{t+T}\Big[f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)$$
$$-a\left(s\right)g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\Big]G\left(t,s\right)ds$$
$$+g\left(t,\varphi(t),\varphi^{[2]}(t),...,\varphi^{[n]}(t)\right),\tag{7}$$

and we will show that it is continuous and compact on $P_T\left(L,M\right)$ and $H\varphi\in P_T\left(L,M\right)$ for all $\varphi\in P_T\left(L,M\right)$.

For ease of exposition, we will adopt the following notations:

$$\delta=\max_{t\in[0,T]}|a(t)|\,,\ \rho_1=\max_{t\in[0,T]}|f(t,0,0,...,0)|\,,\ \rho_2=\max_{t\in[0,T]}|g(t,0,0,...,0)|\,. \tag{8}$$

Lemma 3.1. *Suppose that conditions (2)–(6) hold. If*

$$T\beta(\rho_1+\delta\rho_2)+\rho_2+L\sum_{i=1}^{n}[c_i+T\beta(k_i+\delta c_i)]\sum_{j=0}^{i-1}M^j\le L, \tag{9}$$

and

$$\begin{aligned}&(2\beta+\alpha T\|a\|)L(\rho_1+\delta\rho_2)+K\\&+\sum_{i=1}^{n}[(2\beta+\alpha T\|a\|)L(k_i+\delta c_i)+Mc_i]\sum_{j=0}^{i-1}M^j\\&\le M,\end{aligned} \tag{10}$$

then $H(P_T(L,M))\subset P_T(L,M)$.

Proof. Let $\varphi\in P_T(L,M)$. For having $H(\varphi)\in P_T(L,M)$ we will show that $H(\varphi)\in P_T$, $\|H(\varphi)\|\le L$ and $|(H\varphi)(t_2)-(H\theta)(t_2)|\le M|t_2-t_1|\,,\ \forall t_1,t_2\in\mathbb{R}$. It is easy to show that $(H\varphi)(t+T)=(H\varphi)(t)$. By (8), we have

$$\begin{aligned}&|(H\varphi)(t)|\\&\le\beta\int_t^{t+T}\left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|ds\\&+\beta\delta\int_t^{t+T}\left|g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|ds\\&+\left|g\left(t,\varphi(t),\varphi^{[2]}(t),...,\varphi^{[n]}(t)\right)\right|,\end{aligned}$$

and in view of conditions (4)–(5) and Lemma 2.3, we have

$$\begin{aligned}&\left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|\\&=\left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)-f(s,0,0,...,0)\right|+|f(s,0,0,...,0)|\\&\le\rho_1+\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}M^j\|\varphi\|\\&\le\rho_1+L\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}M^j,\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\left| g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right| \\
&= \left| g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) - g\left(s, 0, 0, ..., 0\right) \right| + \left| g\left(s, 0, 0, ..., 0\right) \right| \\
&\leq \rho_2 + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \left\| \varphi \right\| \\
&\leq \rho_2 + L \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j. \qquad (12)
\end{aligned}$$

Thus, it follows from (11), (12) and (9) that

$$\begin{aligned}
&\left| \left(H\varphi\right)(t) \right| \\
&\leq \beta T \left(\rho_1 + L \sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} M^j \right) + \left(\beta \delta T + 1\right) \left(\rho_2 + L \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \right) \\
&= T\beta \left(\rho_1 + \delta \rho_2\right) + \rho_2 + L \sum_{i=1}^{n} \left[c_i + T\beta \left(k_i + \delta c_i\right) \right] \sum_{j=0}^{i-1} M^j \\
&\leq L.
\end{aligned}$$

Therefore $\left\| H\varphi \right\| = \sup_{t \in [0,T]} \left| \left(H\varphi\right)(t) \right| \leq L$.
Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\left| \left(H\varphi\right)(t_2) - \left(H\varphi\right)(t_1) \right| \\
&\leq \left| \int_{t_2}^{t_2+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G\left(t_2, s\right) ds \right. \\
&\left. - \int_{t_1}^{t_1+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G\left(t_1, s\right) ds \right| \\
&+ \left| \int_{t_2}^{t_2+T} a\left(s\right) g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G\left(t_2, s\right) ds \right. \\
&\left. - \int_{t_1}^{t_1+T} a\left(s\right) g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G\left(t_1, s\right) ds \right| \\
&+ \left| g\left(t_2, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) - g\left(t_1, \varphi(t_1), \varphi^{[2]}(t_1), ..., \varphi^{[n]}(t_1)\right) \right|.
\end{aligned}$$

But,

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G\left(t_2,s\right) ds\right.\\
&\left.-\int_{t_1}^{t_1+T} f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G\left(t_1,s\right) ds\right|\\
&\leq \left|\int_{t_2}^{t_1} f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G\left(t_2,s\right) ds\right.\\
&\left.+\int_{t_1+T}^{t_2+T} f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G\left(t_2,s\right) ds\right|\\
&+\left|\int_{t_1}^{t_1+T} f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) \left(G\left(t_2,s\right)-G\left(t_1,s\right)\right) ds\right|\\
&\leq \int_{t_2}^{t_1} \left|G\left(t_2,s\right)\right| \left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds\\
&+\int_{t_1+T}^{t_2+T} \left|G\left(t_2,s\right)\right| \left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds\\
&+\frac{1}{\left|\exp\left(\int_0^T a\left(u\right) du\right)-1\right|}\int_{t_1}^{t_1+T} \left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|\\
&\times\left|\exp\left(\int_{t_2}^{s} a\left(u\right) du\right)-\exp\left(\int_{t_1}^{s} a\left(u\right) du\right)\right| ds.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_1}^{t_1+T} \left|\exp\left(\int_{t_2}^{s} a\left(u\right) du\right)-\exp\left(\int_{t_1}^{s} a\left(u\right) du\right)\right| ds\\
&=\int_{t_1}^{t_1+T} \exp\left(\int_{t_2}^{s} a\left(u\right) du\right)\left|1-\exp\left(\int_{t_1}^{t_2} a\left(u\right) du\right)\right| ds\\
&\leq T\left\|a\right\|\left|t_2-t_1\right|\exp\left(-\int_0^T a\left(u\right) du\right),
\end{aligned}$$

so

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G\left(t_2,s\right) ds\right.\\
&\left.-\int_{t_1}^{t_1+T} f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G\left(t_1,s\right) ds\right|\\
&\leq 2\beta\left|t_2-t_1\right| L\left(\rho_1+\sum_{i=1}^{n} k_i\sum_{j=0}^{i-1} M^j\right)+T\alpha\left\|a\right\|\left|t_2-t_1\right| L\left(\rho_1+\sum_{i=1}^{n} k_i\sum_{j=0}^{i-1} M^j\right)\\
&\leq \left|t_2-t_1\right| L\left(\rho_1+\sum_{i=1}^{n} k_i\sum_{j=0}^{i-1} M^j\right)\left(2\beta+\alpha T\left\|a\right\|\right).
\end{aligned} \tag{13}$$

Similarly, we have

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} a(s)\, g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G(t_2,s)\, ds\right. \\
&\left. -\int_{t_1}^{t_1+T} a(s)\, g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G(t_1,s)\, ds\right| \\
&\le \left|\int_{t_2}^{t_1} a(s)\, g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G(t_2,s)\, ds\right. \\
&\left. +\int_{t_1+T}^{t_2+T} a(s)\, g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G(t_2,s)\, ds\right| \\
&+\left|\int_{t_1}^{t_1+T} a(s)\, g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ((G(t_2,s)-G(t_1,s)))\, ds\right| \\
&\le \int_{t_2}^{t_1} |a(s)|\,|G(t_2,s)| \left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds \\
&+\int_{t_1+T}^{t_2+T} |a(s)|\,|G(t_2,s)| \left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds \\
&+\frac{1}{\exp\left(\int_0^T a(u)\, du\right)-1}\int_{t_1}^{t_1+T} |a(s)| \left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| \\
&\times\left(\exp\left(\int_{t_2}^{s} a(u)\, du\right)-\exp\left(\int_{t_1}^{s} a(u)\, du\right)\right) ds \\
&\le |t_2-t_1|\, L\delta\left(\rho_2+\sum_{i=1}^{n} c_i\sum_{j=0}^{i-1} M^j\right)(2\beta+\alpha T\,\|a\|). \qquad (14)
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\left|g\left(t_2,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_1),\varphi^{[2]}(t_1),...,\varphi^{[n]}(t_1)\right)\right| \\
&=\left|g\left(t_2,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)\right. \\
&\left.+\, g\left(t_1,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_1),\varphi^{[2]}(t_1),...,\varphi^{[n]}(t_1)\right)\right| \\
&\le\left|g\left(t_2,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)\right| \\
&+\left|g\left(t_1,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_1),\varphi^{[2]}(t_1),...,\varphi^{[n]}(t_1)\right)\right|.
\end{aligned}$$

By (4)-(6) and Lemma 2.3, we have

$$\begin{aligned}
&\left|g\left(t_2,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_1),\varphi^{[2]}(t_1),...,\varphi^{[n]}(t_1)\right)\right| \\
&\le K\,|t_2-t_1|+\sum_{i=1}^{n} c_i\left\|\varphi^{[i]}(t_2)-\varphi^{[i]}(t_1)\right\| \\
&\le\left(K+\sum_{i=1}^{n} c_i\sum_{j=0}^{i-1} M^{j+1}\right)|t_2-t_1|. \qquad (15)
\end{aligned}$$

Thus, it follows from (13)–(15) and (10) that

$$\begin{aligned}
&|(H\varphi)(t_2) - (H\varphi)(t_1)| \\
&\leq \left((2\beta + \alpha T \|a\|) L \left(\rho_1 + \delta\rho_2 + \sum_{i=1}^{n} (k_i + \delta c_i) \sum_{j=0}^{i-1} M^j \right) \right. \\
&\left. + \left(M \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \right) + K \right) |t_2 - t_1| \\
&\leq M |t_2 - t_1| .
\end{aligned}$$

Therefore,

$$|(H\varphi)(t_2) - (H\theta)(t_2)| \leq M |t_2 - t_1| ,$$

for all $t_1, t_2 \in \mathbb{R}$. Consequently, $H(P_T(L, M)) \subset P_T(L, M)$. □

Lemma 3.2. *Suppose that conditions (2)–(6), (9)and (10) hold. Then the operator $H : P_T(L, M) \longrightarrow P_T(L, M)$ given by (7) is continuous and compact.*

Proof. Since $P_T(L, M)$ is a uniformly bounded and equicontinuous subset of the space of continuous functions on the compact $[0, T]$ we can apply the Arzela-Ascoli theorem to confirm that $P_T(L, M)$ is a compact subset from this space. Also, and since any continuous operator maps compact sets into compact sets, then to show that H is a compact operator it's suffices to show that it is continuous.
For $\varphi, \theta \in P_T(L, M)$, we have

$$\begin{aligned}
&|(H\varphi)(t) - (H\theta)(t)| \\
&\leq \int_t^{t+T} |G(t,s)| \left| f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) - f\left(s, \theta(s), \theta^{[2]}(s), ..., \theta^{[n]}(s)\right) \right| ds \\
&+ \int_t^{t+T} |G(t,s)| |a(s)| \left| g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right. \\
&\left. - g\left(s, \theta(s), \theta^{[2]}(s), ..., \theta^{[n]}(s)\right) \right| ds \\
&+ \left| g\left(t, \varphi(t), \varphi^{[2]}(t), ..., \varphi^{[n]}(t)\right) - g\left(t, \theta(t), \theta^{[2]}(t), ..., \theta^{[n]}(t)\right) \right| .
\end{aligned}$$

In view of conditions (3)–(4) and notations (8), we have

$$\begin{aligned}
&|(H\varphi)(t) - (H\theta)(t)| \\
&\leq \beta T \sum_{i=1}^{n} k_i \left\| \varphi^{[i]} - \theta^{[i]} \right\| + \beta\delta T \sum_{i=1}^{n} c_i \left\| \varphi^{[i]} - \theta^{[i]} \right\| + \sum_{i=1}^{n} c_i \left\| \varphi^{[i]} - \theta^{[i]} \right\| .
\end{aligned}$$

From Lemma 2.3 it follows that

$$\begin{aligned}
&|(H\varphi)(t) - (H\theta)(t)| \\
&\leq \beta T \sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} M^j \|\varphi - \psi\| + (\beta\delta T + 1) \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|\varphi - \psi\| \\
&= \sum_{i=1}^{n} (\beta T k_i + (\beta\delta T + 1) c_i) \sum_{j=0}^{i-1} M^j \|\varphi - \psi\| ,
\end{aligned}$$

which shows that the operator H is continuous. Therefore, H is compact. □

Theorem 3.3. *Suppose (2)–(6), (9) and (10) hold, then equation (1) has at last a solution* $x \in P_T(L, M)$.

Proof. From Lemma 2.1, we see that equation (1) has at last a solution x on $P_T(L, M)$ if and only if the operator H defined by (7) has a fixed point. From Lemma 3.2 and Lemma 3.1 all the conditions of Schauder's theorem 2.4 are satisfied. Consequently H has a fixed point on $P_T(L, M)$ and this fixed point is a solution of equation (1). □

3.2. **Uniqueness of the solution of equations (1).** In this section we use the principle of contraction mappings to prove the uniqueness of the solution of the equation (1).

Theorem 3.4. *Suppose (2)–(6), (9) and (10) hold. If*

$$\sum_{i=1}^{n} (\beta T k_i + (\beta \delta T + 1) c_i) \sum_{j=0}^{i-1} M^j < 1, \tag{16}$$

then equation (1) has a unique solution $x \in P_T(L, M)$.

Proof. Similarly as in the proof of lemma 3.2 and by using (16), for $\varphi, \theta \in P_T(L, M)$ we have

$$\begin{aligned} &|(H\varphi)(t) - (H\theta)(t)| \\ &\leq \sum_{i=1}^{n} (\beta T k_i + (\beta \delta T + 1) c_i) \sum_{j=0}^{i-1} M^j \|\varphi - \psi\|. \end{aligned}$$

By the principle of contraction mapping, H has a unique fixed point on $P_T(L, M)$ and in view of Lemma 2.1 this fixed point is a solution of equation (1). □

Acknowledgement. The authors would like to thank the anonymous referee for his valuable comments.

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