Electronic Journal of Mathematical Analysis and Applications
Vol. 7(1) Jan. 2019, pp. 202-216.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

A NABLA CONFORMABLE FRACTIONAL CALCULUS ON TIME SCALES

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Abstract. In this article, we define and study the nabla conformable fractional derivative and nabla conformable fractional integral on time scales. Many basic properties of the theory are proved.

1. Introduction

In 2014, Khalil et al. in [17], defined a new well-behaved simple fractional derivative which is called "the conformable fractional derivative" depending just on the basic limit definition of the derivative. In particular, Benkhettou et al. [9] extended this definition to an arbitrary time scale, which is a natural extension of the conformable fractional calculus, then developed later in [19, 20].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was born in 1988 with the Ph.D. thesis of Hilger [15]. The reader interested on the subject of time scales is referred in [7, 10, 11, 12, 13, 14].

Motivated by results in [9, 19, 20], we introduce definitions of nabla conformable fractional derivative and integral on time scales and study of their important properties. This article is organized as follows. In Section 2, we review some basic concepts and notations of calculus on time scales. In Section 3, we introduce the definition of nabla conformable fractional derivative and their important properties. In Section 4, we introduce and develop the notion of nabla conformable fractional integral on time scales.

2. Preliminaries

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma,\ \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively. We say that t is right-scattered (resp., left-scattered) if $\sigma(t) > t$ (resp., if $\rho(t) < t$); that t is isolated if it is right-scattered and left-scattered. Also,

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2010 *Mathematics Subject Classification.* 26A33, 26E70, 26A24.

Key words and phrases. Fractional calculus, nabla conformable fractional derivatives and integrals, nabla derivative and integral, calculus on time scales.

Submitted Jan. 11, 2018. Revised May 21, 2018 .

if $t < \sup \mathbb{T}$ and $t = \sigma(t)$, we say that t is right-dense. If $t > \inf \mathbb{T}$ and $t = \rho(t)$, we say that t is left dense. Points that are right dense and left dense are called dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^{\kappa} = \mathbb{T} \backslash \{M\}$, otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. The backward graininess $\nu : \mathbb{T} \to [0, \infty)$ is defined by $\nu(t) := t - \rho(t)$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_{\kappa} = \mathbb{T} \backslash \{m\}$, otherwise, $\mathbb{T}_{\kappa} = \mathbb{T}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$. If $f : \mathbb{T} \to \mathbb{R}$, is a function, then we define the function f^{ρ} (resp., f^{σ}) by $f^{\rho}(t) = (fo\rho)(t) = f(\rho(t))$ (resp., $f^{\sigma}(t) = (fo\sigma)(t) = f(\sigma(t))$) for all $t \in \mathbb{T}$.
A function $f : \mathbb{T} \to \mathbb{R}$ is called *ld*-continuous provided it is continuous at left-dense point in $\mathbb{T}$ and has a right-sided limits exist at right-dense points in $\mathbb{T}$, write $f \in C_{ld}(\mathbb{T}, \mathbb{R})$.
A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.
Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_{\kappa}$, the nabla derivative of f at t, denoted by $f^{\nabla}(t)$, is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t) \left(\rho(t) - s\right) \right| \leq \varepsilon \left| \rho(t) - s \right|, \quad \text{for all } s \in U.$$

If $t \in \mathbb{T}^{\kappa}$, then the delta derivative of f at the point t is the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t) \left(\sigma(t) - s\right) \right| \leq \varepsilon \left| \sigma(t) - s \right|, \qquad \text{for all } s \in U.$$

Definition 1 (Conformable fractional derivative [9]) Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^{\kappa}$ and $\alpha \in]0, 1]$. For $t > 0$, we define $T_{\alpha}(f)(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ-neighborhood $\mathcal{V}_t \subset \mathbb{T}$ (i.e., $\mathcal{V}_t :=]t - \delta, t + \delta[\cap \mathbb{T}$) of t, $\delta > 0$, such that

$$\left| [f(\sigma(t)) - f(s)]\, t^{1-\alpha} - T_{\alpha}(f)(t) \left[\sigma(t) - s\right] \right| \leq \epsilon \left| \sigma(t) - s \right|, \quad \text{for all } s \in \mathcal{V}_t.$$

We call $T_{\alpha}(f)(t)$ the conformable fractional derivative of f of order α at t, and we define the conformable fractional derivative at 0 as $T_{\alpha}(f)(0) = \lim_{t \to 0^+} T_{\alpha}(f)(t)$.

The ∇-measure and ∇-integration are defined the same as those in [6, 12, 14].

3. Nabla Conformable Fractional Derivative

We begin by introducing the notion of nabla conformable fractional derivative of order $\alpha \in]0, 1]$ for function defined on arbitrary time scale $\mathbb{T}$.
Definition 2 Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}_{\kappa}$, and $\alpha \in]0, 1]$. For $t > 0$, we define $T_{\nabla,\alpha}(f)(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ-neighborhood $\mathcal{V}_t \subset \mathbb{T}$ (i.e., $\mathcal{V}_t =]t - \delta, t + \delta[\cap \mathbb{T}$) of t, $\delta > 0$, such that

$$\left| \left(f(\rho(t)) - f(s)\right) t^{1-\alpha} - T_{\nabla,\alpha}(f)(t) \left(\rho(t) - s\right) \right| \leq \epsilon \left| \rho(t) - s \right|,$$

for all $s \in \mathcal{V}_t$. We call $T_{\nabla,\alpha}(f)(t)$ the nabla conformable fractional derivative of f of order α at t, and we define the nabla conformable fractional derivative at 0 as $T_{\nabla,\alpha}(f)(0) = \lim_{t \to 0^+} T_{\nabla,\alpha}(f)(t)$. Note that If $\alpha = 1$, and f is nabla conformable fractional derivative of order α, then $T_{\nabla,\alpha}(f) = f^{\nabla}(t)$.
We denote:

(i) $T_{\nabla,\alpha}(f) = f_{\nabla}^{(\alpha)}$.

(ii) $$C^{\alpha}([a,b]_{\mathbb{T}},\mathbb{R}) = \Big\{ f : [a,b]_{\mathbb{T}} \to \mathbb{R},\ f \text{ is nabla conformal fractional differentiable of order } \alpha \text{ on } [a,b]_{\mathbb{T}} \text{ and } T_{\nabla,\alpha}(f) \in C([a,b]_{\mathbb{T}},\mathbb{R}) \Big\}.$$

(iii) $$C_{ld}^{\alpha}([a,b]_{\mathbb{T}},\mathbb{R}) = \Big\{ f : [a,b]_{\mathbb{T}} \to \mathbb{R},\ f \text{ is nabla conformal fractional differentiable of order } \alpha \text{ on } [a,b]_{\mathbb{T}} \text{ and } T_{\nabla,\alpha}(f) \in C_{ld}([a,b]_{\mathbb{T}},\mathbb{R}) \Big\}.$$

Some useful properties of the nabla conformable fractional derivative of f of order α are given in the following theorem.

Theorem 1 Let $\alpha \in]0,1]$ and $\mathbb{T}$ be a time scale. Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}_{\kappa}$. The following properties hold.

(i) If f is nabla conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.

(ii) If f is continuous at t and t is left-scattered, then f is nabla conformable fractional differentiable of order α at t with
$$T_{\nabla,\alpha}(f)(t) = \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}. \tag{1}$$

(iii) If t is left-dense, then f is nabla conformable fractional differentiable of order α at t if, and only if, the limit $\lim_{s\to t} \frac{f(t)-f(s)}{(t-s)} t^{1-\alpha}$ exists as a finite number. In this case,
$$T_{\nabla,\alpha}(f)(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}. \tag{2}$$

(iv) If f is nabla conformable fractional differentiable of order α at t, then
$$f(\rho(t)) = f(t) - (\nu(t)) t^{\alpha-1} T_{\nabla,\alpha}(f)(t).$$

Proof. (i) Assume that f is nabla conformable fractional differentiable at t. Then, there exists a neighborhood $\mathcal{V}_t$ of t such that
$$\left| (f(\rho(t)) - f(s))\, t^{1-\alpha} - T_{\nabla,\alpha}(f)(t)\, (\rho(t) - s) \right| \le \epsilon\, |\rho(t) - s|$$
for $s \in \mathcal{V}_t$. Therefore,
$$\begin{aligned} |f\,(t) - f\,(s)| &\le \left| (f(\rho(t) - f(s)) - T_{\nabla,\alpha}(f)(t)\, (\rho(t) - s)\, t^{\alpha-1} \right| + |(f(\rho(t)) - f(t))| \\ &\quad + \left| f_{\nabla}^{(\alpha)}(t) \right| |(\rho(t) - s)|\, |t^{\alpha} - 1|, \end{aligned}$$
for all $s \in \mathcal{V}_t \cap]t - \epsilon, t + \epsilon[$ and, since t is a left-dense point,
$$\begin{aligned} |f\,(t) - f\,(s)| &\le \left| (f^{\rho}(t) - f(s)) - f^{(\alpha)}(t)\, (\rho(t) - s)^{\alpha} \right| + \left| f_{\nabla}^{(\alpha)}(t)\, (t - s)^{\alpha} \right| \\ &\le \epsilon\delta + \left| t^{\alpha-1} \right| |T_{\nabla,\alpha}(f)(t)|\, \delta. \end{aligned}$$

Since $\delta \to 0$ when $s \to t$, and $t > 0$, it follows the continuity of f at t.

(ii) Assume that f is continuous at t and t is left-scattered. By continuity,
$$\lim_{s\to t} \frac{f(\rho(t)) - f(s)}{\rho(t) - s} t^{1-\alpha} = \frac{f(\rho(t)) - f(t)}{\rho(t) - t} t^{1-\alpha} = \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}.$$
Hence, given $\epsilon > 0$ and $\alpha \in]0,1]$, there is a neighborhood $\mathcal{V}_t$ of t such that
$$\left| \frac{f(\rho(t)) - f(s)}{\rho(t) - s} t^{1-\alpha} - \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha} \right| \le \epsilon$$

for all $s \in \mathcal{V}_t$. It follows that

$$\left|[f(\rho(t)) - f(s)]\, t^{1-\alpha} - \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}(\rho(t) - s)\right| \leq \epsilon|\rho(t) - s|$$

for all $s \in \mathcal{V}_t$. The desired equality (1) follows from Definition 2.

(iii) Assume that f is nabla conformable fractional differentiable of order α at t and t is left-dense. Let $\epsilon > 0$ be given. Since f is nabla conformable fractional differentiable of order α at t, there is a neighborhood $\mathcal{V}_t$ of t such that $\left|[f(\rho(t)) - f(s)]t^{1-\alpha} - T_{\nabla,\alpha}(f)(t)(\rho(t) - s)\right| \leq \epsilon|\rho(t) - s|$ for all $s \in \mathcal{V}_t$. Because $\rho(t) = t$,

$$\left|\frac{f(t) - f(s)}{t - s} t^{1-\alpha} - T_{\nabla,\alpha}(f)(t)\right| \leq \epsilon$$

for all $s \in \mathcal{V}_t$, $s \neq t$. Therefore, we get the desired result (2). Now, assume that the limit on the right-hand side of (2) exists and is equal to L, and t is left-dense. Then, there exists $\mathcal{V}_t$ such that $\left|(f(t) - f(s))t^{1-\alpha} - L(t - s)\right| \leq \epsilon|t - s|$ for all $s \in \mathcal{V}_t$. Because t is left-dense,

$$\left|(f(\rho(t)) - f(s))t^{1-\alpha} - L(\rho(t) - s)\right| \leq \epsilon|\rho(t) - s|,$$

which lead us to the conclusion that f is nabla conformable fractional differentiable of order α at t and $T_{\nabla,\alpha}(f)(t) = L$.

(iv) If t is left-dense, i.e., $\rho(t) = t$, then $\nu(t) = 0$ and $f(\rho(t)) = f(t) = f(t) - \nu(t)T_{\nabla,\alpha}(f)(t)t^{1-\alpha}$. On the other hand, if t is left-scattered, i.e., $\rho(t) < t$, then by (iii)

$$f(\rho(t)) = f(t) - \nu(t)t^{\alpha-1} \cdot \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha} = f(t) - (\nu(t))t^{\alpha-1}T_{\nabla,\alpha}(f)(t).$$

The proof is complete.

Example 1 (i) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = c$ for all $t \in \mathbb{T}$, $c \in \mathbb{R}$, then

$$T_{\nabla,\alpha}(f)(t) = (c)_{\nabla}^{(\alpha)} = 0.$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then

$$T_{\nabla,\alpha}(f)(t) = (t)_{\nabla}^{(\alpha)} = \begin{cases} t^{1-\alpha} & \text{if } \alpha \neq 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

(iii) Let $p > 0$, fix $t_0 \in \mathbb{T}$ and $f(t) = \hat{e}_p(t, t_0)\ for\ t \in \mathbb{T},$ the nabla exponential functiont given in Definition 3.10. of [12]. Then

$$T_{\nabla,\alpha}(f)(t) = t^{1-\alpha} p\hat{e}_p(t, t_0).$$

Example 2 (i) Function $f : \mathbb{R} \to \mathbb{R}$ is nabla conformable fractional differentiable of order α at point $t \in \mathbb{R}$ if, and only if, the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s} t^{1-\alpha}$ exists as a finite number. In this case,

$$T_{\nabla,\alpha}(f)(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}. \tag{3}$$

If $\alpha = 1$, then $T_{\nabla,\alpha}(f) = f^{\nabla}(t) = f'(t)$.

The identity (3) corresponds to the conformable derivative introduced in [17].

(ii) Let $h > 0$. If $f : h\mathbb{Z} \to \mathbb{R}$, then f is nabla conformable fractional differentiable of order α at $t \in h\mathbb{Z}$ with

$$T_{\nabla,\alpha}(f)(t) = \frac{f(t) - f(t - h)}{h} t^{1-\alpha}.$$

If $\alpha = 1$ and $h = 1$, then $T_{\nabla,\alpha}(f) = \nabla f(t) = f(t) - f(t-1)$, where ∇ is the backward difference operator.

Next, we would like to be able to find the derivatives of sums, products, and quotients of nabla conformable fractional differentiable functions. This is possible according to the following theorem.

Theorem 2 Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla conformable fractional differentiable of order α. Then,

(i) the sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla conformable fractional differentiable with

$$T_{\nabla,\alpha}(f+g) = T_{\nabla,\alpha}(f) + T_{\nabla,\alpha}(g);$$

(ii) for any $\lambda \in \mathbb{R}$, $\lambda f : \mathbb{T} \to \mathbb{R}$ is nabla conformable fractional differentiable with

$$T_{\nabla,\alpha}(\lambda f) = \lambda T_{\nabla,\alpha}(f);$$

(iii) if f and g are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is nabla conformable fractional differentiable with

$$T_{\nabla,\alpha}(fg) = T_{\nabla,\alpha}(f)g + f^{\rho}T_{\nabla,\alpha}(g) = T_{\nabla,\alpha}(f)g^{\rho} + fT_{\nabla,\alpha}(g);$$

(iv) if f is continuous, then $1/f$ is nabla conformable fractional differentiable with

$$T_{\nabla,\alpha}\left(\frac{1}{f}\right) = -\frac{T_{\nabla,\alpha}(f)}{ff^{\rho}},$$

valid at all points $t \in \mathbb{T}_{\kappa}$ for which $f(t)f^{\rho}(t) \neq 0$;

(v) if f and g are continuous, then f/g is nabla conformable fractional differentiable with

$$T_{\nabla,\alpha}\left(\frac{f}{g}\right) = \frac{T_{\nabla,\alpha}(f)g - fT_{\nabla,\alpha}(g)}{gg^{\rho}},$$

valid at all points $t \in \mathbb{T}_{\kappa}$ for which $g(t)g^{\rho}(t) \neq 0$.

Proof. Let us consider that $\alpha \in]0,1]$, and let us assume that f and g are nabla conformable fractional differentiable at $t \in \mathbb{T}_{\kappa}$.

(i) Let $\epsilon > 0$. Then there exist neighborhoods $\mathcal{V}_t$ and $\mathcal{U}_t$ of t for which

$$\left|[f(\rho(t)) - f(s)]t^{1-\alpha} - T_{\nabla,\alpha}(f)(t)\,(\rho(t) - s)\right| \leq \frac{\epsilon}{2}|\rho(t) - s| \quad \text{for all} s \in \mathcal{V}_t$$

and

$$\left|[g(\rho(t)) - g(s)]t^{1-\alpha} - T_{\nabla,\alpha}(g)(t)(\rho(t) - s)\right| \leq \frac{\epsilon}{2}|\rho(t) - s| \quad \text{for all} s \in \mathcal{U}_t.$$

Let $\mathcal{W}_t = \mathcal{V}_t \cap \mathcal{U}_t$. Then

$$\left|[(f+g)(\rho(t)) - (f+g)(s)]t^{1-\alpha} - [T_{\nabla,\alpha}(f)(t) + T_{\alpha}(g)(t)]\,(\rho(t) - s)\right| \leq \epsilon|\rho(t) - s|$$

for all $s \in \mathcal{W}_t$. Thus, $f + g$ is nabla conformable differentiable at t and

$$T_{\nabla,\alpha}(f+g)(t) = T_{\nabla,\alpha}(f)(t) + T_{\nabla,\alpha}(g)(t).$$

(ii) Let $\epsilon > 0$. Then $\left|[f(\rho(t)) - f(s)]t^{1-\alpha} - T_{\nabla,\alpha}(f)(t)(\rho(t) - s)\right| \leq \epsilon|\rho(t) - s|$ for all s in a neighborhood $\mathcal{V}_t$ of t. It follows that

$$\left|[(\lambda f)(\rho(t)) - (\lambda f)(s)]t^{1-\alpha} - \lambda T_{\nabla,\alpha}(f)(t)(\rho(t) - s)\right| \leq \epsilon|\lambda|\,|\rho(t) - s| \text{for all} s \in \mathcal{V}_t.$$

Therefore, λf is nabla conformable fractional differentiable at t and $T_{\nabla,\alpha}(\lambda f) = \lambda T_{\nabla,\alpha}(f)$ holds at t.
(iii) If t is left-scattered, then

$$\begin{aligned} T_{\nabla,\alpha}(fg)(t) &= \left[\frac{f(t)-f(\rho(t))}{\nu(t)}t^{1-\alpha}\right] g(\rho(t)) + \left[\frac{g(t)-g(\rho(t))}{\nu(t)}t^{1-\alpha}\right] f(t) \\ &= T_{\nabla,\alpha}(f)(t)g(\rho(t)) + f(t)T_{\nabla,\alpha}(g)(t). \end{aligned}$$

If t is left-dense, then

$$\begin{aligned} T_{\nabla,\alpha}(fg)(t) &= \lim_{s\to t}\left[\frac{f(t)-f(s)}{t-s}t^{1-\alpha}\right] g(t) + \lim_{s\to t}\left[\frac{g(t)-g(s)}{t-s}t^{1-\alpha}\right] f(s) \\ &= T_{\nabla,\alpha}(f)(t)g(t) + T_{\nabla,\alpha}(g)(t)f(t) = T_{\nabla,\alpha}(f)(t)g(\rho(t)) + T_{\nabla,\alpha}(g)(t)f(t). \end{aligned}$$

The other product rule formula follows by interchanging the role of functions f and g.
(iv) From Example 1 (i), we know that $T_{\nabla,\alpha}\left(f\cdot\frac{1}{f}\right)(t) = (1)_{\nabla}^{(\alpha)} = 0$. Therefore, by (iii)

$$T_{\nabla,\alpha}\left(\frac{1}{f}\right)(t)f(\rho(t)) + T_{\nabla,\alpha}(f)(t)\frac{1}{f(t)} = 0.$$

Since we are assuming $f(\rho(t)) \neq 0$, $T_{\nabla,\alpha}\left(\frac{1}{f}\right)(t) = -\dfrac{T_{\nabla,\alpha}(f)(t)}{f(t)f(\rho(t))}$.
(v) We use (ii) and (iv) to obtain

$$\begin{aligned} T_{\nabla,\alpha}\left(\frac{f}{g}\right)(t) &= T_{\nabla,\alpha}\left(f\cdot\frac{1}{g}\right)(t) = f(t)T_{\nabla,\alpha}\left(\frac{1}{g}\right)(t) + T_{\nabla,\alpha}(f)(t)\frac{1}{g(\rho(t))} \\ &= \frac{T_{\nabla,\alpha}(f)(t)g(t) - f(t)T_{\nabla,\alpha}(g)(t)}{g(t)g(\rho(t))}. \end{aligned}$$

The proof is complete.
Theorem 3 Let c be a constant, $m \in \mathbb{N}$, $\alpha \in]0,1]$ and $f(t) = (t-c)^m$. Then

$$T_{\nabla,\alpha}(f)(t) = t^{1-\alpha}\sum_{i=0}^{m-1}(t-c)^{m-1-i}(\rho(t)-c)^i. \tag{4}$$

If $c = 0$, then $T_{\nabla,\alpha}(f)(t) = (t^m)_{\nabla}^{(\alpha)} = t^{1-\alpha}\sum_{i=0}^{m-1}(t)^{m-1-i}(\rho(t))^i$.
Proof. We prove the first formula by induction. If $m = 1$, then $f(t) = t - c$ and $T_{\nabla,\alpha}(f)(t) = t^{1-\alpha}$ holds from Example 1 and Theorem 2 (i). Now assume that

$$T_{\nabla,\alpha}(f)(t) = t^{1-\alpha}\sum_{i=0}^{m-1}(t-c)^{m-1-i}(\rho(t)-c)^i$$

holds for $f(t) = (t-c)^m$ and let $F(t) = (t-c)^{m+1} = (t-c)f(t)$. We use Theorem 2 (iii) to obtain

$$(F(t))^{(\alpha)} = T_{\nabla,\alpha}(t-c)f(\rho(t)) + T_{\nabla,\alpha}(f)(t)(t-c) = t^{1-\alpha}\sum_{i=0}^{m}(t-c)^{m-p}(\rho(t)-c)^i.$$

Hence, by mathematical induction, (4) holds. If $c = 0$, then we have

$$T_{\nabla,\alpha}(f)(t) = t^{1-\alpha}\sum_{i=0}^{m-1}(t)^{m-1-i}(\rho(t))^i.$$

Note that if t is left-dense, then $T_{\nabla,\alpha}(f)(t) = mt^{m-\alpha}$.

Theorem 4 (Chain rule) Let $\alpha \in]0,1]$. Assume $g : \mathbb{T} \to \mathbb{R}$ is continuous and nabla conformable fractional differentiable of order α at $t \in \mathbb{T}_\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[\rho(t), t]$ with

$$T_{\nabla,\alpha}(f \circ g)(t) = f'(g(c))\, T_{\nabla,\alpha}(g)(t). \tag{5}$$

Proof. Let $t \in \mathbb{T}_\kappa$. First we consider t to be left-scattered. In this case,

$$T_{\nabla,\alpha}(f \circ g)(t) = \frac{f(g(t)) - f(g(\rho(t)))}{\nu(t)} t^{1-\alpha}.$$

If $g(\rho(t)) = g(t)$, then we get $T_{\nabla,\alpha}(f \circ g)(t) = 0$ and $T_{\nabla,\alpha}(g)(t) = 0$. Therefore, (5) holds for any c in the real interval $[\rho(t), t]$. Now assume that $g(\rho(t)) \neq g(t)$. By the mean value theorem we have

$$T_{\nabla,\alpha}(f \circ g)(t) = \frac{f(g(\rho(t))) - f(g(t))}{g(\rho(t)) - g(t)} \cdot \frac{g(t) - g(\rho(t))}{\nu(t)} t^{1-\alpha} = f'(\xi) T_{\nabla,\alpha}(g)(t),$$

where ξ between $g(\rho(t))$ and $g(t)$. Since $g : \mathbb{T} \to \mathbb{R}$ is continuous, there is a $c \in [\rho(t), t]$ such that $g(c) = \xi$, which gives the desired result. Now let us consider the case when t is left-dense. In this case

$$T_{\nabla,\alpha}(f \circ g)(t) = \lim_{s \to t} \frac{f(g(t)) - f(g(s))}{g(t) - g(s)} \cdot \frac{g(t) - g(s)}{t - s} t^{1-\alpha}.$$

By the mean value theorem, there exist ξ_s between $g(\rho(t))$ and $g(t)$ such that

$$T_{\nabla,\alpha}(f \circ g)(t) = \lim_{s \to t} \left\{ f'(\xi_s) \cdot \frac{g(t) - g(s)}{t - s} t^{1-\alpha} \right\}.$$

By the continuity of g, we get that $\lim\limits_{s \to t} \xi_s = g(t)$. Then $T_{\nabla,\alpha}(f \circ g)(t) = f'(g(t)) \cdot T_{\nabla,\alpha}(g)(t)$. Since t is left-dense, we conclude that $c = t = \rho(t)$, which gives the desired result.

We define the nabla conformable fractional derivative $T_{\nabla,\alpha}$ for $\alpha \in (m,\ m+1]$, where m is some natural number.

Definition 3 Let $\mathbb{T}$ be a time scale, $\alpha \in (m,\ m+1]$, $m \in \mathbb{N}$, and let f be m times nabla differentiable at $t \in \mathbb{T}_{\kappa^m}$. We define the nabla conformable fractional derivative of f of order α as $T_{\nabla,\alpha}(f)(t) = T_{\nabla,\alpha-m}\left(f^{\nabla^m}\right)(t)$.

Theorem 5 Let $\alpha \in (m,\ m+1]$, $m \in \mathbb{N}$. The following relation holds:

$$T_{\nabla,\alpha}(f)(t) = t^{1+m-\alpha} f^{\nabla^{1+m}}(t). \tag{6}$$

Proof. Let f be a function m times nabla-differentiable. For $\alpha \in (m, m+1]$, there exist $\beta \in (0,1]$ such that $\alpha = m + \beta$. Using Definition 3, $T_{\nabla,\alpha}(f) = T_{\nabla,\beta}\left(f^{\nabla^m}\right)$. From the definition of (higher-order) nabla derivative and Theorem 1 (ii) and (iii), it follows that $T_{\nabla,\alpha}(f)(t) = t^{1-\beta}\left(f^{\nabla^m}\right)^{\nabla}(t)$.

Remark 1 In(6), when $m = 0$, we have $T_{\nabla,\alpha}(f)(t) := t^{1-\alpha} f^{\nabla}(t),\ \alpha \in (0,1]$.

Next, we introduce the nabla conformable fractional derivative on time scales for vector-valued functions and study some of their important properties.

Definition 4 The function $f : \mathbb{T} \to \mathbb{R}^n$ is called *ld*-continuous provided it is continuous at left-dense point in $\mathbb{T}$ and has a right-sided limits exist at right-dense points in $\mathbb{T}$, write $f \in C_{ld}\left(\mathbb{T}, \mathbb{R}^n\right)$.

Definition 5 Assume $f : \mathbb{T} \to \mathbb{R}^n$, is a function, $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))$ and let $t \in \mathbb{T}_\kappa$. Then one defines

$$T_{\nabla,\alpha}(f)(t) = \Big(T_{\nabla,\alpha}(f_1)(t), T_{\nabla,\alpha}(f_2)(t), \cdots, T_{\nabla,\alpha}(f_n)(t)\Big)$$

provided it exists. One calls $T_{\nabla,\alpha}(f)(t)$ the nabla conformable fractional derivative of f of order α at $t > 0$. The function f is nabla conformal fractional differentiable of order α provided $T_{\nabla,\alpha}(f)(t)$ exists for all t in $\mathbb{T}_\kappa$. The function $T_{\nabla,\alpha}(f) : \mathbb{T}_\kappa \to \mathbb{R}^n$ is then called the nabla conformable fractional derivative of f of order α, and we define the nabla conformable fractional derivative at 0 as $T_{\nabla,\alpha}(f)(0) = \lim\limits_{t\to 0^+} T_{\nabla,\alpha}(f)(t)$.

Definition 6 Let $\mathbb{T}$ be a time scale, $\alpha \in (m,\ m+1]$, $m \in \mathbb{N}$, and let $f : \mathbb{T} \to \mathbb{R}^n$ be m times nabla differentiable at $t \in \mathbb{T}_{\kappa^m}$. We define the nabla conformable fractional derivative of f of order α as $T_{\nabla,\alpha}(f)(t) := T_{\nabla,\alpha-m}\left(f^{\nabla^m}\right)(t)$.

Combining Definition 5 and Theorems 1, 2 we have the following theorems.

Theorem 6 Let $\alpha \in]0, 1]$. Assume $f : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}_\kappa$. The following properties hold:

(i) If f is nabla conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.

(ii) If f is continuous at t and t is left-scattered, then f is nabla conformable fractional differentiable of order α at t with

$$T_{\nabla,\alpha}(f)(t) = \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}. \tag{7}$$

(iii) If t is left-dense, then f is nabla conformable fractional differentiable of order α at t if and only if the limit $\lim_{s\to t} \frac{f(t)-f(s)}{(t-s)} t^{1-\alpha}$ exists as a finite number. In this case,

$$T_{\nabla,\alpha}(f)(t) = \lim_{s\to t} \frac{f(t) - f(s)}{(t - s)} t^{1-\alpha}. \tag{8}$$

(iv) If f is nabla conformable fractional differentiable of order α at t, then

$$f(\rho(t)) = f(t) - (\nu(t))t^{\alpha-1} T_{\nabla,\alpha}(f)(t).$$

Theorem 7 Assume $f, g : \mathbb{T} \to \mathbb{R}^n$ are nabla conformable fractional differentiable of order α. Then,

(i) the sum $f + g : \mathbb{T} \to \mathbb{R}^n$ is nabla conformable fractional differentiable with

$$T_{\nabla,\alpha}(f + g) = T_{\nabla,\alpha}(f) + T_{\nabla,\alpha}(g);$$

(ii) for any $\lambda \in \mathbb{R}$, $\lambda f : \mathbb{T} \to \mathbb{R}^n$ is nabla conformable fractional differentiable with $T_{\nabla,\alpha}(\lambda f) = \lambda T_{\nabla,\alpha}(f)$;

(iii) if f and g are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}^n$ is nabla conformable fractional differentiable with

$$T_{\nabla,\alpha}(fg) = T_{\nabla,\alpha}(f)g + (f \circ \rho)T_{\nabla,\alpha}(g) = T_{\nabla,\alpha}(f)(g \circ \rho) + fT_{\nabla,\alpha}(g).$$

4. Nabla Conformable Fractional Integral

Now we introduce the nabla conformable fractional integral (or nabla α-fractional integral) on time scales.

Definition 7 Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then the nabla α-fractional integral of f, $0 < \alpha \le 1$, is defined by $\int f(t)\nabla_\alpha t = \int f(t)t^{\alpha-1}\nabla t$.

Note that If $\alpha = 1$, then $\int f(t)\nabla_\alpha t = \int f(t)\nabla t$ is the integral given in [12]. If $\mathbb{T} = \mathbb{R}$, then $\int f(t)\nabla_\alpha t = \int t^{\alpha-1} f(t)dt$ is the conformable fractional integral given in [17].

Definition 8 Suppose $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Denote the indefinite nabla α-fractional integral of f of order α, $\alpha \in (0,\ 1]$, as follows: $F_{\nabla,\alpha}(t) = \int f(t)\nabla_\alpha t$. Then, for all $a, b \in \mathbb{T}$, we define the Cauchy nabla α-fractional integral by $\int_a^b f(t)\nabla_\alpha t = F_{\nabla,\alpha}(b) - F_{\nabla,\alpha}(a)$.

Definition 9 Assume $f : \mathbb{T} \to \mathbb{R}$, is a function. Let A is a ∇-measurable subset of $\mathbb{T}$. f is nabla α-integrable on A if and only if $t^{\alpha-1} f(t)$ is integrable on A, and $\int_A f(t)\nabla_\alpha t = \int_A t^{\alpha-1} f(t)\nabla t$.

Theorem 8 Let $\alpha \in (0,\ 1]$. Then, for any ld-continuous function $f : \mathbb{T} \to \mathbb{R}$, there exist a function $F_{\nabla,\alpha} : \mathbb{T} \to \mathbb{R}$ such that $T_{\nabla,\alpha}\left(F_{\nabla,\alpha}\right)(t) = f(t)$ for all $t \in \mathbb{T}_\kappa$. Function $F_{\nabla,\alpha}$ is said to be an nabla α-antiderivative of f.

Proof. The case $\alpha = 1$ is proved in [11]. Let $\alpha \in (0,\ 1)$. Suppose f is ld-continuous. By Theorem 1.16 of [12], f is regulated. Then, $F_{\nabla,\alpha}(t) = \int f(t)\nabla_\alpha t$ is nabla conformable fractional differentiable on $\mathbb{T}_\kappa$. Using (6) and Definition 7, we obtain that

$$T_{\nabla,\alpha}\left(F_{\nabla,\alpha}\right)(t) = t^{1-\alpha}\left(F_{\nabla,\alpha}(t)\right)^\nabla = f(t),\ t \in \mathbb{T}_\kappa.$$

Theorem 9 Let $\alpha \in (0,\ 1]$, $a, b, c \in \mathbb{T}$, $\lambda, \gamma \in \mathbb{R}$, and f, g be two ld-continuous functions. Then,

(i) $\displaystyle\int_a^b [\lambda f(t) + \gamma g(t)]\nabla_\alpha t = \lambda \int_a^b f(t)\nabla_\alpha t + \gamma \int_a^b g(t)\nabla_\alpha t;$

(ii) $\displaystyle\int_a^b f(t)\nabla_\alpha t = -\int_b^a f(t)\nabla_\alpha t;$

(iii) $\displaystyle\int_a^b f(t)\nabla_\alpha t = \int_a^c f(t)\nabla_\alpha t + \int_c^b f(t)\nabla_\alpha t;$

(iv) $\displaystyle\int_a^a f(t)\nabla_\alpha t = 0;$

(v) if there exist $g : \mathbb{T} \to \mathbb{R}$ with $|f(t)| \le g(t)$ for all $t \in [a,\ b]$, then

$$\left|\int_a^b f(t)\nabla_\alpha t\right| \le \int_a^b g(t)\nabla_\alpha t;$$

(vi) if $f(t) > 0$ for all $t \in [a,\ b]$, then $\displaystyle\int_a^b f(t)\nabla_\alpha t \ge 0.$

Proof. The relations follow from Definitions 7 and 8, analogous properties of the nabla-integral, and the properties of Section 3 for the nabla conformable fractional derivative on time scales.

Theorem 10 If $f : \mathbb{T}_\kappa \to \mathbb{R}$ is a ld-continuous function and $t \in \mathbb{T}_\kappa$, then

$$\int_{\rho(t)}^t f(s)\nabla_\alpha s = \nu(t) f(t) t^{\alpha-1}.$$

Proof. Let f be a ld-continuous function on $\mathbb{T}_\kappa$. Then f is a regulated function. By Definition 8 and Theorem 8, there exist an antiderivative $F_{\nabla,\alpha}$ of f satisfying

$$\int_{\rho(t)}^t f(s)\nabla_\alpha s = F_{\nabla,\alpha}(t) - F_{\nabla,\alpha}(\rho(t)) = T_{\nabla,\alpha}\left(F_{\nabla,\alpha}\right)(t)\nu(t)t^{1-\alpha} = \nu(t) f(t) t^{1-\alpha}.$$

This concludes the proof.

Theorem 11 Let $a, b \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ be ld-continuous function. Then we have

the following.
(i) If $\mathbb{T}=\mathbb{R}$, then $\int_a^b f(t)\nabla_\alpha t=\int_a^b f(t)t^{\alpha-1}dt$ where the integral on the right is the conformable fractional integral given in [17]. If $\alpha=1$, then it reduces to the usual Riemann integral.
(ii) If $[a,b]_{\mathbb{T}}$ consists of only isolated points, then

$$\int_a^b f(t)\nabla_\alpha t=\begin{cases}\sum_{t\in(a,b]_{\mathbb{T}}}\nu(t)t^{\alpha-1}f(t) & if\ a<b\\ 0 & if\ \ a=b\\ -\sum_{t\in(b,a]_{\mathbb{T}}}\nu(t)t^{\alpha-1}f(t) & if\ a>b\end{cases}$$

(iii) If $\mathbb{T}=h\mathbb{Z}=\{hk:k\in\mathbb{Z}\}$, where $h>0$, then

$$\int_a^b f(t)\nabla_\alpha t=\begin{cases}\sum_{k=\frac{a}{h}+1}^{\frac{b}{h}}h(kh)^{\alpha-1}f(kh) & if\ a<b\\ 0 & if\ \ a=b\\ -\sum_{k=\frac{b}{h}+1}^{\frac{a}{h}}h(kh)^{\alpha-1}f(kh) & if\ a>b\end{cases}$$

(iv) If $\mathbb{T}=\mathbb{Z}$ then

$$\int_a^b f(t)\nabla_\alpha t=\begin{cases}\sum_{t=a+1}^{b}t^{\alpha-1}f(t) & if\ a<b\\ 0 & if\ \ a=b\\ -\sum_{t=b+1}^{a}t^{\alpha-1}f(t) & if\ a>b\end{cases}$$

Proof. Part (i). It follows from Example 2 (i).
Part (ii). First, note that $[a,b]_{\mathbb{T}}$ contains only finitely many points since each point in $[a,b]_{\mathbb{T}}$ is isolated. Assume that $a<b$ and let $[a,b]=\{t_0,t_1,\cdots,t_n\}$, where

$$a=t_0<t_1<t_2<\cdots<t_n=b$$

By virtue of Theorem 9 (iii),

$$\int_a^b f(t)\nabla_\alpha t=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}f(t)\nabla_\alpha t=\sum_{i=0}^{n-1}\int_{\rho(t_{i+1})}^{\sigma(t_{i+1})}f(t)\nabla_\alpha t=\sum_{i=0}^{n-1}\nu(t_{i+1})f(t_{i+1})t_{i+1}^{\alpha-1}.$$

Consequently,

$$\int_a^b f(t)\nabla_\alpha t=\sum_{t\in(a,b]_{\mathbb{T}}}\nu(t)t^{\alpha-1}f(t).$$

If $a>b$, then the result follows from what we just proved and Theorem 9 (ii). If $a=b$, then the result follows from Theorem 9 (vi). Part (iii) and (iv) are special cases of Part (ii). The proof is complete.
Example 3 (i) If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)=ct^{1-\alpha}$ for all $t\in\mathbb{T}$, $c\in\mathbb{R}$, then

$$\int_a^b f(t)\nabla_\alpha t=c(b-a)$$

(ii) If $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(t)=t$ for all $t\in\mathbb{R}$, then

$$\int_a^b f(t)\nabla_\alpha t=\int_a^b t^\alpha dt=\frac{1}{\alpha+1}(b^{\alpha+1}-a^{\alpha+1})$$

(iii) If $f : \frac{1}{2}\mathbb{N} \to \mathbb{R}$ is defined by $f(t) = 2^t$ and $\alpha = \frac{1}{2}$, then

$$\int_1^3 2^t \nabla_{\frac{1}{2}} t = \frac{1}{2} \sum_{t\in(1,3]_{\frac{1}{2}\mathbb{N}}} \sqrt{\frac{1}{t}}\, 2^t = \frac{1}{2}\Big(\sqrt{\frac{2}{3}}\, 2^{\frac{3}{2}} + \sqrt{\frac{1}{2}}\, 2^2 + \sqrt{\frac{2}{5}}\, 2^{\frac{5}{2}} + \sqrt{\frac{1}{3}}\, 2^3\Big) = \sqrt{2} + \frac{6}{\sqrt{3}} + \frac{4}{\sqrt{5}}.$$

Lemma 1 Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$. If $T_{\nabla,\alpha}(f)(t) \geq 0$ for all $t \in [a, b]_{\mathbb{T}}$, then f is an increasing function on $[a, b]_{\mathbb{T}}$.
Proof. Assume $T_{\nabla,\alpha}(f)$ exist on $[a, b]_{\mathbb{T}}$ and $T_{\nabla,\alpha}(f)(t) \geq 0$ for all $t \in [a, b]_{\mathbb{T}}$. Then, by (i) of Theorem 1, $T_{\nabla,\alpha}(f)$ is continuous on $[a, b]_{\mathbb{T}}$ and, therefore, by Theorem 9 (vi),

$$\int_s^t T_{\nabla,\alpha} f(\xi)\nabla_\alpha \xi \geq 0 \quad \text{for } s, t \text{ such that } a \leq s \leq t \leq b.$$

From Definition 8, $f(t) = f(s) + \int_s^t T_{\nabla,\alpha} f(\xi)\nabla_\alpha \xi \geq f(s)$.
Theorem 12 Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function on $[a, b]_{\mathbb{T}}$ that is nabla conformal fractional differentiable of order α on $(a, b]_{\mathbb{T}}$ and satisfies $f(a) = f(b)$. Then there exist $\xi, \eta \in [a, b]_{\kappa,\mathbb{T}}$ such that

$$T_\alpha(f)(\xi) \leq 0 \leq T_\alpha(f)(\eta).$$

Proof. Since the function f is continuous on the compact set $[a, b]_{\mathbb{T}}$, f assumes its minimum m and its maximum M Therefore there exist $\xi, \eta \in [a, b]_{\mathbb{T}}$? such that $m = f(\xi)$ and $M = f(\eta)$. Since $f(a) = f(b)$, we may assume that $\xi, \eta \in [a, b]_{\kappa,\mathbb{T}}$. By Lemma 1, we have

$$T_{\nabla,\alpha}(f)(\xi) \leq 0 \leq T_{\nabla,\alpha}(f)(\eta).$$

Theorem 13 (Mean value theorem). Let $0 < a < b$ and f be a continuous function on $[a, b]_{\mathbb{T}}$ which is nabla conformal fractional differentiable of order α on $[a, b]_{\kappa,\mathbb{T}}$. Then there exist $\xi, \eta \in [a, b]_{\kappa,\mathbb{T}}$ such that

$$\xi^{\alpha-1} T_{\nabla,\alpha}(f)(\xi) \leq \frac{(f)(b) - (f)(a)}{b - a} \leq \eta^{\alpha-1} T_{\nabla,\alpha}(f)(\eta).$$

Proof. It follows from Theorem 5 that

$$T_{\nabla,\alpha}(t) = (t)^{(\alpha)}_{\nabla} = \begin{cases} t^{1-\alpha} & \text{if } 0 < \alpha < 1, \\ 1 & \text{if } \alpha = 1. \end{cases} \tag{9}$$

Let $h(t) = f(t) - f(b) - \dfrac{(f(b) - f(a))}{(b - a)}(t - b)$. Then, the function h is continuous function on $[a, b]_{\mathbb{T}}$ which is nabla conformal fractional differentiable of order α on $[a, b)_{\mathbb{T}}$ and $h(a) = h(b)$. Combining Theorem 2 and (9), we have

$$T_{\nabla,\alpha}(h)(t) = \begin{cases} T_{\nabla,\alpha}(f)(t) - \frac{(f(b)-f(a))}{(b-a)} & \text{if } \alpha = 1, \\ T_{\nabla,\alpha}(f)(t) - \frac{(f(b)-f(a))}{(b-a)} t^{1-\alpha} & \text{if } 0 < \alpha < 1. \end{cases} \tag{10}$$

Applying Theorem 12 to h, there exist $\xi, \eta \in (a, b]_{\mathbb{T}}$ such that $T_{\nabla,\alpha}(h)(\xi) \leq 0 \leq T_{\nabla,\alpha}(h)(\eta)$. That is

$$\xi^{\alpha-1} T_{\nabla,\alpha}(f)(\xi) \leq \frac{(f)(b) - (f)(a)}{b - a} \leq \eta^{\alpha-1} T_{\nabla,\alpha}(f)(\eta).$$

The proof is complete.

In the next theorems we give a relationship between the nabla conformable fractional differentiable and the conformable fractional differentiable given in Definition 1.

Theorem 14 Assume $f:\mathbb{T}\to\mathbb{R}$ is conformable fractional differentiable (Definition 1) on $\mathbb{T}^\kappa$ and if $T_\alpha(f)$ is continuous on $\mathbb{T}^\kappa$, then f is nabla conformable fractional differentiable on $\mathbb{T}_\kappa$ and

$$T_{\nabla,\alpha}(f)(t)=T_\alpha(f)(\rho(t)) \quad for\ all\ t\in\mathbb{T}_\kappa.$$

Proof. Fix $t\in\mathbb{T}_k$. First we consider the case where t is left-scattered. Since f is conformable fractional differentiable, it will be continuous function. Therefore, f will be nabla conformable fractional differentiable at t and

$$T_{\nabla,\alpha}(f)(t)=\frac{f(\rho(t))-f(t)}{\rho(t)-t}t^{1-\alpha}.$$

On the other hand, since $\rho(t)$ will be right-scattered, we have

$$T_\alpha(f)(\rho(t))=\frac{f(\sigma(\rho(t)))-f(\rho(t))}{\sigma(\rho(t))-\rho(t)}t^{1-\alpha}=\frac{f(t)-f(\rho(t))}{t-\rho(t)}t^{1-\alpha}.$$

Therefore $T_{\nabla,\alpha}(f)(t)=T_\alpha(f)(\rho(t))$ which is the desired result.
Let now t be left-dense and right-dense, simultaneously. In this case from the existence of $T_\alpha(f)(t)$ it follows that the limit

$$\lim_{s\to t}\frac{f(t)-f(s)}{(t-s)}t^{1-\alpha} \tag{11}$$

exists as a finite number and is equal to $T_\alpha(f)(t)$. On the other hand since t is left-dense, from the existence of the limit 11 it follows that $T_{\nabla,\alpha}(f)(t)$ exists and is equal to this limit. Therefore $T_{\nabla,\alpha}(f)(t)=T_\alpha(f)(t)$.
Finally, let t be left-dense and right-scattered. Applying mean value Theorem 15 of [20] to f, we can write

$$\xi^{\alpha-1}T_\alpha(f)(\xi)\leq\frac{(f)(t)-(s)(a)}{t-s}\leq\eta^{\alpha-1}T_\alpha(f)(\eta), \tag{12}$$

where ξ,η are between s and t. Since $\xi\to t,\ \eta\to t\ as\ s\to t$ and since, by the condition, $T_\alpha(f)$ is continuous, it follows from 12 that

$$\lim_{s\to t}\frac{f(t)-f(s)}{(t-s)}=t^{\alpha-1}T_\alpha(f). \tag{13}$$

On the other hand since t is left-dense, the left-hand side of (13) is equal to $t^{\alpha-1}T_{\nabla,\alpha}(f)(t)$. So, $T_{\nabla,\alpha}(f)(t)=T_\alpha(f)$. The theorem is proved.

The following Theorem can be proved in a similar way using an analogous mean value Theorem 13.

Theorem 15 Assume $f:\mathbb{T}\to\mathbb{R}$ is nabla conformable fractional differentiable on $\mathbb{T}_\kappa$ and if $T_{\nabla,\alpha}(f)$ is continuous on $\mathbb{T}_\kappa$, then f is conformable fractional differentiable (Definition 1) on $\mathbb{T}^\kappa$ and

$$T_\alpha(f)(t)=T_{\nabla,\alpha}(f)(\sigma(t)) \quad for\ all\ t\in\mathbb{T}^\kappa.$$

Similar to the Definition 16 in [20], we give the following definition of absolutely continuous function.

Definition 10 A function $f:[a,b]_\mathbb{T}\to\mathbb{R}$ is said to be absolutely continuous on

$[a,b]_{\mathbb{T}}$ (*i.e.,* $f \in AC([a,b]_{\mathbb{T}}, \mathbb{R})$) if for every $\varepsilon > 0$, there exists a $\eta > 0$ such that if $\{(a_k, b_k]_{\mathbb{T}}\}_{k=1}^{m}$, is a finite pairwise disjoint family of subintervals of $[a,b]_{\mathbb{T}}$ satisfying

$$\sum_{k=1}^{k=m} (b_k - a_k) < \eta \ then \ \sum_{k=1}^{k=m} |f(b_k) - f(\sigma(a_k))| < \varepsilon.$$

The following analogue for nabla differentiable of Theorem 4.1 in [13] can be proved in a similar way.

Lemma 2 Assume function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is absolutely continuous on $[a,b]_{\mathbb{T}}$, if and only if f is nabla differentiable ∇-a.e. on $[a,b]_{\mathbb{T}}$ and

$$f(t) = f(a) + \int_{[a,t]_{\mathbb{T}}} f^{\nabla}(s)\nabla s, \quad for\ all\ t \in [a,b]_{\mathbb{T}}.$$

The following analogue for nabla conformable fractional differentiable of Theorem 18 in [20] can be proved in a similar way.

Theorem 16 Assume function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is absolutely continuous on $[a,b]_{\mathbb{T}}$, then f is nabla conformable fractional differentiable of order α ∇-a.e. on $[a,b]_{\mathbb{T}}$ and the following equality is valid:

$$f(t) = f(a) + \int_{[a,t]_{\mathbb{T}}} T_{\nabla,\alpha}(f)(s)\nabla_{\alpha} s \quad for\ all\ t \in [a,b]_{\mathbb{T}}.$$

Next, we introduce the nabla conformable fractional integral (or nabla α-fractional integral) on time scales for vector-valued functions.

Definition 11 Assume $f : \mathbb{T} \to \mathbb{R}^n$, is a function and $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))$. Let A be a ∇-measurable subset of $\mathbb{T}$. Then f is nabla α-integrable on A if and only if $f_i (i = 1, 2, \cdots, n)$ are nabla α-integrable on A, and

$$\int_A f(t)\nabla_{\alpha} t = \Big(\int_A f_1(t)\nabla_{\alpha} t, \int_A f_2(t)\nabla_{\alpha} t, ..., \int_A f_n(t)\nabla_{\alpha} t \Big).$$

Combining Definition 11 and Theorem 9, we have the following theorem.

Theorem 17 Let $\alpha \in (0,\ 1]$, $a, b, c \in \mathbb{T}$, $\lambda, \gamma \in \mathbb{R}$, and $f, g : \mathbb{T} \to \mathbb{R}^n$ be two ld-continuous functions. Then,

(i) $\displaystyle\int_a^b [\lambda f(t) + \gamma g(t)]\nabla_{\alpha} t = \lambda \int_a^b f(t)\nabla_{\alpha} t + \gamma \int_a^b g(t)\nabla_{\alpha} t;$

(ii) $\displaystyle\int_a^b f(t)\nabla_{\alpha} t = -\int_b^a f(t)\nabla_{\alpha} t;$

(iii) $\displaystyle\int_a^b f(t)\nabla_{\alpha} t = \int_a^c f(t)\nabla_{\alpha} t + \int_c^b f(t)\nabla_{\alpha} t;$

(iv) $\displaystyle\int_a^a f(t)\nabla_{\alpha} t = 0;$

(v) if there exist $g : \mathbb{T} \to \mathbb{R}$ with $\|f(t)\| \le g(t)$ for all $t \in [a,\ b]$, then

$$\Big\| \int_a^b f(t)\nabla_{\alpha} t \Big\| \le \int_a^b g(t)\nabla_{\alpha} t.$$

Similar to the Definition 37 in [20], we give the following definition of absolutely continuous function.

Definition 12 A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$, $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))$. We say f absolutely continuous on $[a,b]_{\mathbb{T}}$ (*i.e.,* $f \in AC([a,b]_{\mathbb{T}}, \mathbb{R}^n)$), if for every $\varepsilon > 0$, there exists a $\eta > 0$ such that if $\{(a_k, b_k]_{\mathbb{T}}\}_{k=1}^{m}$, is a finite pairwise disjoint family of subintervals of $[a, b]_{\mathbb{T}}$ satisfying

$$\sum_{k=1}^{k=m}(b_k - a_k) < \eta \ then \ \sum_{k=1}^{k=m} \|f(b_k) - f(\sigma(a_k))\| < \varepsilon.$$

Combining Definitions 5, 8 and Theorem 16, we have the following theorem.

Theorem 18 Assume function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ is absolutely continuous on $[a, b]_{\mathbb{T}}$, then f is nabla conformable fractional differentiable of order α ∇-a.e. on $[a, b]_{\mathbb{T}}$ and the following equality is valid:

$$f(t) = f(a) + \int_{[a,t]_{\mathbb{T}}} T_{\nabla,\alpha}(f)(s)\nabla_\alpha s \quad for\ all\ t \in [a, b]_{\mathbb{T}}.$$

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