

CERTAIN INTEGRAL ASSOCIATED WITH GENERALIZED WHITTAKER FUNCTION

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ABSTRACT. In this draft, we establish certain unified integral formulae involving the product of the generalized Whittaker functions $M_{\kappa,(\mu_j)}[z_1, \dots, z_r]$, $W_{\kappa,(\mu_j)}[z_1, \dots, z_r]$ and the Srivastava polynomials $S_n^m[x]$. The result presented here is much more fabricating from the previous well known results.

1. OVERTURE

In this write-up, we consider $\mathbb{Z}_0^-, \mathbb{R}_0^+, \mathbb{N}$ and \mathbb{C} to be the sets of non-positive integers, non-negative real numbers, positive integers and complex numbers respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. At the beginning of the twentieth century (1904), an English mathematician Edmund Taylor Whittaker introduced a pair of confluent hypergeometric functions that now displays his name. The Whittaker functions arise as solutions of the confluent hypergeometric equation after a transformation to Liouville's standard form of the differential equation [2]. Whittaker function $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ [23] defined by

$$M_{\kappa,\mu}(z) = z^{\mu+1/2} \exp\left(-\frac{1}{2}z\right) \Phi\left(\frac{1}{2} - \kappa + \mu, 2\mu + 1; z\right), |arg| < \pi \quad (1)$$

$$W_{\kappa,\mu}(z) = z^{\mu+1/2} \exp\left(-\frac{1}{2}z\right) \Psi\left(\frac{1}{2} - \kappa + \mu, 2\mu + 1; z\right), |arg| < \pi \quad (2)$$

where $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are Whittaker's functions of first kind and second kind respectively. Further, in 1920 a French mathematician Humbert, P. developed a multivariable extension of Whittaker's function [8, p. 429] defined by

$$M_{\kappa,\mu_1,\dots,\mu_r}(x_1, \dots, x_r) = x_1^{\mu_1+1/2} \dots x_r^{\mu_r+1/2} \exp\left[-\frac{1}{2}(x_1 + \dots + x_r)\right] \\ \times \Psi_2^{(r)}\left[\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2}r; 2\mu_1 + 1, \dots, 2\mu_r + 1; x_1, \dots, x_r\right], \quad (3)$$

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where $\Psi_2^{(r)}$ denotes Humbert's confluent hypergeometric function of r variables.

In (1966, 1967) a French mathematician Hervé Jacquet established Whittaker functions of reductive groups over local fields where the functions studied by Whittaker are essentially the case where the local field is the real numbers and the group is $SL_2(\mathbf{R})$, i.e., special linear group [9]. Whittaker functions occur naturally in different disciplines in physical, biological, mathematical modelling and social sciences, such as input-output situations in Econometric problems, storage-consumption situations, Growth-decay situations, when dealing with bilinear forms in random variables, and so on [13]. In particular, it has various applications in many areas of mathematical physics including studies of the Coulomb Greens function [4], analysis of the Schrodinger equations [7], studies of the spectral evolution resulting from the Compton scattering of radiation by hot electrons [10, 21, 3], modeling of the structure of the hydrogen atom [24] and analysis of fluctuations in financial markets [12] and statistics. It also arises in Equations (6.6) (6.9) (6.15) (6.16) (6.18) of Tungs (1983) paper on the study of fluid flow [22]. We also look forward in the book [1] in which some functions are particular cases of Whittaker's function, e.g., error function, incomplete Gamma function, logarithmic integral, integral sine and cosine, Fresnel integrals and exponential integral and so on has been described. Parabolic cylinder functions are also particular cases of the Whittaker functions [23].

Srivastava [17, p. 1, Eq. (1)] established the following very general polynomials $S_n^m[x]$ defined by

$$S_n^m[x] = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}}{k!} A_{n,k} \quad (n \in \mathbb{N}_0; m \in \mathbb{N}), \quad (4)$$

where the coefficients $A_{n,k}$ ($n, k \in \mathbb{N}_0$) are arbitrary constants, real or complex.

In fact, by suitably specializing the coefficients $A_{n,k}$ the polynomials $S_n^m[x]$ can be reduced to the classical orthogonal polynomials including, for example, the Hermite polynomials, Jacobi polynomials, the Laguerre polynomials. On the other hand, the polynomials $S_n^m[x]$ defined by (4) can be reduced to several special cases of the Jacobi polynomials, e.g., the Gegenbauer (or ultraspherical) polynomials, the relatively more familiar Legendre polynomials and the Tchebycheff polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds. [20].

Other interesting special cases of the polynomials $S_n^m[x]$ include such generalized hypergeometric polynomials as the Bessel polynomials, the Gould-Hopper polynomials, Brafman polynomials (see, e.g., [20]).

The following quite well known integral formula is employed in this write up (see, e.g., [19, p. 275, Eq. (3)]):

$$\int \cdots \int u_1^{\alpha_1-1} \cdots u_n^{\alpha_n-1} (1 - u_1 - \cdots - u_n)^{\beta-1} du_1 \cdots du_n = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n) \Gamma(\beta)}{\Gamma(\alpha_1 + \cdots + \alpha_n + \beta)} \quad (5)$$

($u_j \in \mathbb{R}_0^+$ with $u_1 + \cdots + u_n \leq 1$ and $\Re(\alpha_j) > 0$ $j \in \{1, \dots, n\}$; $\Re(\beta) > 0$).

The Psi (or Digamma) function $\psi(z)$ is defined by (see, e.g., [18, Section 1.3])

$$\psi(z) = \frac{d}{dz} \{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_1^z \psi(t) dt. \quad (6)$$

Integral formulas associated with Whittaker functions have been developed by many authors (see, e.g., [5, 11, 6, 15]). In this draft, we establish certain new

generalized integral formulae including the product of the generalized Whittaker functions $M_{\kappa,(\mu_j)}[x_1^{\rho_1}(1-x_1-\dots-x_r)^{\delta_1}, \dots, x_r^{\rho_r}(1-x_1-\dots-x_r)^{\delta_r}]$ and the Srivastava's polynomials $S_n^m[x]$.

The main result presented here is general enough to be specialized to yield many interesting integral formulas involving the product of familiar polynomials and known hypergeometric series in one and several variables, some of which are demonstrated.

2. MAIN RESULTS

This section demonstrates our main result of the draft by presenting the integral formula stated in Theorem 2.1 underneath

Theorem 2.1. *Let $t, s_j, \lambda_j, \rho_j, \delta_j, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$ and*

$$\min\{s_j, \lambda_j, \rho_j, \delta_j, \mu_j\} \text{ and } j \in \{1, \dots, r\}.$$

Also, let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned} & \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \dots x_r^{s_r-1} (1-x_1-\dots-x_r)^{t-1} \\ & \times S_n^m[x_1^{\lambda_1} \dots x_r^{\lambda_r} (1-x_1-\dots-x_r)^\lambda] M_{\kappa,(\mu_j)}[x_1^{\rho_1}(1-x_1-\dots-x_r)^{\delta_1}, \\ & \dots, x_r^{\rho_r}(1-x_1-\dots-x_r)^{\delta_r}] dx_r \dots dx_1 \\ & = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2}r)_{k_1+\dots+k_r}}{(2\mu_1+1)_{k_1} \dots (2\mu_r+1)_{k_r}} \\ & \times \frac{1}{k_1! \dots k_r!} \exp \left[-\frac{1}{2} \sum_{j=1}^r x_j^{\rho_j} (1-x_1-\dots-x_r)^{\delta_j} \right] \\ & \times \left[\Gamma(s_1 + \lambda_1 k + \rho_1 k_1 + \rho_1 \mu_1 + \frac{\rho_1}{2}) \dots \Gamma(s_r + \lambda_r k + \rho_r k_r + \rho_r \mu_r + \frac{\rho_r}{2}) \right. \\ & \left. \frac{\Gamma(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2})}{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)} \right] \end{aligned} \quad (7)$$

where $A_{n,k}$ are arbitrary constants, real or complex, and $M_{\kappa,(\mu_j)}[\cdot]$ and $S_n^m[\cdot]$ are given as in (3) and (4), respectively.

Proof. Substituting the defined series of $M_{\kappa,(\mu_j)}[\cdot]$ and $S_n^m[\cdot]$ in the integrand and then interchanging integral signs and summations, which may be verified under the given conditions, and finally, evaluating the following integral:

$$\begin{aligned} & \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \dots x_r^{s_r-1} (1-x_1-\dots-x_r)^{t-1} \\ & \times S_n^m[x_1^{\lambda_1} \dots x_r^{\lambda_r} (1-x_1-\dots-x_r)^\lambda] M_{\kappa,(\mu_j)}[x_1^{\rho_1}(1-x_1-\dots-x_r)^{\delta_1}, \dots \\ & \dots, x_r^{\rho_r}(1-x_1-\dots-x_r)^{\delta_r}] dx_r \dots dx_1 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \dots x_r^{s_r-1} (1-x_1-\dots-x_r)^{t-1} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} \\
 &\quad \times x_1^{\lambda_1 k} \dots x_r^{\lambda_r k} (1-x_1-\dots-x_r)^{\lambda k} x_1^{\rho_1 \mu_1 + \frac{\rho_1}{2}} (1-x_1-\dots-x_r)^{\delta_1 \mu_1 + \frac{\delta_1}{2}} \dots \\
 &\quad \dots x_r^{\rho_r \mu_r + \frac{\rho_r}{2}} (1-x_1-\dots-x_r)^{\delta_r \mu_r + \frac{\delta_r}{2}} \exp \left[\frac{-1}{2} \left\{ x_1^{\rho_1} (1-x_1-\dots-x_r)^{\delta_1} + \right. \right. \\
 &\quad \left. \left. \dots + x_r^{\rho_r} (1-x_1-\dots-x_r)^{\delta_r} \right\} \right] \Psi_2^{(r)} \left[\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2} r; 2\mu_1 + 1, \dots, 2\mu_r + 1; \right. \\
 &\quad \left. x_1^{\rho_1} (1-x_1-\dots-x_r)^{\delta_1}, \dots, x_r^{\rho_r} (1-x_1-\dots-x_r)^{\delta_r} \right] dx_r \dots dx_1 \\
 &= \sum_{k=0}^{[n/m]} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2} r)_{k_1 + \dots + k_r}}{(2\mu_1 + 1)_{k_1} \dots (2\mu_r + 1)_{k_r}} \\
 &\quad \times \frac{1}{k_1! \dots k_r!} \exp \left[-\frac{1}{2} \sum_{j=1}^r x_j^{\rho_j} (1-x_1-\dots-x_r)^{\delta_j} \right] \\
 &\quad \times \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1 + \lambda_1 k + \rho_1 \mu_1 + \rho_1 k_1 + \frac{\rho_1}{2} - 1} \dots x_r^{s_r + \lambda_r k + \rho_r \mu_r + \rho_r k_r + \frac{\rho_r}{2} - 1} \\
 &\quad \times (1-x_1-\dots-x_r)^{t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2} - 1} dx_r \dots dx_1 \\
 &= \sum_{k=0}^{[n/m]} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2} r)_{k_1 + \dots + k_r}}{(2\mu_1 + 1)_{k_1} \dots (2\mu_r + 1)_{k_r}} \frac{1}{k_1! \dots k_r!} \\
 &\quad \times \exp \left[-\frac{1}{2} \sum_{j=1}^r x_j^{\rho_j} (1-x_1-\dots-x_r)^{\delta_j} \right] \\
 &\quad \times \left[\Gamma(s_1 + \lambda_1 k + \rho_1 \mu_1 + \rho_1 k_1 + \frac{\rho_1}{2}) \dots \Gamma(s_r + \lambda_r k + \rho_r \mu_r + \rho_r k_r + \frac{\rho_r}{2}) \right. \\
 &\quad \left. \frac{\Gamma(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2})}{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)} \right] \tag{8}
 \end{aligned}$$

On differentiating partially of both sides of (8) with respect to the variables which are suitably chosen and combined from the parameters s_1, \dots, s_r , and t , among many integral formulas given in Corollaries (2.2) and (2.3). \square

Corollary 2.2. *Let $t, s_j, \lambda_j, \rho_j, \delta_j, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$ and*

$$\min\{s_j, \lambda_j, \rho_j, \delta_j, \mu_j\} \text{ and } j \in \{1, \dots, r\}.$$

Also, let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1-1} \cdot \log(x_1) \cdot x_2^{s_2-1} \dots x_r^{s_r-1} (1-x_1-\dots-x_r)^{t-1}$$

$$\begin{aligned}
& \times S_n^m [x_1^{\lambda_1} \dots x_r^{\lambda_r} (1 - x_1 - \dots - x_r)^\lambda] M_{\kappa, (\mu_j)} [x_1^{\rho_1} (1 - x_1 - \dots - x_r)^{\delta_1}, \dots \\
& \quad \dots, x_r^{\rho_r} (1 - x_1 - \dots - x_r)^{\delta_r}] dx_r \dots dx_1 \\
& = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2}r)_{k_1 + \dots + k_r}}{(2\mu_1 + 1)_{k_1} \dots (2\mu_r + 1)_{k_r}} \\
& \quad \times \frac{1}{k_1! \dots k_r!} \exp \left[-\frac{1}{2} \sum_{j=1}^r x_j^{\rho_j} (1 - x_1 - \dots - x_r)^{\delta_j} \right] \\
& \quad \times \left[\frac{\Gamma(s_1 + \lambda_1 k + \rho_1 \mu_1 + \rho_1 k_1 + \frac{\rho_1}{2}) \dots \Gamma(s_r + \lambda_r k + \rho_r \mu_r + \rho_r k_r + \frac{\rho_r}{2})}{\Gamma(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2})} \right. \\
& \quad \left. \frac{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)}{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)} \right] \\
& \quad \times \left[\Psi \left(s_1 + \lambda_1 k + \rho_1 \mu_1 + \rho_1 k_1 + \frac{\rho_1}{2} \right) \right. \\
& \quad \left. - \Psi \left(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k \right) \right], \tag{9}
\end{aligned}$$

where $A_{n,k}$ are arbitrary constants, real or complex, and $M_{\kappa, (\mu_j)}[\cdot]$ and $S_n^m[\cdot]$ are given as in (3) and (4), respectively and $\Psi(\cdot)$ function in (6).

Corollary 2.3. Let $t, s_j, \lambda_j, \rho_j, \delta_j, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$ and

$$\min\{s_j, \lambda_j, \rho_j, \delta_j, \mu_j\} \text{ and } j \in \{1, \dots, r\}.$$

Also, let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned}
& \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \dots x_r^{s_r-1} (1 - x_1 - \dots - x_r)^{t-1} \\
& \times \log(1 - x_1 - \dots - x_r) \cdot S_n^m [x_1^{\lambda_1} \dots x_r^{\lambda_r} (1 - x_1 - \dots - x_r)^\lambda] \\
& \times M_{\kappa, (\mu_j)} [x_1^{\rho_1} (1 - x_1 - \dots - x_r)^{\delta_1}, \dots, x_r^{\rho_r} (1 - x_1 - \dots - x_r)^{\delta_r}] dx_r \dots dx_1 \\
& = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2}r)_{k_1 + \dots + k_r}}{(2\mu_1 + 1)_{k_1} \dots (2\mu_r + 1)_{k_r}} \\
& \quad \times \frac{1}{k_1! \dots k_r!} \exp \left[-\frac{1}{2} \sum_{j=1}^r x_j^{\rho_j} (1 - x_1 - \dots - x_r)^{\delta_j} \right] \\
& \quad \times \left[\frac{\Gamma(s_1 + \lambda_1 k + \rho_1 \mu_1 + \rho_1 k_1 + \frac{\rho_1}{2}) \dots \Gamma(s_r + \lambda_r k + \rho_r \mu_r + \rho_r k_r + \frac{\rho_r}{2})}{\Gamma(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2})} \right. \\
& \quad \left. \frac{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)}{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)} \right]
\end{aligned}$$

$$\begin{aligned} & \times \left[\Psi \left(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2} \right) \right. \\ & \quad \left. - \Psi \left(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k \right) \right], \end{aligned}$$

where $A_{n,k}$ are arbitrary constants, real or complex, and $M_{\kappa,(\mu_j)}[\cdot]$ and $S_n^m[\cdot]$ are given as in (3) and (4), respectively and $\Psi(\cdot)$ function in (6).

Proof. Differentiating both sides of (8) with respect to the parameters s_1 and t , respectively, gives the formulas (2.2) and (2.3). □

3. PARTICULAR CASES AND REMARKS

Since the polynomials $A_{n,k}$ in (4) and the generalized Whittaker function $M_{\kappa,(\mu_j)}[\cdot]$ in (3) are very general, the main result (2.1) can be specialized to yield a large number of integral formulas involving familiar polynomials and special functions, five of which are demonstrated as in the following examples. Choosing $m = 2$ and $A_{n,k} = (-1)^k$ in (4), the polynomials $S_n^2(x)$ become the Hermite polynomials $H_n(x)$ (see [20]; see also [14, p. 187]):

$$S_n^2(x) = x^{n/2} H_n \left(\frac{1}{2\sqrt{x}} \right). \tag{10}$$

Example 3.1. Let $t, s_j, \lambda_j, \rho_j, \delta_j, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$

and $\min\{s_j, \lambda_j, \rho_j, \delta_j, \mu_j\}$ and $j \in \{1, \dots, r\}$.

Also, let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned} & \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \dots x_r^{s_r-1} (1-x_1-\dots-x_r)^{t-1} \\ & \times [x_1^{\lambda_1} \dots x_r^{\lambda_r} (1-x_1-\dots-x_r)^\lambda] H_n \left(\frac{1}{x_1^{\lambda_1} \dots x_r^{\lambda_r} (1-x_1-\dots-x_r)^\lambda} \right) \\ & \times M_{\kappa,(\mu_j)} [x_1^{\rho_1} (1-x_1-\dots-x_r)^{\delta_1}, \dots, x_r^{\rho_r} (1-x_1-\dots-x_r)^{\delta_r}] dx_r \dots dx_1 \\ & = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-1)^k (-n)_{2k}}{k!} A_{n,k} \frac{(\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2}r)_{k_1+\dots+k_r}}{(2\mu_1+1)_{k_1} \dots (2\mu_r+1)_{k_r}} \\ & \quad \times \frac{1}{k_1! \dots k_r!} \exp \left[-\frac{1}{2} \sum_{j=1}^r x_j^{\rho_j} (1-x_1-\dots-x_r)^{\delta_j} \right] \\ & \quad \times \left[\Gamma(s_1 + \lambda_1 k + \rho_1 \mu_1 + \rho_1 k_1 + \frac{\rho_1}{2}) \dots \Gamma(s_r + \lambda_r k + \rho_r \mu_r + \rho_r k_r + \frac{\rho_r}{2}) \right. \\ & \quad \left. \frac{\Gamma(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2})}{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)} \right] \end{aligned}$$

where $M_{\kappa,(\mu_j)}[\cdot]$ are given as in (3) and $H_n(x)$ are Hermite polynomials.

Setting $m = 1$ and $A_{n,k} = \binom{n+\alpha}{n} \frac{1}{(\alpha+1)_k}$ in (4), we have

$$S_n^1(x) = L_n^{(\alpha)}(x) \quad (11)$$

where $L_n^{(\alpha)}(x)$ are Laguerre polynomial (see[20], see also [14, p. 200]). Applying (11) in the main result (2.1) yields an integral formula asserted by the following example:

Example 3.2. Let $t, s_j, \lambda_j, \rho_j, \delta_j, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$

and $\min\{s_j, \lambda_j, \rho_j, \delta_j, \mu_j\}$ and $j \in \{1, \dots, r\}$.

Also, let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned} & \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \dots x_r^{s_r-1} (1-x_1-\dots-x_r)^{t-1} \\ & \times L_n^{(\alpha)}(x_1^{\lambda_1} \dots x_r^{\lambda_r} (1-x_1-\dots-x_r)^\lambda) M_{\kappa,(\mu_j)}[x_1^{\rho_1} (1-x_1-\dots-x_r)^{\delta_1}, \dots \\ & \dots x_r^{\rho_r} (1-x_1-\dots-x_r)^{\delta_r}] dx_r \dots dx_1 \\ & = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k k!} \binom{n+\alpha}{\alpha} \frac{(\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2}r)_{k_1+\dots+k_r}}{(2\mu_1+1)_{k_1} \dots (2\mu_r+1)_{k_r}} \\ & \times \frac{1}{k_1! \dots k_r!} \exp \left[-\frac{1}{2} \sum_{j=1}^r x_j^{\rho_j} (1-x_1-\dots-x_r)^{\delta_j} \right] \\ & \times \left[\frac{\Gamma(s_1 + \lambda_1 k + \rho_1 \mu_1 + \rho_1 k_1 + \frac{\rho_1}{2}) \dots \Gamma(s_r + \lambda_r k + \rho_r \mu_r + \rho_r k_r + \frac{\rho_r}{2})}{\Gamma(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2})} \right] \\ & \left[\frac{\Gamma(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2})}{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)} \right] \end{aligned}$$

where $M_{\kappa,(\mu_j)}[\cdot]$ are given as in (3) and $L_n^{(\alpha)}(x)$ are Laguerre polynomials.

Substituting the value of $m = 1$ and $A_{n,k} = \binom{n+\alpha}{n} \frac{(\alpha+\beta+n+1)_k}{(\alpha+1)_k}$ in (4), we have

$$S_n^1[x] = P_n^{(\alpha,\beta)}(1-2x), \quad (12)$$

where $P_n^{(\alpha,\beta)}(1-2x)$ are Jacobi polynomials (see [20]; see also [14, p. 254]). Applying equation (12) to the main result (2.1) yields an integral formula claimed by the following exemplification:

Example 3.3. Let $t, s_j, \lambda_j, \rho_j, \delta_j, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$

and $\min\{s_j, \lambda_j, \rho_j, \delta_j, \mu_j\}$ and $j \in \{1, \dots, r\}$.

Also, let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned} & \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \dots x_r^{s_r-1} (1-x_1-\dots-x_r)^{t-1} \\ & \times P_n^{(\alpha,\beta)} \left(1-2 \{ x_1^{\lambda_1} \dots x_r^{\lambda_r} (1-x_1-\dots-x_r)^\lambda \} \right) M_{\kappa,(\mu_j)}[x_1^{\rho_1} (1-x_1-\dots-x_r)^{\delta_1}, \dots \end{aligned}$$

$$\begin{aligned} & \dots, x_r^{\rho_r} (1 - x_1 - \dots - x_r)^{\delta_r} dx_r \dots dx_1 \\ = & \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-n)_k}{k!} \binom{n + \alpha}{\alpha} \frac{(\alpha + \beta + n + 1)_k}{(\alpha + 1)_k} \frac{(\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2}r)_{k_1 + \dots + k_r}}{(2\mu_1 + 1)_{k_1} \dots (2\mu_r + 1)_{k_r}} \\ & \times \frac{1}{k_1! \dots k_r!} \exp \left[-\frac{1}{2} \sum_{j=1}^r x_j^{\rho_j} (1 - x_1 - \dots - x_r)^{\delta_j} \right] \\ & \times \left[\Gamma(s_1 + \lambda_1 k + \rho_1 \mu_1 + \rho_1 k_1 + \frac{\rho_1}{2}) \dots \Gamma(s_r + \lambda_r k + \rho_r \mu_r + \rho_r k_r + \frac{\rho_r}{2}) \right. \\ & \quad \left. \frac{\Gamma(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2})}{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)} \right] \end{aligned}$$

where $M_{\kappa, (\mu_j)}[\cdot]$ are given as in (3) and $P_n^{(\alpha, \beta)}(x)$ are Jacobi polynomials.

Also, we can get Bessel polynomials by setting $m = 1$ and $A_{n,k} = (\alpha + n - 1)_k$ in (3), we have

$$S_n^1[x] = y_n(-\beta x, \alpha, \beta), \tag{13}$$

where $y_n(-\beta x, \alpha, \beta)$ are Bessel polynomials (see [20]; see also [14, p. 294]). Applying (13) to the main result (2.1) yields an integral formula claimed by the following exemplification:

Example 3.4. Let $t, s_j, \lambda_j, \rho_j, \delta_j, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$

and $\min\{s_j, \lambda_j, \rho_j, \delta_j, \mu_j\}$ and $j \in \{1, \dots, r\}$.

Also, let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned} & \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \dots x_r^{s_r-1} (1 - x_1 - \dots - x_r)^{t-1} \\ & \times y_n(-\beta \{x_1^{\lambda_1} \dots x_r^{\lambda_r} (1 - x_1 - \dots - x_r)^\lambda\}, \alpha, \beta) M_{\kappa, (\mu_j)}[x_1^{\rho_1} (1 - x_1 - \dots - x_r)^{\delta_1}, \dots \\ & \dots, x_r^{\rho_r} (1 - x_1 - \dots - x_r)^{\delta_r}] dx_r \dots dx_1 \\ = & \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-n)_k}{k!} (\alpha + n - 1)_k \frac{(\mu_1 + \dots + \mu_r - \kappa + \frac{1}{2}r)_{k_1 + \dots + k_r}}{(2\mu_1 + 1)_{k_1} \dots (2\mu_r + 1)_{k_r}} \\ & \times \frac{1}{k_1! \dots k_r!} \exp \left[-\frac{1}{2} \sum_{j=1}^r x_j^{\rho_j} (1 - x_1 - \dots - x_r)^{\delta_j} \right] \\ & \times \left[\Gamma(s_1 + \lambda_1 k + \rho_1 \mu_1 + \rho_1 k_1 + \frac{\rho_1}{2}) \dots \Gamma(s_r + \lambda_r k + \rho_r \mu_r + \rho_r k_r + \frac{\rho_r}{2}) \right. \\ & \quad \left. \frac{\Gamma(t + \lambda k + \delta_1 \mu_1 + \dots + \delta_r \mu_r + \delta_1 k_1 + \dots + \delta_r k_r + \frac{\delta_1}{2} + \dots + \frac{\delta_r}{2})}{\Gamma(\sum_{j=1}^r (s_j + \lambda_j k + \rho_j k_j + \rho_j \mu_j + \frac{\rho_j}{2} + \frac{\delta_j}{2} + \delta_j \mu_j + \delta_j k_j) + t + \lambda k)} \right] \end{aligned}$$

where $M_{\kappa, (\mu_j)}[\cdot]$ are given as in (3) and $y_n(-\beta x, \alpha, \beta)$ are Bessel polynomials.

Substituting $r = 1$ in the main result (2.1), after somewhat simplification, yields a simpler integral formula as in Example(3.5).

Example 3.5. Consider $t, s, \alpha, \rho, \delta, \mu \in \mathbb{C}$ with

$$\min\{\Re(t), \Re(s), \Re(\alpha), \Re(\rho), \Re(\delta), \Re(\mu)\}$$

and $l \in \mathbb{N}$. Aso, suppose that $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned} & \int_0^1 x^{s-1}(1-x)^{t-1} S_n^m [x^\alpha(1-x)^\lambda] M_{\kappa, (\mu_j)} [x^\rho(1-x)^\delta] dx \\ &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(-\frac{1}{2})(\mu - \kappa + \frac{1}{2})}{(2\mu + 1)_j j! l!} \\ & \quad \times \beta \left(t + \lambda k + \delta \mu + \delta l + \delta j + \frac{\delta}{2}, s + \alpha k + \rho l + \rho \mu + \rho j + \frac{\rho}{2} \right), \end{aligned}$$

where $A_{n,k}$ are arbitrary constants, real or complex, $S_n^m[\cdot]$ are given in (4), and $\beta(\cdot, \cdot)$ is the Beta function (see, e.g., [19], p. 7).

The main result (2.1) is sufficient to be further specialized to derive many integral formulas involving the product of diverse known polynomials and various known hypergeometric series in one or several variables.

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