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ASYMPTOTIC BEHAVIOR OF A CLASS OF IMPULSIVE PARTIAL STOCHASTIC FUNCTIONAL NEUTRAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY

KORA HAFIZ BETE, AZIZ MANE,CARLOS OGOUYANDJOU AND MAMADOU ABDOUL DIOP

Abstract

This paper is devoted to the existence and asymptotic behavior in p-th moment of the mild solution to a class of impulsive neutral stochastic functional integro-differential equations with infinite delay in Hilbert spaces. A new and sufficient set of conditions are formulated concerning the existence of solutions and the stability of the nonlinear stochastic system. To obtain the desired result, the theory of the resolvent operator in the sense of Grimmer, the stochastic analysis theory, the fixed point theorem and the Hausdorff measure of non-compactness are used. However, it is very important to specify that in this paper, we have left the classical framework in which the nonlinear terms are assumed to be Lipschitz continuous. At the end of this paper, an illustration is also given to show the application of our results.

1. Introduction

Stochastic differential equations have attracted great interest because of their applications in characterizing many problems in physics, mecanics, electrical engineering, biology, ecology and so on. On this matter, we refer the reader to [19, 20, 23] and references therein. In particular, integro-differential equations arise in the mathematical modeling of several natural phenomena and various investigations led to the exploration of their different aspects. The theory of semigroups of bounded linear operators is closely related to the solution of differential and integro-differential equations in Banach spaces. In recent years, this theory has been applied to a large class of nonlinear differential equations in Banach spaces. The existence, uniqueness, stability, invariant measures and other quantitative and qualitative properties of solutions to stochastic partial integrodifferential equations have been extensively considered by many authors see [19, 23] for details. Based on the method of semigroups, the existence and uniqueness of mild, strong and classical

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solutions of semilinear integro-differential evolution equations were discussed by Ezzinbi et al. [6] in the deterministic case and Diop et al. [5] in the stochastic case.

In addition to stochastic effects, impulsive effects have become more important in some mathematical models of real phenomena. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes. That is why the perturbations are considered to take place instantaneously in the form of impulses. The theory of impulsive differential and integro-differential equations have seen considerable development, see [15, 30]. With regard to this topic, see the monographs of Lakshmikanthan and al. [15], Hale and Lunel [13]. On the other hand, there has been intense interest in the study of impulsive neutral stochastic partial integro-differential equations with resolvent operator in the sense of Grimmer. Moreover, it is very important to note that, compared to the finite delay, the problems with infinite delay is clearly more complicated, because the properties of solutions depend on the choice of the phase space $\mathcal{B}$ which is proposed by Hale and Kato [11]. For the fundamental theory related to functional differential equations with infinite delay, one can see [14].

In recent years, the existence of solution for partial neutral integro-differential equation with infinite delay in infinite dimensional spaces has been extensively studied by many authors. Ezzinbi and al. [7] investigated the existence and regularity of solutions for some partial functional integrodifferential equations in Banach spaces. More recently, Cui and Yan [4] investigated the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces.

In particular, the stability theory of stochastic differential equations has been popularly applied in variety fields of science and technology. Several authors have established the stability results of mild solutions for these equations by using various techniques. Govindan [8] considered the existence and stability for mild solution of stochastic partial differential equations by applying the comparison theorem. Caraballo and Liu [3] proved the exponential stability for mild solution to stochastic partial differential equations with delays by utilizing the well-known Gronwall inequality. The exponential stability of the mild solutions for semilinear stochastic delay evolution equations have been discussed by using Lyapunov functionals in [17]. The author in [16] considered the exponential stability for stochastic partial functional differential equations by means of the Razuminkhin-type theorem. Taniguchi [6] discussed the exponential stability for stochastic delay differential equations by the energy inequality. Using fixed point approach, Luo [18] studied the asymptotic stability of mild solutions of stochastic partial differential equations with infinite delays. Further, Sakthivel et al.[27, 28, 29] established the asymptotic stability and exponential stability of second-order stochastic evolution equations in Hilbert spaces.

But it seem that little is known about asymptotic behavior of impulsive partial stochastic functional neutral integro-differential equations with infinite delay, and the aim of this paper is to fill this gap.

Motivated by the above works, the purpose of this paper is to study the existence and asymptotic stability of mild solutions for a class of impulsive neutral partial stochastic functional integro-differential equations with infinite delay in Hilbert spaces of the form

$$
\begin{aligned}
d[x(t) - g(t, x(t-\rho_1(t)))] = & \Bigg[A[x(t) - g(t, x(t-\rho_1(t)))] \\
& + \int_0^t B(t-s)[x(s) - g(s, x(s-\rho_1(s)))]ds \\
& + h\left(t, x(t-\rho_2(t)), \int_0^t a(t, s, x(s-\rho_3(s)))ds\right)\Bigg] dt \qquad (1) \\
& + f\left(t, x(t-\rho_4(t)), \int_0^t b(t, s, x(s-\rho_5(s)))ds\right) dw(t), \quad t \geq 0,\ t \neq t_k, \\
\Delta x(t_k) = & I_k(x(t_k^-)), \quad t = t_k,\ k = 1, \cdots, m, \\
x_0(\cdot) = & \varphi \in \mathcal{B}_{\mathcal{F}_0}([\tilde{m}(0), 0], H),
\end{aligned}
$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$; and norm $\| \cdot \|_H$. The operator $A : D(A) \to H$ is the infinitesimal generator of a strongly continuous semigroup on H, and $B(t-s)$ is a closed linear operator with domain at least $D(A)$. Let K be another separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_K$ and norm $\| \cdot \|_K$. Suppose $\{w(t),\ t \geq 0\}$ is a given K-valued Wiener process with a covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$, which is generated by the Wiener process w, and $g : [0, +\infty) \times H \to H$; $a, b : [0, +\infty) \times [0, +\infty) \times H \to H$; $f : [0, +\infty) \times H \times H \to L(K, H)$, are all Borel measurable, where $L(K; H)$ denotes the space of all bounded linear operators from K into H; $I_k : H \to H$ $(k = 1, \cdots, m)$, is a given function. Moreover, $t_k,\ k \in \overline{1, m}$ are fixed moments of time and satisfy $0 < t_1 < \cdots < t_m < \lim_{k \to \infty} t_k = \infty,\ x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump in the state x at time t_k with I_k, J_k determining the size of the jump; let $\rho_i(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy $t - \rho_i(t) \to \infty$ as $t \to \infty$, and $\tilde{m}(0) = \max\{\inf_{s \geq 0}(s - \rho_i(s)),\ i = 1, 2, 3, 4, 5\}$. Here $\mathcal{B}_{\mathcal{F}_0}([\tilde{m}(0), 0], H)$ denote the family of all almost surely bounded, $\mathcal{F}_0$-measurable, continuous random variables $\varphi(t) : [\tilde{m}(0), 0] \to H$ with norm $\|\varphi\|_{\mathcal{B}} = \sup_{\tilde{m}(0) \leq t \leq 0} \mathbb{E}\|\varphi(t)\|_H$.

To the best of the authors knowledge, there is no result about the existence and asymptotic stability of mild solutions for this class of impulsive neutral partial stochastic functional integrodifferential equations with infinite delay, which is expressed in the form (1). Using mainly the theory of resolvent operator in the sense of Grimmer, the Hölder's inequality, stochastic analysis, the Darbo fixed point theorem combined with techniques of the Hausdorff measure of noncompactness, we get the existence and asymptotic stability of mild solutions for system (1).

It is also very important to clarify that the most common and easily verified conditions to guarantee the existence and stability of mild solutions are the impulsive stochastic systems for which the nonlinear function is a Lipschitz function, but in this paper, we will not suppose these classical Lipschitz conditions. In fact, we assume that the nonlinear items f, h are continuous functions while the neutral item g satisfies the generally Lipschitz continuity condition, and some suitable conditions on the above-defined functions, which can make the solution operator satisfies all conditions of the Darbo fixed point theorem.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and the theory of the resolvent operator in the sense of Grimmer which will be used to establish the existence of our mild solution. In Section 3, we give the main results of this paper. In Section 4, an illustration is given to show the application of the obtained theory.

2. Preliminaries

Let K and H be two real separable Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_K$ and $\langle\cdot,\cdot\rangle_H$ and $\|\cdot\|_K$ and $\|\cdot\|_H$, their vector norms respectively.

Let $(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{F})$ $(\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0})$ be a complete probability space satisfying that $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued Brownian motion with a trace class operator Q, denote $Tr(Q) = \sum\limits_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum\limits_{i=1}^{\infty} \sqrt{\lambda_i} w_i(t) e_i$, where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Brownian motions. Then, the above K-valued stochastic process $w(t)$ is called a Q-Wiener process. Let $L(K, H)$ be the space of bounded linear operators mapping K into H equipped with the usual norm $\|\cdot\|_H$ and $L(H)$ denotes the Hilbert space of bounded linear operators from H to H. For $\tilde{\psi} \in L(K, H)$ we define

$$\|\tilde{\psi}\|^2_{L_2^0} = Tr(\tilde{\psi} Q \tilde{\psi}^*) = \sum_{i=1}^{\infty} \|\sqrt{\lambda_i}\tilde{\psi} e_i\|^2.$$

If $\|\tilde{\psi}\|^2_{L_2^0} < \infty$ then $\tilde{\psi}$ is called a Q-Hilbert-Schmidt operator, and let $L_2^0(K, H)$ denote the space of all Q-Hilber-Schmidt operators $\tilde{\psi} : K \to H$.

Let Y be the space of all $\mathcal{F}_0$-adapted process $\psi(t, \tilde{w})\ :\ [\tilde{m}(0), \infty) \times \Omega \to \mathbb{R}$ which is almost certainly continuous in t for fixed $\tilde{w} \in \Omega$. Moreover $\psi(s; \tilde{w}) = \varphi(s)$ for $s \in [\tilde{m}(0), 0]$ and $\mathbb{E}\|(t, \tilde{w})\|^p_H \to 0$ as $t \to \infty$.
Also Y is a Banach space when it is equipped with a norm defined by

$$\|\psi\|^p_Y = \sup_{t\geq 0} \mathbb{E}\|\psi(t)\|^p_H.$$

The notation $B_r(x, H)$ stands for the closed ball with center at x and radius $r > 0$ in H.

3. Partial integrodifferential equation in Banach space

Now for the question of existence of mild solutions of the integrodifferential equation (1), we recall some needed fundamental results. Regarding the theory of resolvent operator, we refer the reader to [9]. Let $\mathbb{Y}$ be the Banach space $D(A)$ equipped with the graph norm defined by $\|y\|_{\mathbb{Y}} = \|T(0)y\| + \|y\| \quad \text{for} \quad y \in \mathbb{Y}$, (where $\|\cdot\|$ is the norm on H). The notation $\mathcal{C}(\mathbb{R}_+, \mathbb{Y})$ stands for the space of all continuous functions from $\mathbb{R}_+$ into $\mathbb{Y}$. We consider the following Cauchy problem:

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds \text{ for } t \geq 0, \\ v(0) = v_0 \in H. \end{cases} \tag{2}$$

The process $\{v(t),\ t \geq 0\}$ satisfying the above equation takes values in the space H.

The notation $\mathcal{C}([0,+\infty),\mathbb{Y})$ stands for the space of all continuous function from $[0,+\infty)$ into $\mathbb{Y}$.

Definition 3.1. [9] *A resolvent operator of* (2) *is a bounded linear operator valued function $R(t) \in \mathcal{L}(H)$ for $t \geq 0$, satisfying the following properties:*

1. $R(0) = I$ *and* $\|R(t)\|_{\mathcal{L}(H)} \leq Me^{-\beta t}$ *for some constants M and β,*
2. *for each $x \in H$, $R(t)x$ is strongly continuous for $t \geq 0$,*
3. $R(t) \in \mathcal{L}(\mathbb{Y})$ *for $t \geq 0$. For $x \in \mathbb{Y}$, $R(\cdot)x \in \mathcal{C}^1(\mathbb{R}_+, H) \cap \mathcal{C}(\mathbb{R}_+, \mathbb{Y})$.*

$$R'(t)x = AR(t)x + \int_0^t B(t-r)R(s)xds$$
$$R'(t)x = R(t)Ax + \int_0^t R(t-r)B(s)xds, \quad t \geq 0.$$

For additional details on resolvent operators, we refer the reader to [9, 10]. The resolvent operator plays an important role to study the existence of solutions and to etablish a variation of constants for non-linear systems. For this reason, we need to know when the linear system (1) possesses a resolvent operator. Theorem 3.1 below provides a satisfactory answer to this problem. In what follows we suppose the following assumptions:

(A1) The operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on H.

(A2) For all $t \geq 0$, $B(t)$ is closed linear operator from $D(A)$ to H and $B(t) \in L(\mathbb{Y}, H)$. For any $y \in H$, the map $t \mapsto B(t)y$ is bounded, differentiable and the derivative $t \mapsto B(t)y$ is bounded uniformly continuous on $\mathbb{R}_+$. Moreover, there is an integrable function $\mu : J \to \mathbb{R}_+$ such that for each $z \in H$, the map $t \to B(t)z$ belongs to $W^{1,1}(J,H)$ and $\left\|\frac{dB(t)z}{dt}\right\| \leq \mu(t)\|z\|,\ z \in H,\ t \in J.$

Theorem 3.1. [9] *Assume that (A1)-(A2) hold. Then there exists a unique resolvent operator to the Cauchy problem* (2)*.*

In the following, we give some results for the existence of solutions for the following integro-differential equation.

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds + q(t) \text{ for } t \geq 0, \\ v(0) = v_0 \in H. \end{cases} \tag{3}$$

where $q:\ [0,+\infty[\to H$ is continuous function.

Definition 3.2. [9] *A continuous function $v:\ [0,+\infty[\to H$ is said to be a strict solution of the Eq.* (3) *if*

1. $v \in \mathcal{C}^1([0,+\infty[, H) \cap \mathcal{C}([0,+\infty[, \mathbb{Y})$,
2. *v satisfies Eq.* (3) *for $t \geq 0$.*

The next theorem provides sufficient conditions ensuring the regularity of solutions of the Eq. (2).

Theorem 3.2. [9] *Assume that hypotheses* $(A1)$ *and* $(A2)$ *hold. If* v *is a strict solution of the Eq.*(2)*, then the variation of constant formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds, \quad for\ t \geq 0. \tag{4}$$

Accordingly, we can establish the following definition.

Theorem 3.3. [9] *Let* $q \in \mathcal{C}^1([0,+\infty), \mathbb{H})$ *and* v *be defined by* (4)*. If* $v_0 \in \mathcal{D}(A)$*, then* v *is a strict solution of the Eq.*(2)*.*

Now we will derive the appropriate definition of mild solutions of (2).

Definition 3.3. [9] *A function* $v: [0,+\infty) \to \mathbb{H}$ *is called mild solution of the Eq.* (2)*, for* $v_0 \in H$*, if* v *satisfies the variation of constants formula* (4)*.*

Accordingly, we can establish the following definition of the equation (1).

Definition 3.4. *A stochastic process* $\{x(t),\ t \in [0,T]\}$ $(0 \leq T < \infty)$ *is called a mild solution of system* (1) *if*

(i) $x(t)$ *is adapted to* $\mathcal{F}_t, t \geq 0$.
(ii) $x(t) \in H$ *has càdlàg paths on* $t \in [0,T]$ *a.s and for each* $t \in [0,T]$, $x(t)$ *satisfies the integral equation*

$$\begin{aligned} x(t) =& R(t)[\varphi - g(0,\varphi(-\rho_1(0)))] + g(t, x(t-\rho_1(t))) \\ &+ \int_0^t R(t-s)h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau))d\tau\right)ds \\ &+ \int_0^t R(t-s)f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_2(\tau))d\tau\right)dw(s) \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)), \end{aligned} \tag{5}$$

and

$$x_0(\cdot) = \varphi \in \mathcal{B}_{\mathcal{F}_0}([\tilde{m}(0),0],H).$$

Definition 3.5. *Let* $p \geq 2$ *be an integer. Eq.* (5) *is said to be stable in p-th moment if for arbitrarily given* $\varepsilon > 0$*, there exists a* $\tilde{\delta} > 0$ *such that* $\|\varphi\|_{\mathcal{B}} < \tilde{\delta}$ *guarantees that*

$$\mathbb{E}\left[\sup_{t\geq 0}\|x(t)\|_H^p\right] < \varepsilon.$$

Definition 3.6. *Let* $p \geq 2$ *be an integer. Eq.* (5) *is said to be asymptotically stable in p-th moment if it stable in p-th moment and for any* $\varphi \in \mathcal{B}_{\mathcal{F}_0}([\tilde{m}(0),0],H)$,

$$\lim_{T\to\infty} \mathbb{E}\left[\sup_{t\geq T}\|x(t)\|_H^p\right] = 0.$$

Now, we introduce the Hausdorff measure of noncompactness $\chi_{\widetilde{Y}}$ defined by

Definition 3.7. *The Hausdorff measure of non-compactness of a nonempty and bounded subset* Γ *of* $\widetilde{Y}$*, denoted by* $\chi(\Gamma)$*, is the infimum of all numbers* $\varepsilon > 0$ *such that* Γ *has a finite* ε*-net in* Y*, i.e.,*

$$\chi_{\widetilde{Y}}(\Gamma) = \inf\{\varepsilon > 0;\ \Gamma \subset S + \varepsilon\Gamma_{\widetilde{Y}},\ S \subset \widetilde{Y},\ S\ is\ finite\ \}.$$

for bounded set Γ in any Hilbert space $\widetilde{Y}$. Some basic properties of $\chi_{\widetilde{Y}}(\cdot)$ are given in the following lemma.

Lemma 3.4. *([2]) Let $\widetilde{Y}$ be a real Hilbert space and $\Gamma,\ C \subseteq \widetilde{Y}$ be bounded, the following properties are satisfied:*

1. *Γ is pre-compact if and only if $\chi_{\widetilde{Y}}(\Gamma) = 0$;*
2. *$\chi_{\widetilde{Y}}(\Gamma) = \chi_{\widetilde{Y}}(\overline{\Gamma}) = \chi_{\widetilde{Y}}(conv\Gamma)$, where $\overline{\Gamma}$ and $conv\Gamma$ are the closure and the convex hull of Γ respectively;*
3. *$\chi_{\widetilde{Y}}(\Gamma) \leq \chi_{\widetilde{Y}}(C)$ when $\Gamma \subseteq C$;*
4. *$\chi_{\widetilde{Y}}(\Gamma + C) \leq \chi_{\widetilde{Y}}(\Gamma) + \chi_{\widetilde{Y}}(C)$ where $\Gamma + C = \{x + y : x \in \Gamma,\ y \in C\}$;*
5. *$\chi_{\widetilde{Y}}(\Gamma \cup C) = \max\{\chi_{\widetilde{Y}}(\Gamma), \chi_{\widetilde{Y}}(C)\}$;*
6. *$\chi_{\widetilde{Y}}(\lambda\Gamma) \leq |\lambda|\chi_{\widetilde{Y}}(\Gamma)$ for any $\lambda \in \mathbb{R}$;*
7. *If the map $D(\Phi) \subseteq Y \to Z$ is Lipschitz continuous with constant κ, then $\chi_Z(\Phi\Gamma) \leq \kappa\chi_{\widetilde{Y}}(\Gamma)$ for any bounded subset $\Gamma \subseteq D(\Phi)$, where Z is a Banach space;*

Definition 3.8. [26] *The map $\Phi : V \subseteq \widetilde{Y} \to Y$ is said to be a $\chi_{\widetilde{Y}}$-contraction if there exists a positive constant $\kappa < 1$ such that $\chi_{\widetilde{Y}}(\Phi(\Gamma)) \leq \kappa\chi_{\widetilde{Y}}(\Gamma)$ for any bounded close subset $\Gamma \subseteq V$ where $\widetilde{Y}$ is a Banach space.*

In this paper we denote by χ_C the Hausdorff's measure of noncompactness of $C([0, b], H)$ and by χ_Y, the Hausdorff's measure of noncompactness of $\widetilde{Y}$.

Lemma 3.5. [23] *For any $p \geq 1$ and for arbitrary $L_2^0(K, H)$-valued predictable process $\phi(\cdot)$, we have the following inequality*

$$\sup_{s\in[o,t]} \mathbb{E}\left\|\int_0^s \phi(v)dw(v)\right\|_H^{2p} \leq (p(2p-1))^p \left(\int_0^t (\mathbb{E}\|\phi(s)\|_{L_2^0}^{1/p})ds\right)^p,\ t \in [0, \infty).$$

In the rest of this paper, we denote by $C_p = (p(p-1)/2)^{p/2}$.

Lemma 3.6. *([1] Darbo) If $V \subseteq Y$ is closed and convex and $0 \in V$, the continuous map $\Phi : V \to V$ is a χ_Y-contraction, if the set $\{x \in V : x = \lambda\Phi x\}$ is bounded for $0 < \lambda < 1$, then the map Φ has at least one fixed point in V.*

4. Main Results

In this section we present our main results on the existence and asymptotic stability in the p-th moment of mild solutions of system (1). To do this, we make the following hypotheses:

(H1) A is the infinitesimal generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ on H. We will also suppose that the resolvent operator $R(t),\ t > 0$ of (1) is compact and there exists $M > 0$, $\beta > 0$ such that $\|R(t)\|_{\mathcal{L}(H)} \leq Me^{-\beta t}$, that is R is exponentially stable.

(H2) The function $g : [0, \infty) \times H \to H$ is continuous and there exists $L_g > 0$ such that

$$\mathbb{E}\|g(t, \psi_1) - g(t, \psi_2)\|_H^p \leq L_g\|\psi_1 - \psi_2\|_H^p, \quad t \geq 0,\ \psi_1,\ \psi_2 \in H,$$

and

$$\mathbb{E}\|g(t, \psi)\|_h^p \leq L_g\mathbb{E}\|\psi\|_H^p, \quad t \geq 0,\ \psi \in H$$

with $M^p L_g < 1$.

(H3) There exists a continuous function $m_a : [0,\infty) \to [0,\infty)$ such that

$$\mathbb{E}\left\|\int_0^t a(t,s,\psi)ds\right\|_H^p \leq m_a(t)\Theta_a(\mathbb{E}\|\psi\|_H^p)$$

for a.e. $t \geq 0$ and all $\psi \in H$, where $\Theta_a : [0,\infty) \to (0,\infty)$ is a continuous and nondecreasing function.

(H4) The function $h : [0,\infty) \times H \times H \to H$ satisfies the following conditions:
   (i) The function $h : [0,\infty) \times H \times H \to H$ is continuous.
   (ii) There exists a continuous function $m_h : [0,\infty) \to [0,\infty)$ and a continuous nondecreasing function $\Theta_h : [0,\infty) \to (0,\infty)$ such that

$$\mathbb{E}\|h(t,\psi,x)\|_H^p \leq m_h(t)\Theta_h(\mathbb{E}\|\psi\|_H^p) + \mathbb{E}\|x\|_{H'}^p, \quad t \geq 0, \ \psi, x \in H.$$

   (iii) The set $\left\{R(t-s)h\left(s,\psi,\int_0^s a(s,\tau,\psi)d\tau\right) : \ t,s \in [0,b], \ \psi \in B_r(0,H)\right\}$ is relatively compact in H.

(H5) There exists a function continuous function $m_b : [0,\infty) \to [0,\infty)$ such that

$$\mathbb{E}\left\|\int_0^t b(t,s,\psi)ds\right\|_H^p \leq m_b(t)\Theta(\mathbb{E}\|\psi\|_H^p)$$

for a.e. $t \geq 0$ and all $\psi \in H$, where $\Theta_b : [0,\infty) \to (0,\infty)$ is a continuous and nondecreasing function.

(H6) The function $f : [0,\infty) \times H \times H \to L(K,H)$ satisfies the following conditions:
   (i) The function $f : [0,\infty) \times H \times H \to L(K,H)$ is continuous.
   (ii) There exists a continuous function $m_f : [0,\infty) \to [0,\infty)$ and a continuous nondecreasing function $\Theta_f : [0,\infty) \to (0,\infty)$ such that

$$\mathbb{E}\|f(t,\psi,x)\|_H^p \leq m_f(t)\Theta_f(\mathbb{E}\|\psi\|_H^p) + \mathbb{E}\|x\|_{H'}^p, \quad t \geq 0, \ \psi, x \in H.$$

   (iii) The set $\left\{R(t-s)f\left(s,\psi,\int_0^s b(s,\tau,\psi)d\tau\right) : \ t,s \in [0,b], \ \psi \in B_r(0,H)\right\}$ is relatively compact in H.

(H7) The functions $I_k, \ J_k : H \to H$ are completely continuous and that there are constants $d_k^{(j)}, \ k \in \overline{1,m}, \ j = 1,2,3,4$ such that $\mathbb{E}\|I_k(x)\|_H^p \leq d_k^{(1)}\mathbb{E}\|x\|_H^p + d_k^{(2)}$, for every $x \in H$.

In the proof of the main results, we need the following lemmas.

Lemma 4.1. *Assume that conditions (H1), (H2) hold. Let Φ_1 be the operator defined by: for each $x \in Y$,*

$$(\Phi_1 x)(t) = g(t, x(t-\rho_1(t))). \tag{6}$$

Then Φ_1 is continuous on $[0,\infty)$ in p-th mean and maps Y into itself.

Proof. The continuity in p-th moment of Φ_1 on $[0,\infty)$ follows from $(H2)$.

Next we show that $\Phi_1(Y) \subset Y$. By (H1) and (H2), from the equation (6), we have for $t \in [\tilde{m}(0),\infty)$,

$$\begin{aligned}\mathbb{E}\|(\Phi_1 x)(t)\|_H^p &\leq \mathbb{E}\|g(t,x(t-\rho_1(t)))\|_H^p \\ &\leq L_g \mathbb{E}\|x(t-\rho_1(t))\|_H^p.\end{aligned}$$

That is to say $\mathbb{E}\|(\Phi_1 x)(t)\|_H^p \to 0$ as $t\to\infty$. So we conclude that $\Phi_1(Y)\subset Y$. •

Lemma 4.2. *Assume that conditions $(H1)$, $(H3)$, $(H4)(i)-(ii)$ hold. Let Φ_2 be the operator defined by: for each $x \in Y$,*

$$(\Phi_2 x)(t) = \int_0^t R(t-s)h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds. \tag{7}$$

Then Φ_2 is continuous and maps Y into itself.

Proof. We first prove that Φ_2 is continuous in p-th moment on $[0,\infty)$. Let $x\in Y, \tilde{t}\geq 0$ and $|\xi|$ be sufficiently small, we have

$$\begin{aligned}
&\mathbb{E}\|(\Phi_2 x)(\tilde{t}+\xi)-(\Phi_2 x)(\tilde{t})\|_H^p \\
&\leq 2^{p-1}\mathbb{E}\left\|\int_0^{\tilde{t}}[R(\tilde{t}+\xi-s)-R(\tilde{t}-s)]h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds\right\|_H^p \\
&+2^{p-1}\mathbb{E}\left\|\int_{\tilde{t}}^{\tilde{t}+\xi}R(\tilde{t}+\xi-s)h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds\right\|_H^p \\
&\leq 2^{p-1}\mathbb{E}\left[\int_0^{\tilde{t}}\left\|[R(\tilde{t}+\xi-s)-R(\tilde{t}-s)]h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H ds\right]^p \\
&+2^{p-1}M^p\mathbb{E}\left[\int_{\tilde{t}}^{\tilde{t}+\xi}e^{-\beta(\tilde{t}+\xi-s)}\left\|h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H ds\right]^p \\
&\leq 2^{p-1}\left[\int_0^{\tilde{t}}\|R(\tilde{t}+\xi-s)-R(\tilde{t}-s)\|_{\mathcal{L}_2(H)}^{(p/p-1)}ds\right]^{p-1}\int_0^{\tilde{t}}\mathbb{E}\left\|h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H^p ds \\
&+2^{p-1}M^p\left[\int_{\tilde{t}}^{\tilde{t}+\xi}e^{-(p\beta/p-1)(\tilde{t}+\xi-s)}ds\right]^{p-1} \\
&\times\int_{\tilde{t}}^{\tilde{t}+\xi}\mathbb{E}\left\|h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H^p ds \to 0 \text{ as } \xi\to 0.
\end{aligned}$$

Thus Φ_2 is continuous in p-th moment on $[0,\infty)$.

Next we show that $\Phi_2(Y) \subset Y$. By $(H1)$, $(H3)$ and $(H4)(i)-(ii)$, from the equation (7), we have for $t \in [\tilde{m}(0), \infty)$,

$$
\begin{aligned}
&\mathbb{E}\|(\Phi_2 x)(t)\|_H^p \\
&\le \mathbb{E}\left[\int_0^t \left\| R(t-s)h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H ds\right]^p \\
&\le M^p \mathbb{E}\left[\int_0^t e^{-\beta(t-s)} \left\| h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H ds\right]^p \\
&\le M^p \left[\int_0^t e^{-\beta(t-s)} ds\right]^{p-1} \int_0^t e^{-\beta(t-s)} \mathbb{E}\left\| h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H^p ds \\
&\le M^p \beta^{1-p} \int_0^t e^{-\beta(t-s)} [m_h(s)\Theta(\mathbb{E}\|x(s-\rho_2(s))\|_H^p) + m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_3(s))\|_H^p)] ds \\
&= K_2 \int_0^t e^{-\beta(t-s)} [m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_2(s))\|_H^p) + m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_3(s))\|_H^p)] ds.
\end{aligned}
$$

However, for any any $\varepsilon > 0$ there exists a $\tilde{\tau}_1 > 0$ such that $\mathbb{E}\|x(s-\rho_2(s))\|_H^p < \varepsilon$ and $\mathbb{E}\|x(s-\rho_3(s))\|_H^p < \varepsilon$ for $t \ge \tilde{\tau}_1$. Thus, we obtain

$$
\begin{aligned}
\mathbb{E}\|(\Phi_2 x)(t)\|_H^p &\le K_2 \int_0^t e^{-\beta(t-s)} [m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_2(s))\|_H^p) + m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_3(s))\|_H^p)] ds \\
&\le K_2 \int_0^{\tilde{\tau}_1} e^{-\beta(t-s)} [m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_2(s))\|_H^p) + m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_3(s))\|_H^p)] ds \\
&+ K_2 \int_{\tilde{\tau}_1}^t e^{-\beta(t-s)} [m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_2(s))\|_H^p) + m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_3(s))\|_H^p)] ds \\
&\le K_2 e^{-\beta t} \int_0^{\tilde{\tau}_1} e^{\beta s} [m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_2(s))\|_H^p) + m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_3(s))\|_H^p)] ds \\
&+ K_2 \int_{\tilde{\tau}_1}^t e^{-\beta(t-s)} [m_h(s)\Theta_h(\varepsilon) + m_a(s)\Theta_a(\varepsilon)] ds \\
&\le K_2 e^{-\beta t} \int_0^{\tilde{\tau}_1} e^{\beta s} [m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_2(s))\|_H^p) + m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_3(s))\|_H^p)] ds \\
&+ K_2 L_{h,a} [\Theta_h(\varepsilon) + \Theta_a(\varepsilon)],
\end{aligned}
$$

where $L_{h,a} = \sup_{t\ge 0} \int_{\tilde{\tau}_1}^t e^{-\beta(t-s)}[m_h(s) + m_a(s)]ds$. As $e^{-\beta t} \to 0$ as $t \to \infty$ and, there exists $\tilde{\tau}_2 \ge \tilde{\tau}_1$ such that for any $t \ge \tilde{\tau}_2$ we have

$$
K_2 e^{-\beta t} \int_0^{\tilde{\tau}_1} e^{\beta s} [m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_2(s))\|_H^p) + m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_3(s))\|_H^p)] ds < \varepsilon - K_2 L_{h,a}[\Theta_h(\varepsilon) + \Theta_a(\varepsilon)].
$$

From the above inequality, for any $t \ge \tilde{\tau}_2$, we obtain $\mathbb{E}\|(\Phi_2 x)(t)\|_H^p < \varepsilon$. That is to say $E\|(\Phi_2 x)(t)\|_H^p \to 0$ as $t \to \infty$. So we conclude that $\Phi_2(Y) \subset Y$. •

Lemma 4.3. *Assume that conditions $(H1)$, $(H5)$, $(H6)(i)-(ii)$ hold. Let Φ_3 be the operator defined by: for each $x \in Y$,*

$$(\Phi_3 x)(t) = \int_0^t R(t-s) f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right) dw(s). \quad (8)$$

Then Φ_3 is continuous and maps Y into itself.

Proof. We first prove that Φ_3 is continuous in p-th moment on $[0,\infty)$. Let $x \in Y$, $\tilde{t} \geq 0$ and $|\varepsilon|$ be sufficiently small, we have

$$\begin{aligned}
&\mathbb{E}\|(\Phi_3 x)(\tilde{t}+\xi) - (\Phi_3 x)(\tilde{t})\|_H^p \\
&\leq 2^{p-1}\mathbb{E}\left\|\int_0^{\tilde{t}} [R(\tilde{t}+\xi-s) - R(\tilde{t}-s)] f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right) dw(s)\right\|_H^p \\
&+ 2^{p-1}\mathbb{E}\left\|\int_{\tilde{t}}^{\tilde{t}+\xi} R(\tilde{t}+\xi-s) f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right) dw(s)\right\|_H^p \\
&\leq 2^{p-1}C_p\left[\int_0^{\tilde{t}} \left(\mathbb{E}\left\|[R(\tilde{t}+\xi-s) - R(\tilde{t}-s)] f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p\right)^{2/p} ds\right]^{p/2} \\
&+ 2^{p-1}C_p\left[\int_{\tilde{t}}^{\tilde{t}+\xi} \left(\mathbb{E}\left\|R(\tilde{t}+\xi-s) \quad \times f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p\right)^{2/p} ds\right]^{p/2} \to 0 \\
&\hspace{20em} \text{as } \xi \to 0
\end{aligned}$$

Thus Φ_3 is continuous in p-th moment on $[0,\infty)$.
Next we show that $\Phi_3(Y) \subset Y$. By $(H1)$,$(H5)$ and $(H6)(i)-(ii)$, from the equation (8), we have for $t \in [\tilde{m}(0),\infty)$,

$$\begin{aligned}
&\mathbb{E}\|(\Phi_3 x)(t)\|_H^p \\
&\leq C_p\left[\int_0^t \left(\mathbb{E}\left\|R(t-s) f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p\right)^{2/p} ds\right]^{p/2} \\
&\leq C_p M^p\left[\int_0^t \left(e^{-p\beta(t-s)}\mathbb{E}\left\|f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p\right)^{2/p} ds\right]^{p/2} \\
&\leq C_p M^p\left[\int_0^t \left[e^{-p\beta(t-s)}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p) + m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p)]\right]^{2/p} ds\right]^{p/2} \\
&\leq C_p M^p\left[\int_0^t e^{-\left[\frac{2(p-1)}{p-2}\right]\beta(t-s)} ds\right]^{p/2-1} \int_0^t e^{-p\beta(t-s)}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p) + m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p \\
&\leq C_p M^p\left[\frac{2\beta(p-1)}{p-2}\right]^{1-p/2} \int_0^t e^{-p\beta(t-s)}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p) + m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p)]ds \\
&= K_3\int_0^t e^{-p\beta(t-s)}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p) + m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p)]ds.
\end{aligned}$$

However, for any any $\varepsilon > 0$ there exists a $\tilde{\theta}_1 > 0$ such that $\mathbb{E}\|x(s-\rho_4(s))\|_H^p < \varepsilon$ and $\mathbb{E}\|x(s-\rho_5(s))\|_H^p < \varepsilon$ for $t \geq \tilde{\theta}_1$. Thus, we obtain

$$\begin{aligned}\mathbb{E}\|(\Phi_3x)(t)\|_H^p &\leq K_3\int_0^t e^{-p\beta(t-s)}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p) + m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p)]ds\\ &\leq K_3\int_0^{\tilde{\theta}_1} e^{\beta s}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p)\\ &+ m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p)]ds + K_3L_{f,b}[\Theta_f(\varepsilon)+\Theta_b(\varepsilon)],\end{aligned}$$

where $L_{f,b} = \sup_{t\geq 0}\int_{\tilde{\theta}_1}^t e^{-\beta(t-s)}[m_f(s)+m_b(s)]ds$. As $e^{-\beta t} \to 0$ as $t \to \infty$ and there exists $\tilde{\theta}_2 \geq \tilde{\theta}_1$ such that for any $t \geq \tilde{\theta}_2$ we have

$$K_3e^{-\delta t}\int_0^{t_1} e^{\delta s}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p)+m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p)]ds < \varepsilon - K_3L_{f,b}[\Theta_f(\varepsilon)+\Theta_b(\varepsilon)].$$

From the above inequality, for any $t \geq \tilde{\theta}_2$, we obtain $\mathbb{E}\|(\Phi_3x)(t)\|_H^p < \varepsilon$. That is to say $\mathbb{E}\|(\Phi_3x)(t)\|_H^p \to 0$ as $t \to \infty$. So we conclude that $\Phi_3(Y) \subset Y$. •

Now, we are ready to present our main result.

Theorem 4.4. *Assume the conditions $(H1)-(H7)$ hold. Let $p \geq 2$ be an integer. Then the impulsive stochastic differential equations (1) is asymptotically stable in p-th moment, provided that $(14m)^{p-1}M^p\sum_{k=1}^m(d_k^{(1)}+d_k^{(3)}) < 1$, and*

$$\int_1^\infty \frac{1}{s+\Theta_h(s)+\Theta_a(s)+\Theta_f(s)+\Theta_b(s)}ds = \infty.$$

Proof. Let the nonlinear operator $\Psi : Y \to Y$ be defined as $(\Psi x)(t) = \varphi(t)$ for $t \in [\tilde{m}(0), 0]$ and for $t \geq 0$,

$$\begin{aligned}(\Psi x)(t) &= R(t)[\varphi - g(0,\varphi(-\rho_1(0)))] + g(t, x(t-\rho_1(t)))\\ &+ \int_0^t R(t-s)h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds\\ &+ \int_0^t R(t-s)f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)dw(s)\\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)).\end{aligned}$$

Using $(H1)-(H7)$, and the proof of the Lemmas 4.1-4.3, it is clear that the nonlinear operator Ψ is well defined and continuous. Moreover, for each $t \geq 0$ we have

$$
\begin{aligned}
\mathbb{E}\|(\Psi x)(t)\|_H^p \leq\ & 5^{p-1}\mathbb{E}\|R(t)[\phi - g(0,\varphi(-\rho_1(0)))]\|_H^p + 5^{p-1}\mathbb{E}\left\|g(t,x(t-\rho_1(t)))\right\|_H^p \\
& + 5^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds\right\|_H^p \\
& + 5^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)f\left(s, x(s-\rho_4(s)), \int_0^s a(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)dw(s)\right\|_H^p \\
& + 5^{p-1}\mathbb{E}\left\|\sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-))\right\|_H^p .
\end{aligned} \tag{9}
$$

Using $(H1)$ and $(H2)$,we have

$$
5^{p-1}\mathbb{E}\|R(t)[\phi - g(0,\varphi(-\rho_1(0)))]\|_H^p \leq 10^{p-1}M^p e^{-p\beta t}[\mathbb{E}\|\phi\|_H^p + L_g\mathbb{E}\|\varphi(-\rho_1(0))\|_H^p] \to 0 \text{ as } t\to\infty.
$$

By $(H1)$-$(H7)$ and the proof of the Lemmas 4.1-4.3 again, we obtain

$$
\begin{aligned}
& 5^{p-1}\mathbb{E}\left\|g(s,x(s-\rho_1(s)))\right\|_H^p \to 0 \text{ as } t\to\infty, \\
& 5^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds\right\|_H^p \to 0 \text{ as } t\to\infty, \\
& 5^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)f\left(s, x(s-\rho_4(s)), \int_0^s a(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)dw(s)\right\|_H^p \to 0 \text{ as } t\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
5^{p-1}\mathbb{E}\left\|\sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-))\right\|_H^p &\leq 5^{p-1}\sum_{0<t_k<t}\mathbb{E}\|R(t-t_k)I_k(x(t_k^-))\|_H^p \\
&\leq 5^{p-1}M^p e^{-p\beta t}\mathbb{E}\|I_k(x(t_k^-))\|_H^p \to 0 \text{ as } t\to\infty
\end{aligned}
$$

so Ψ maps Y into itself.

Next we prove that the operator Ψ has a fixed point, which is a mild solution of the problem (1). We shall employ Lemma 3.6. For better readability, we break the proof into a sequence of steps.

Step 1. For $0<\lambda<1$, set $\{x\in Y : x=\lambda\Psi x\}$ is bounded.

Let $x\in Y$ be a possible solution of $x=\lambda\Psi(x)$ for some $0<\lambda<1$. Then, by $(H1)$-$(H7)$, we have for each $t\in[0,T]$

$$
\begin{aligned}
\mathbb{E}\|x(t)\|_H^p &\leq 5^{p-1}\mathbb{E}\|R(t)[\phi - g(0,\varphi(-\rho_1(0)))]\|_H^p + 5^{p-1}\mathbb{E}\left\|g(t,x(t-\rho_1(t)))\right\|_H^p \\
&+ 5^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds\right\|_H^p \\
&+ 5^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)f\left(s, x(s-\rho_4(s)), \int_0^s a(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)dw(s)\right\|_H^p \\
&+ 5^{p-1}\mathbb{E}\left\|\sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-))\right\|_H^p \\
&\leq 10^{p-1}e^{-\beta pt}M^p[\mathbb{E}\|\phi\|_H^p + L_g\mathbb{E}\|\varphi(-\rho_1(0))\|_H^p] + 5^{p-1}\mathbb{E}\|g(s,x(s-\rho_1(s)))\|_H^p \\
&+ 5^{p-1}M^p\mathbb{E}\left[\int_0^t e^{-\beta(t-s)}\left\|h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H ds\right]^p \\
&+ 5^{p-1}C_p\left[\int_0^t \left[e^{-p\beta(t-s)}\mathbb{E}\left\|f\left(s,x(s-\rho_4(s)),\int_0^s a(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p\right]^{2/p}\right]^{p/2} \\
&+ (5m)^{p-1}M^p\sum_{k=1}^m e^{-\beta p(t-t_k)}\mathbb{E}\|I_k(x(t_k^-))\|_H^p \\
&\leq 10^{p-1}e^{-\beta pt}M^p[\mathbb{E}\|\phi\|_H^p + L_g\mathbb{E}\|\varphi(-\rho_1(0))\|_H^p] + 5^{p-1}L_g\mathbb{E}\|x(t-\rho_1(t))\|_H^p \\
&+ 5^{p-1}M^pT^{p-1}\int_0^t e^{-\beta p(t-s)}[m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_2(s))\|_H^p) + m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_3(s))\|_H^p)]ds \\
&+ 5^{p-1}C_pM^pT^{p/2-1}\int_0^t e^{-\beta p(t-s)}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p) + m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p)]ds \\
&+ (5m)^{p-1}M^p\sum_{k=1}^m e^{-\beta p(t-t_k)}[d_k^{(1)}\mathbb{E}\|x(t_k^-)\|_H^p + d_k^{(2)}].
\end{aligned}
$$

By the definition of Y, it follows that

$$
\mathbb{E}\|x(s-\rho_i(s))\|_H^p \leq 2^{p-1}\|\varphi\|_{\mathcal{B}}^p + 2^{p-1}\sup_{s\in[0,t]}\|x(s)\|_H^p,\ \ i=1,2,3,4,5.
$$

If $\mu(t) = 2^{p-1}\|\varphi\|_{\mathcal{B}}^p + 2^{p-1}\sup\limits_{s\in[0,t]}\|x(s)\|_H^p$, we obtain that

$$
\begin{aligned}
\mu(t) &\leq 2^{p-1}\|\varphi\|_{\mathcal{B}}^p + 20^{p-1}e^{-\beta pt}M^p[\mathbb{E}\|\phi\|_H^p + L_g\mathbb{E}\|\varphi(-\rho_1(0))\|_H^p] + 10^{p-1}L_g\mu(t) \\
&+ 10^{p-1}M^pT^{p-1}e^{-\beta pt}\int_0^t e^{-\beta pt}\int_0^t e^{\beta ps}[m_h(s)\Theta_h(\mu(s)) + m_a(s)\Theta_a(\mu(s))]ds \\
&+ 10^{p-1}C_pM^pT^{p/2-1}e^{-\beta pt}\int_0^t e^{\beta ps}[m_f(s)\Theta_f(\mu(s)) + m_b(s)\Theta_b(\mu(s))]ds \\
&+ (10m)^{p-1}M^pe^{-\beta pt}\sum_{k=1}^m e^{-\beta pt}\sum_{k=1}^m e^{\beta pt_k}[d_k^{(1)}\mu(t) + d_k^{(2)}].
\end{aligned}
$$

Since $\tilde{L} = (14m)^{p-1}M^p\sum\limits_{k=1}^m(d_k^{(1)}) + 10^{p-1}L_g < 1$, we obtain

$$e^{\beta pt}\mu(t) \leq \frac{1}{1-\tilde{L}}\left[\tilde{M} + 10^{p-1}M^pT^{p-1}\int_0^t e^{\beta ps}[m_h(s)\Theta_h(\mu(s)) + m_a(s)\Theta_a(\mu(s))]ds\right.$$
$$\left.+10^{p-1}C_pM^pT^{p/2-1}\int_0^t e^{\beta ps}[m_f(s)\Theta_f(\mu(s)) + m_b(s)\Theta_b(\mu(s))]ds\right],$$

where

$$\tilde{M} = 2^{p-1}\|\varphi\|_{\mathcal{B}}^p + 20^{p-1}M^p[\mathbb{E}\|\phi\|_H^p + L_g\mathbb{E}\|\varphi(-\rho_1(0))\|_H^p]$$
$$+ (10m)^{p-1}M^p\sum_{k=1}^m d_k^{(2)}.$$

Denoting by $\zeta(t)$ the right-hand side of the above inequality, we have

$$e^{\beta pt}\mu(t) \leq \zeta(t) \text{ for all } t \in [0,T],$$

and $\zeta(0) = \frac{1}{1-\tilde{L}}\tilde{M}$,

$$\zeta'(t) = \frac{1}{1-\tilde{L}}\left[14^{p-1}M^pT^{p-1}e^{\beta pt}[m_h(s)\Theta_h(\mu(s)) + m_a(s)\Theta_a(\mu(s))]\right.$$
$$\left.+14^{p-1}C_pM^pT^{p/2-1}e^{\beta pt}[m_f(s)\Theta_f(\mu(s)) + m_b(s)\Theta_b(\mu(s))]\right]$$
$$\leq \frac{1}{1-\tilde{L}}\left[14^{p-1}M^pT^{p-1}e^{\beta pt}[m_h(s)\Theta_h(e^{-\beta pt}\zeta(t)) + m_a(s)\Theta_a(e^{-\beta pt}\zeta(t))]\right.$$
$$\left.+14^{p-1}C_pM^pT^{p/2-1}e^{\beta pt}[m_f(s)\Theta_f(e^{-\beta pt}\zeta(t)) + m_b(s)\Theta_b(e^{-\beta pt}\zeta(t))]\right].$$

If $\xi(t) = e^{-\beta pt}\zeta(t)$, then $\xi(0) = \zeta(0) \leq \xi(t)$, and

$$\zeta'(t) \leq \frac{1}{1-\tilde{L}}\left[14^{p-1}M^pT^{p-1}e^{\beta pt}[m_h(s)\Theta_h(\xi(t)) + m_a(s)\Theta_a(\xi(t))]\right.$$
$$\left.+14^{p-1}C_pM^pT^{p/2-1}e^{\beta pt}[m_f(s)\Theta_f(\xi(t)) + m_b(s)\Theta_b(\xi(t))]\right],$$

and we have

$$\xi'(t) = (-\beta p)e^{-\beta pt}\zeta(t) + e^{-\beta pt}\zeta'(t)$$
$$\leq (-\beta p)\xi(t) + \frac{1}{1-\tilde{L}}\left[14^{p-1}M^pT^{p-1}[m_h(s)\Theta_h(\xi(t)) + m_a(s)\Theta_a(\xi(t))]\right.$$
$$\left.+14^{p-1}C_pM^pT^{p/2-1}e^{\beta pt}[m_f(s)\Theta_f(\xi(t)) + m_b(s)\Theta_b(\xi(t))]\right]$$
$$\leq m^*(t)[\xi(t) + \Theta_h(\xi(t)) + \Theta_a(\xi(t)) + \Theta_f(\xi(t)) + \Theta_b(\xi(t))],$$

where

$$m^*(t) = \max\left\{(-\beta p), \frac{1}{1-\tilde{L}}14^{p-1}M^pT^{p-1}m_h(t), \frac{1}{1-\tilde{L}}14^{p-1}M^pT^{p-1}m_a(t),\right.$$
$$\left.\frac{1}{1-\tilde{L}}14^{p-1}C_pM^pT^{p/2-1}m_f(t), \frac{1}{1-\tilde{L}}14^{p-1}C_pM^pT^{p-1}m_b(t)\right\}.$$

This implies for each $t \in [0,T]$ that

$$\int_{\xi(0)}^{\xi(t)} \frac{du}{u+\Theta_h(u)+\Theta_a(u)+\Theta_f(u)+\Theta_b(u)} \leq \int_0^T m^*(s)ds < \infty.$$

This inequality shows that there is a constant $\tilde{K}$ such that $\xi(t) \leq \tilde{K},\ t \in [0,T]$, and hence $\|x\|_Y^p \leq \mu(t) \leq \tilde{K}$, where $\tilde{K}$ depends only on $p,\ \beta,\ M,\ T$ and on the functions $m_h(\cdot), \Theta_a(\cdot), m_f(\cdot), \Theta_b(\cdot)$. This indicates that $x(\cdot)$ are bounded on $[0,T]$.

Step 2. $\Psi : Y \to Y$ is continuous.

Let $\{x_n(t)\}_{n=0}^{\infty} \subseteq Y$ with $x_n \to x$ $(n \to \infty)$ in Y. Then there is a number $r > 0$ such that $\mathbb{E}\|x_n(t)\|_H^p \leq r$ for all n and a.e. $t \in [0,T]$, so $x_n \in Br(0,Y) = \{x \in Y : \|x\|_Y^p \leq r\}$ and $x \in Br(0,Y)$. By the assumptions (H3)-(H7), we have

$$\begin{aligned}
&\mathbb{E}\left\| h\left(s, x_n(s-\rho_2(s)), \int_0^s a(s,\tau,x_n(\tau-\rho_3(\tau)))d\tau\right)\right.\\
&\quad\left. -h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H^p \to 0 \text{ as } n \to \infty,\\
&\mathbb{E}\left\| f\left(s, x_n(s-\rho_4(s)), \int_0^s b(s,\tau,x_n(\tau-\rho_5(\tau)))d\tau\right)\right.\\
&\quad\left. -f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p \to 0 \text{ as } n \to \infty,
\end{aligned}$$

for each $s \in [0,T]$, and since

$$\begin{aligned}
&E\left\| h\left(s, x_n(s-\rho_2(s)), \int_0^s a(s,\tau,x_n(\tau-\rho_3(\tau)))d\tau\right) - h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H^p\\
&\quad \leq 2\max\{\Theta_h(r^*), \Theta_a(r^*)\}[m_h(s)+m_a(s)],\\
&E\left\| f\left(s, x_n(s-\rho_4(s)), \int_0^s b(s,\tau,x_n(\tau-\rho_5(\tau)))d\tau\right) - f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p\\
&\quad \leq 2\max\{\Theta_f(r^*), \Theta_b(r^*)\}[m_f(s)+m_b(s)].
\end{aligned}$$

Then by the dominated convergence theorem and $I_k,\ J_k,\ k = 1, 2, \cdots, m$; are completely continuous, we have for $t \in [0,T]$,

$$
\begin{aligned}
&\mathbb{E}\|(\Psi x_n)(t)-(\Psi x)(t)\|_H^p \\
&\quad \le 4^{p-1}\mathbb{E}\left\|g(s,x_n(s-\rho_1(s)))-g(s,x(s-\rho_1(s)))\right\|_H^p \\
&\quad + 4^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)\left[h\left(s,x_n(s-\rho_2(s)),\int_0^s a(s,\tau,x_n(\tau-\rho_3(\tau)))d\tau\right)\right.\right. \\
&\quad \left.\left. -h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right]ds\right\|_H^p \\
&\quad + 4^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)\left[f\left(s,x_n(s-\rho_4(s)),\int_0^s b(s,\tau,x_n(\tau-\rho_5(\tau)))d\tau\right)\right.\right. \\
&\quad \left.\left. -f\left(s,x(s-\rho_4(s)),\int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right]dw(s)\right\|_H^p \\
&\quad + 4^{p-1}\mathbb{E}\left\|\sum_{0<t_k<t} R(t-t_k)[I_k(x_n(t_k^-))-I_k(x(t_k^-))]\right\|_H^p \\
&\quad \le 4^{p-1}\mathbb{E}\|g(s,x_n(s-\rho_1(s)))-g(s,x(s-\rho_1(s)))\|_H^p \\
&\quad + 4^{p-1}T^{p-1}\int_0^t e^{-\beta p(t-s)}\mathbb{E}\left\|h\left(s,x_n(s-\rho_2(s)),\int_0^s a(s,\tau,x_n(\tau-\rho_3(\tau)))d\tau\right)\right. \\
&\quad \left. -h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H^p ds \\
&\quad + 4^{p-1}M^pC_pT^{p/2-1}\int_0^t e^{-\beta p(t-s)}\mathbb{E}\left\|f\left(s,x_n(s-\rho_4(s)),\int_0^s b(s,\tau,x_n(\tau-\rho_5(\tau)))d\tau\right)\right. \\
&\quad \left. -f\left(s,x(s-\rho_4(s)),\int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p ds \\
&\quad + (4m)^{p-1}\sum_{0<t_k<t} e^{-\beta(t-t_k)}\mathbb{E}\|I_k(x_n(t_k^-))-I_k(x(t_k^-))\|_H^p \to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Then, we have for all $t\in[0,T]$,

$$\|\Psi x_n-\Psi x\|_Y^p \to 0 \text{ as } n\to\infty.$$

Therefore, Ψ is continuous.

Step 3. Ψ is χ-contraction.
To see this, we decompose Ψ as $\Psi_1+\Psi_2$ for $t\in[0,T]$, where

$$(\Psi_1 x)(t)=R(t)[\phi-g(0,\varphi(-\rho_1(0)))]+g(s,x(s-\rho_1(s))),$$

and

$$\begin{aligned}(\Psi_2 x)(t) &= \int_0^t R(t-s)h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau))d\tau\right)ds \\ &+ \int_0^t R(t-s)f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_2(\tau))d\tau\right)dw(s) \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)),\end{aligned}$$

(1) Ψ_1 is a contraction on Y.

Let $t \in [0,T]$ and $x, y \in Y$. From (H1) and (H2), we have

$$\begin{aligned}&\mathbb{E}\|(\Psi_1 x)(t) - (\Psi_1 y)(t)\|_H^p \\ &= \mathbb{E}\,\|g(s, x(s-\rho_1(s))) - g(s, y(s-\rho_1(s)))\|_H^p \\ &\le L_g \mathbb{E}\|x(s-\rho_1(s)) - y(s-\rho_1(s))\|_H^p \\ &\le L_g \sup_{s\in[0,T]} \mathbb{E}\|x(s) - y(s)\|_H^p \\ &\le L_g \|x-y\|_Y^p.\end{aligned}$$

Taking supremum over t

$$\|\Psi_1 x - \Psi_1 y\|_Y^p \le L_0 \|x-y\|_Y^p.$$

where $L_0 = L_g < 1$. Hence, Ψ_1 is a contraction on Y.

(2) Ψ_2 is a compact operator.

For this purpose, we decompose Ψ_2 by $\Psi_2 = \Upsilon_1 + \Upsilon_2$, where

$$\begin{aligned}(\Upsilon_1 x)(t) &= \int_0^t R(t-s)h\left(s, x(s-\rho_2(s)), \int_0^s a(s,\tau,x(\tau-\rho_3(\tau))d\tau\right)ds \\ &+ \int_0^t R(t-s)f\left(s, x(s-\rho_4(s)), \int_0^s b(s,\tau,x(\tau-\rho_5(\tau))d\tau\right)dw(s),\end{aligned}$$

and

$$(\Upsilon_2 x)(t) = \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)).$$

(i) Υ_1 is a compact operator.

We now prove that $\Upsilon_1(B_r(0,Y))(t) = \{(\Upsilon_1 x)(t) : x \in B_r(0,Y)\}$ is relatively compact for every $t \in [0,T]$. If $x \in B_r(0,Y)$, from the definition of Y, it follows that

$$\begin{aligned}\mathbb{E}\|x(s-\rho_i(s))\|_H^p &\le 2^{p-1}\|\varphi\|_{\mathcal{B}}^p + 2^{p-1}\sup_{s\in[0,T]}\mathbb{E}\|x(s)\|_H^p \\ &\le 2^{p-1}\|\varphi\|_{\mathcal{B}}^p + 2^{p-1}r := r^*,\ i = 1,2,3,4,5.\end{aligned}$$

It follows from conditions (H4)(iii) and (H6)(iii) that the sets $\Big\{R(t-s)h\left(s,\psi,\int_0^s a(s,\tau,\psi)d\tau\right) :$ $t,s \in [0,T],\ \|\psi\|_H^p \le r^*\Big\}$ and $\left\{R(t-s)f(s,\psi,\int_0^s b(s,\tau,\psi)d\tau) :\ t,s\in[0,T],\ \|\psi\|_H^p \le r^*\right\}$ are relatively compact in H. Moreover, for $x \in B_r(0,Y)$, from the mean value theorem for the Bochner integral, we can infer that

$$(\Upsilon_1 x)(t) \in t\overline{conv}\left\{R(t-s)h\left(s,\psi,\int_0^s a(s,\tau,\psi)d\tau\right):\ t,s\in[0,T],\ \|\psi\|_H^p\leq r^*\right\}$$
$$+ t^{1/2}\overline{conv}\left\{R(t-s)f\left(s,\psi,\int_0^s b(s,\tau,\psi)d\tau\right):\ t,s\in[0,T],\ \|\psi\|_H^p\leq r^*\right\}$$

for all $t\in[0,T]$, and $\overline{conv}$ denotes the convex hull. As a result we conclude that the set $\{(\Upsilon_1 x)(t) : x\in B_r(0,Y)\}$ is relatively compact in H for every $t\in[0,T]$.

Next we show that Υ_1 maps bounded sets into equicontinuous sets of Y. Let $0<\varepsilon<t<T$. From $(\Upsilon_1 B_r(0,Y))(t)$ is relatively compact for each t and by the strong continuity of $R(t)$, we can choose $0<\xi<T-t$ with

$$\|R(t+\xi)x - R(t)x\|_H \leq \varepsilon$$

for $x\in(\Phi_2 B_r(0,Y))(t)$. For any $x\in B_r(0,Y)$. Using (H1)-(H5) and Hölder's inequality, it follows that

$$\begin{aligned}
&\mathbb{E}\|(\Upsilon_1 x)(t+\xi)-(\Upsilon_1 x)(t)\|_H^p\\
&\leq 6^{p-1}\mathbb{E}\left\|\int_0^{t-\varepsilon}[R(t+\xi-s)-R(t-s)]h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds\right\|_H^p\\
&+6^{p-1}\mathbb{E}\left\|\int_{t-\varepsilon}^{t}[R(t+\xi-s)-R(t-s)]h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds\right\|_H^p\\
&+6^{p-1}\mathbb{E}\left\|\int_t^{t+\xi}R(t+\xi-s)h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)ds\right\|_H^p\\
&+6^{p-1}\mathbb{E}\left\|\int_0^{t-\varepsilon}[R(t+\xi-s)-R(t-s)]f\left(s,x(s-\rho_4(s)),\int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)dw(s)\right\|_H^p\\
&+6^{p-1}\mathbb{E}\left\|\int_{t-\varepsilon}^{t}[R(t+\xi-s)-R(t-s)]f\left(s,x(s-\rho_4(s)),\int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)dw(s)\right\|_H^p\\
&+6^{p-1}\mathbb{E}\left\|\int_t^{t+\varepsilon}[R(t+\xi-s)-R(t-s)]f\left(s,x(s-\rho_4(s)),\int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)dw(s)\right\|_H^p\\
&\leq 6^{p-1}(t-\varepsilon)^{p-1}\int_0^{t-\varepsilon}\mathbb{E}\left\|[R(t+\xi-s)-R(t-s)]h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H^p ds\\
&+6^{p-1}\mathbb{E}\left[\int_{t-\varepsilon}^{t}\|R(t+\xi-s)-R(t-s)\|_H\left\|h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H ds\right]^p\\
&+6^{p-1}\mathbb{E}\left[\int_t^{t+\xi}\|R(t+\xi-s)\|_H\left\|h\left(s,x(s-\rho_2(s)),\int_0^s a(s,\tau,x(\tau-\rho_3(\tau)))d\tau\right)\right\|_H ds\right]^p
\end{aligned}$$

$$
\begin{aligned}
&+ 6^{p-1}C_p\left[\int_0^{t-\varepsilon}\left[\mathbb{E}\left\|[S(t+\xi-s)-S(t-s)]f\left(s,x(s-\rho_4(s)),\int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p\right]^{2/p}ds\right]^{p/2}\\
&+ 6^{p-1}C_p\left[\int_{t-\varepsilon}^{t}\left[\|R(t+\xi-s)-R(t-s)\|_H^p\mathbb{E}\left\|f\left(s,x(s-\rho_4(s)),\int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p\right]^{2/p}ds\right]^{p/2}\\
&+ 6^{p-1}C_p\left[\int_{t}^{t+\xi}\left[\|R(t+\xi-s)\|_H^p\mathbb{E}\left\|f\left(s,x(s-\rho_4(s)),\int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p\right]^{2/p}ds\right]^{p/2}\\
&\le 6^{p-1}(t-\varepsilon)^p\varepsilon^p + 12^{p-1}M^p\left[\int_{t-\varepsilon}^{t}e^{-\beta(t-s)}ds\right]^{p-1}\int_{t-\varepsilon}^{t}e^{-\beta(t-s)}[m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_1(s))\|_H^p)\\
&+ m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_2(s))\|_H^p)]ds\\
&+ 6^{p-1}M^p\left[\int_{t}^{t+\xi}e^{-\beta(t+\xi-s)}ds\right]^{p-1}\int_{t}^{t+\xi}e^{-\beta(t+\xi-s)}[m_h(s)\Theta_h(\mathbb{E}\|x(s-\rho_1(s))\|)+m_a(s)\Theta_a(\mathbb{E}\|x(s-\rho_2(s))\|_H^p)]\\
&+ 6^{p-1}C_p(t-\varepsilon)^{p/2-1}\int_0^{t-\varepsilon}\mathbb{E}\left\|[S(t+\xi-s)-S(t-s)]f\left(s,x(s-\rho_4(s)),\int_0^s b(s,\tau,x(\tau-\rho_5(\tau)))d\tau\right)\right\|_H^p ds\\
&+ 12^{p-1}C_pM^p\left[\int_{t-\varepsilon}^{t}[e^{-p\beta(t-s)}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p)+m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p)]]^{2/p}ds\right]^{p/2}\\
&+ 6^{p-1}C_pM^p\left[\int_{t}^{t+\varepsilon}[e^{-p\beta(t+\xi-s)}[m_f(s)\Theta_f(\mathbb{E}\|x(s-\rho_4(s))\|_H^p)+m_b(s)\Theta_b(\mathbb{E}\|x(s-\rho_5(s))\|_H^p)]]^{2/p}ds\right]^{p/2}\\
&\le 6^{p-1}(t-\varepsilon)^p\varepsilon^p + 12^{p-1}M^p\max\{\Theta_h(r^*),\Theta_a(r^*)\}\beta^{1-p}\int_{t-\varepsilon}^{t}e^{-\beta(t-s)}[m_h(s)+m_a(s)]ds\\
&+ 6^{p-1}M^p\max\{\Theta_h(r^*),\Theta_a(r^*)\}\beta^{1-p}\int_{t}^{t+\varepsilon}e^{-\beta(\tau_2-s)}[m_h(s)+m_a(s)]ds + 6^{p-1}C_p(t-\varepsilon)^{p/2}\varepsilon^p\\
&+ 12^{p-1}C_pM^p\max\{\Theta_f(r^*),\Theta_b(r^*)\}\left[\frac{2\beta(p-1)}{p-2}\right]^{1-p/2}\int_{t-\varepsilon}^{t}e^{-\beta(t-s)}[m_f(s)+m_b(s)]ds\\
&+ 6^{p-1}C_pM^p\max\{\Theta_f(r^*),\Theta_b(r^*)\}\left[\frac{2\beta(p-1)}{p-2}\right]^{1-p/2}\int_{t}^{t+\varepsilon}e^{-\beta(t+\xi-s)}[m_f(s)+m_b(s)]ds.
\end{aligned}
$$

Then the right-hand side of the above inequality is independent of $x \in B_r$ and tends to zero as $\xi \to 0$ and sufficiently small positive number ε. Thus, the set $\{\Upsilon_1 x : x \in B_r(0,Y)\}$ is equicontinuous.

(ii) Υ_2 is a compact operator.

To prove the compactness of Υ_2, note that

$$(\Upsilon_2 x)(t) = \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-))$$
$$= \begin{cases} 0, & t \in [0,t_1], \\ R(t-t_1)I_1(x(t_1^-)), & t \in (t_1,t_2], \\ \cdots \\ \sum_{k=1}^m R(t-t_k)I_k(x(t_k^-)), & t \in (t_m,T], \end{cases}$$

and that the interval $[0,T]$ is divided into finite subintervals by $t_k,\ k=1,2,\cdots,m$, so that we only need to prove that

$$W = \{R(t-t_1)I_1(x(t_1^-)),\ \ t \in [t_1,t_2],\ x \in B_r(0,Y)\}$$

is relatively compact in $C([t_1,t_2],H)$, as the cases for other subintervals are the same. In fact, from $(H1)$ and $(H7)$, it follows that the set $\{R(t-t_1)I_1(x(t_1^-))\ x \in B_r(0,Y)\}$ is relatively compact in H for all $t \in [t_1,t_2]$.

Thus, we see that the functions in W are equicontinuous due to the compactness of I_1 and the strong continuity of the operator $R(t)$ for all $t \in [0,T]$. Now an application of the Arzelá-Ascoli theorem justifies the relatively compactness of W. Therefore, we conclude that operator Υ_2 is also a compact map.

Let arbitrary bounded subset $V \subset Y$. Since the mapping Ψ_2 is a compact operator, we get that $\chi_Y(\Psi_2 V) = 0$. Consequently

$$\chi_Y(\Psi V) = \chi_Y(\Psi_1 V + \Psi_2 V) \le \chi_Y(\Psi_1 V) + \chi_Y(\Psi_2 V) \le L_0 \chi_Y(V) < \chi_Y(V).$$

Therefore, Ψ is χ-contraction. In view of Lemma 3.6, we conclude that Ψ has at least one fixed point $x^* \in V \subset Y$. Then, x is a fixed point of the operator Ψ, which is a mild solution of the system (1) with $x(s) = \varphi(s)$ on $[\tilde{m}(0),0]$ and $\mathbb{E}\|x(t)\|_H^p \to 0$ as $t \to \infty$. This shows the asymptotic stability of the mild solution of (1). In fact, let $\varepsilon > 0$ be given and choose $\tilde{\gamma} > 0$ such that $\tilde{\gamma} < \varepsilon$ and satisfies $[14^{p-1}M^p + 14^{p-1}(K_2L_{h,a} + K_3L_{f,b}]\tilde{\gamma} + (14^{p-1}K_1\beta^{-1} + \tilde{L})\varepsilon < \varepsilon$. If $x(t) = x(t,\varphi)$ is mild solution of (1), with $\|\varphi\|_{\mathcal{B}}^p + \mathbb{E}\|\phi\|_H^p + L_g\mathbb{E}\|\varphi(-\rho_1(0))\|_H^p < \tilde{\gamma}$, then $(\Psi x)(t) = x(t)$ and satisfies $\mathbb{E}\|x(t)\|_H^p < \varepsilon$ for every $t \ge 0$. Notice that $E\|x(t)\|_H^p < \varepsilon$ on $t \in [\tilde{m}(0),0]$. If there exists $\tilde{t}$ such that $\mathbb{E}\|x(\tilde{t})\|_H^p = \varepsilon$ and $\mathbb{E}\|x(s)\|_H^p < \varepsilon$ for $s \in [\tilde{m}(0),\tilde{t}]$. Then (9) show that

$$\mathbb{E}\|x(t)\|_H^p \le [14^{p-1}M^p e^{-p\beta\tilde{t}} + 14^{p-1}(K_2L_{h,a} + K_3L_{f,b})]\tilde{\gamma} + (14^{p-1}K_1\beta^{-1} + \tilde{L})\varepsilon < \varepsilon,$$

which contradicts the definition of $\tilde{t}$. Therefore, the mild solution of (1) is asymptotically stable in p-th moment. •

5. Example

Consider the following impulsive partial stochastic neutral integrodifferential equation of the form

$$\begin{aligned} d[u(t,x)-\vartheta(t,u(t-\rho(t),x))] &= \frac{\partial^2}{\partial t^2}[u(t,x)dt-\vartheta(t,u(t-\rho(t),x))] \\ &+\int_0^t b(t-s)[u(t,x)-\vartheta(t,u(t-\rho(t),x))]ds \\ &+\zeta\left(t,u(t-\rho(t),x),\int_0^t \zeta_1(t,u(t-\rho(t),x)\right)dt, \\ &+\sigma\left(t,u(t-\rho(t),x),\int_0^t \sigma_1(t,u(t-\rho(t),x)\right)dw(t), \\ & t\geq 0,\ 0\leq x\leq\pi,\ t\neq t_k, \\ u(t,0)=u(t,\pi)=0,\ t\geq 0. \\ u(t,x)=\varphi(t,x),\ t\leq 0,\ 0\leq x\leq \pi, \\ \Delta u(t_k,x) &= \int_0^{t_k}\eta_k(t_k-s)u(s,x)ds, \end{aligned} \tag{10}$$

where $(t_k)_k\in\mathbb{N}$ is a strictly increasing sequence of positive numbers, $\rho(t)\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$, and $\eta_k,\ \tilde{\eta}_k\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}_+),\ k=1,2,\cdots,m$. $w(t)$ denotes a one-dimensional standard Wiener process in H defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$. Let $H=L^2([0,\pi])$ with the norm $\|\cdot\|$ and define the operator A by $A\omega=\omega'$ with the domain

$$D(A):=\{\omega(\cdot)\in H:\omega,\ \text{are absolutely continuous},\ \omega'\in H,\ \omega(0)=\omega(\pi)=0\}.$$

It is well-known that A is the infinitesimal generator of a strongly continuous semigroup $\{S(t):t\geq 0\}$ in H. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2,\ n\in\mathbb{N}$, and corresponding normalized eigenfunctions given by $e_n(x)=\sqrt{2/\pi}sin(nx)$.
We assume the following conditions hold.

(1) The function $\vartheta:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a positive constant L_ϑ such that $\vartheta(t,0)=0$, and

$$|\vartheta(t,y)-\vartheta(t,z)|\leq L_\vartheta|y-z|,\ t\geq 0,\ y,z\in\mathbb{R}.$$

(2) The function $\zeta:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a positive continuous function $m_{\zeta_1}(\cdot):[0,\infty)\to\mathbb{R}$ such that

$$\left|\int_0^t \zeta_1(t,s,z)\right|\leq m_{\zeta_1}(t)|z|,\ t\geq 0,\ z\in\mathbb{R}.$$

(3) The function $\zeta_1:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a positive continuous function $m_\zeta(\cdot):[0,\infty)\to\mathbb{R}$ such that

$$|\zeta(t,z,v)|\leq m_\zeta(t)|z|+2^{1-p}|v|,\ t\geq 0,\ z,v\in\mathbb{R}.$$

(4) The function $\vartheta:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a positive continuous function $m_\sigma(\cdot):[0,\infty)\to\mathbb{R}$ such that

$$|\sigma(t,z,v)|\leq m_\sigma(t)|z|+2^{1-p}|v|,\ t\geq 0,\ z,v\in\mathbb{R}.$$

(5) The function $\vartheta_1 : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive continuous function $m_{\sigma_1}(\cdot) : [0,\infty) \to \mathbb{R}$ such that

$$\left| \int_0^t \sigma_1(t,s,z) \right| \leq m_{\sigma_1}(t)|z|,\ t \geq 0,\ z \in \mathbb{R}.$$

Let $B : D(A) \subset H \to H$ be the operator defined by $B(t)(z) = b(t)Az$ for $t \geq 0$ and $z \in D(A)$. Let $u(s)(x) = u(s,x)$. We define respectively $g : [0,\infty) \times H \to H$, $h : [0,\infty) \times H \times H \to H$, $f : [0,\infty) \times H \times H \to L(K,H)$ and $I_k : H \to H$ by

$$g(t,u(t-\rho(t),x))(x) = \vartheta(t,u(t-\rho(t),x)),$$
$$h\left(t,u(t-\rho(t)),\int_0^t a(t,s,u(s-\rho(s)))ds\right)(x) = \zeta\left(t,u(t-\rho(t),x),\int_0^t \zeta_1(t,s,u(s-\rho(s),x),x)ds\right),$$
$$f\left(t,u(t-\rho(t)),\int_0^t b(t,s,u(s-\rho(s)))ds\right)(x) = \sigma\left(t,u(t-\rho(t),x),\int_0^t \sigma_1(t,s,u(s-\rho(s),x),x)ds\right),$$
$$I_k(u)(x) = \int_0^\pi \eta_k(s)u(s,x)ds.$$

Then the problem (10) can be written as (1).
Moreover, if b is bounded and C^1 function such that b' is bounded and uniformly continuous, then (A1) and (A2) are satisfied, and hence, by Theorem 3.1, (2) has a resolvent operator $(R(t))_{t\geq 0}$ on H.Using Lemma 5.2 [12], let $\mu > \delta > 1$ and $b(t) < \frac{1}{a}\exp(-\beta)$, for all $t \geq 0$. Then the above resolvent operator decays exponentially to zero. Specifically $||R(t)|| < \exp(-at)$ where $a = 1 - 1/\delta$.

Else, using (1) we can prove that

$$\begin{aligned}
\mathbb{E}\|g(t,z_1) - g(t,z_2)\|^p &= \mathbb{E}\left[\left(\int_0^\pi |\vartheta(t,z_1(x)) - \vartheta(t,z_2(x))|^2 ds\right)^{1/2}\right]^p \\
&\leq \left[\left(\int_0^\pi L_\vartheta |z_1(x) - z_2(x)|^2 ds\right)^{1/2}\right]^p \\
&\leq L_\vartheta^p \mathbb{E}\|z_1 - z_2\|^p
\end{aligned}$$

for all $(t,z_j) \in [0,\infty) \times H,\ j = 1,2$, and $\mathbb{E}\|g(t,z)\|_H^p \leq L_\vartheta^p \|z\|^p$ for all $(t,z) \in [0,\infty) \times H$. By assumptions (2) and (3) we have

$$\begin{aligned}
\mathbb{E}\|h(t,u,y)\|^p &= \mathbb{E}\left[\left(\int_0^\pi |\zeta(t,u(x),y(x))|^2 dx\right)^{1/2}\right]^p \\
&\leq \mathbb{E}\left[\left(\int_0^\pi [L_\zeta(t)|u(x)| + 2^{1-p}|y(x)|]^2\right)^{1/2}\right]^p \\
&\leq 2^{p-1}[(L_\zeta(t))^p \mathbb{E}\|u(x)\|^p + 2^{1-p}\mathbb{E}\|y\|^p] \\
&= m_h(t)\mathbb{E}\|u\|^p + \mathbb{E}\|y\|^p
\end{aligned}$$

for all $(t,z,y) \in [0,\infty,y) \times H \times H$, and

$$\begin{aligned}\mathbb{E}\left\|\int_0^t a(t,s,z)ds\right\|^p &= \mathbb{E}\left[\left(\int_0^\pi \left|\int_0^t \zeta_1(s,z(x))ds\right|^2 dx\right)^{1/2}\right]^p \\ &\leq \left[\left(\int_0^\pi |m_{\zeta_1}(t)z(x)|^2 dx\right)^{1/2}\right]^p \\ &\leq m_a(t)\mathbb{E}\|z\|^p\end{aligned}$$

for all $(t,s,z) \in [0,\infty) \times [0,\infty) \times H$, where $m_h(t) = m_\zeta^p(t),\ m_a(t) = m_{\zeta_1}^p(t)$. Similarly, by using assumptions (4) and (5) we have

$$\mathbb{E}\|f(t,z,y)\|^p \leq m_f(t)\mathbb{E}\|z\|^p + \mathbb{E}\|y\|^p$$

for all $(t,s,y) \in [0,\infty) \times [0,\infty) \times H$, and

$$\mathbb{E}\left\|\int_0^t b(t,s,z)\right\|^p \leq m_b(t)\mathbb{E}\|z\|^p$$

for all $(t,s,z) \in [0,\infty) \times [0,\infty) \times H$, where $m_f(t) = m_\sigma^p(t),\ m_b(t) = m_{\sigma_1}(t)$. Therefore (H1)-(H6) are all satisfied and condition (3.1) holds with $\Theta_h(s) = \Theta_a(s) = \Theta_f(s) = \Theta_b(s) = s$. It is clear that I_k are bounded linear maps with

$$\mathbb{E}\|I_k(z)\|^p \leq d_k\mathbb{E}\|z\|^p,\ z \in H,\ k \in \overline{1,m},$$

where $d_k = \left(\int_0^\pi |\eta_k(s)|^2 ds\right)^{p/2},\ k = 1,\cdots,m$. Moreover, the map I_k is completely continuous. Further, suppose that $(14m)^{p-1}\sum_{k=1}^m d_k < 1$ holds. Then, from Theorem 4.4, we can conclude that the mild solution of (10) is asymptotically stable in p-th mean.

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Kora Hafiz BETE
Institut de Mathématiques et de Sciences Physiques, URMPM, B.P 613, Porto-Novo, Bénin
E-mail address: betekorahafiz1@yahoo.fr

Aziz MANE
Université Gaston Berger de Saint-Louis, UFR SAT, Département de Mathématiques, B.P. 234, Saint-Louis, Sénégal
E-mail address: azizmanesn@outlook.fr

Carlos OGOUYANDJOU
Institut de Mathématiques et de Sciences Physiques, URMPM, B.P 613, Porto-Novo, Bénin
E-mail address: ogouyandjou@imsp-uac.org

Mamadou Abdoul DIOP
Université Gaston Berger de Saint-Louis, UFR SAT, Département de Mathématiques, B.P. 234, Saint-Louis, Sénégal
UMMISCO UMMISCO UMI209 IRD/UPMC, Bondy, France
E-mail address: mamadou-abdoul.diop@ugb.edu.sn