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HERMITE-HADAMARD TYPE INEQUALITIES FOR QUASI-CONVEX FUNCTIONS VIA CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this paper, we propose and prove some new Hadamard's type inequalities for the recently proposed conformable fractional derivatives and fractional integrals. These results are based on the identities obtained by Khan et al. [21] and Chu et al. [9]. Also, we present applications of the results pertaining to some special means of real numbers, midpoint as well as trapezoidal formulas.

1. Introduction

The concept of derivative is conventionally connected with integers where the order of derivative is considered to be integer. In 1695 L'Hospital, in his letter to Leibnitz, asked, "what does it mean by $\frac{d^n f}{dx^n}$ when $n = \frac{1}{2}$?". In the bid to answer L'Hospital's question, many researchers tried to put a definition on a fractional derivative. Various types of fractional derivatives were introduced Riemann-Liouville, Caputo, Hadamard, Erdélyi-Kober, Grünwald-Letnikov, Marchaud and Riesz are just a few to name [17, 18, 19, 31, 33, 34]. Most of the fractional derivative are defined through fractional integrals [34]. Due to the same reason, those fractional derivatives inherit some non-local behaviors, which lead them to many interesting applications including memory effects and future dependence [28]. In recent time, there are many applications of the fractional derivatives cutting across many fields such as found in control theory of dynamical systems, nanotechnology, viscoelasticity, anomalous transport and anomalous diffusion, financial modeling, random walk see [5, 6, 15, 27, 29, 30, 37, 43]. These recent discoveries of the applications of fractional calculus have drawn the attention of many researchers in other to gain more insight into the field. Existence and uniqueness of solutions, asymptotic behavior, analytical and numerical solutions of some of the fractional differential equations both linear and nonlinear see [3, 13].

We present the two most well-known definitions in the sense that they are mainly

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bring under consideration for mathematical modeling in many applied problems.
Definition 1. The Riemann-Liouville's (RL) fractional integral operator of order $\alpha \in [n-1, n)$, of a function $f \in \mathtt{L}^1([a, b])$ is given as

$$I^{\alpha} w(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x w(\tau)(x-\tau)^{n-\alpha-1} d\tau \tag{1}$$

with Γ as the Gamma function and $I^0 w(x) = w(x)$.
The Riemann-Liouville's (RL) fractional derivative operator is then given as

$$D^{\alpha} w(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^x w(\tau)(x-\tau)^{n-\alpha-1} d\tau. \tag{2}$$

Definition 2. The Caputo fractional derivative operator of order $\alpha \in [n-1, n)$, of a function $f \in \mathtt{L}^1([a, b])$ is given as

$$^{c}D^{\alpha} w(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x w^n(\tau)(x-\tau)^{n-\alpha-1} d\tau. \tag{3}$$

Now, all the definitions including (2) and (3) above satisfy the property that the fractional derivative is linear. This is the only property inherited from the first derivative by all of the definitions. However, there are some inconsistencies of many existing definitions that limit the extent of applications in so many fields. Properties such as the derivative of constant should be zero, the product rule, quotient rule, chain rule, Rolle's theorem, mean value theorem and composition rule and so on are lacking in almost all fractional derivatives. These inconsistencies and many more have posed a lot of problems in real life applications and have limited how far these fractional calculus could be explored. To overcome some of these and other difficulties, Khalil et al. in [20] came up with an interesting idea that extends the familiar limit definition of the derivatives of a function called conformable fractional derivative. The simple nature of this definition allows for many extensions of some classical theorems in calculus for which the applications are indispensable in the fractional differential models that the existing definitions do not permit. In [20], the extended mean value theorem and the Racetrack type principle are proven for the class of functions which are α-differentiable in the context of conformable fractional derivatives and fractional integral. These definitions are incorporated in the following lines:
Definition 3.([20]) Given a function $\Phi : [0, \infty) \to \mathbb{R}$. Then the **conformable fractional derivative** of Φ of order α is defined by

$$\mathtt{D}_{\alpha}(\Phi)(t) = \lim_{\epsilon \to 0} \frac{\Phi(t + \epsilon t^{1-\alpha}) - \Phi(t)}{\epsilon} \tag{4}$$

for all $t > 0, \alpha \in (0, 1)$. If the conformable fractional derivative of Φ of order α exists, then we say that Φ is α-differentiable. Let Φ be α-differentiable in $(0, a)$, and $\lim_{t \to 0^+} \Phi^{\alpha}(t)$ exists, then define

$$\Phi^{\alpha}(0) = \lim_{t \to 0^+} \Phi^{\alpha}(t). \tag{5}$$

We will, sometimes, write $\Phi^{\alpha}(t)$ and $\frac{d_{\alpha}}{d_{\alpha} t}(\Phi)$ for $\mathtt{D}_{\alpha}(\Phi)(t)$, to denote the conformable fractional derivatives of Φ of order α.
Definition 4.([4])(**Conformable fractional integral**). Let $\alpha \in (0, 1)$ and $0 \leq$

$a < b$. A function $\Phi : [a, b] \to \mathbb{R}$ is α-fractional integrable on $[a, b]$ if the integral

$$\int_a^b \Phi(s) d_\alpha s := \int_a^b \Phi(s) s^{\alpha-1} ds \tag{6}$$

exists and is finite. All α-fractional integrable functions on $[a, b]$ is indicated by $\mathtt{L}^1_\alpha([a, b])$.

Remark 1.

$$\mathtt{I}^a_\alpha(\Phi)(t) = \mathtt{I}^a_1(t^{\alpha-1}\Phi) = \int_a^t \frac{\Phi(s)}{s^{1-\alpha}} ds, \tag{7}$$

where the integral is the usual Riemann improper integral, $\alpha \in (0, 1]$.

The inequalities obtained by Hermite and Hadamard for convex functions are very important in the literature (see, e.g., [32, p. 137]).

Suppose that $\mathtt{I} \subseteq \mathbb{R}$ and $\Phi : \mathtt{I} \to \mathbb{R}$ is a convex function defined on $\mathtt{I}$, then for any $a < b$ with $a, b \in \mathtt{I}$, the following inequalities hold:

$$\Phi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \Phi(s) ds \leq \frac{\Phi(a) + \Phi(b)}{2}. \tag{8}$$

Both the inequalities hold in reversed direction if the function Φ is concave on the interval $\mathtt{I}$. Due to its rich geometry, involvement in many branches of mathematics and other areas of sciences, this inequality has remained a center of interest for many well-known mathematicians and researchers. Also, it has been modified and updated in many directions, by providing various refinements, improvements, extensions, and generalizations etc. For example, see ([22, 12]).

A function $\Phi : \mathtt{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on $\mathtt{I}$ if the inequality

$$\Phi(\alpha x + (1 - \alpha)y) \leq \alpha\Phi(x) + (1 - \alpha)\Phi(y) \tag{9}$$

holds for any two points $x, y \in \mathtt{I}$ and $\alpha \in [0, 1]$. If the inequality (9) is reversed in the foresaid definition, then the function Φ is known as concave function on the interval $\mathtt{I}$.

The concept of quasi-convex functions generalizes the notion of convex functions. More precisely, we say that a function $\Phi : [a, b] \to \mathbb{R}$ is quasi-convex on $[a, b]$ if

$$\Phi(\alpha x + (1 - \alpha)y) \leq \max\{\Phi(x), \Phi(y)\} \tag{10}$$

holds for any two points $x, y \in \mathtt{I}$ and $\alpha \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [10]).

The following results were proved by Dragomir and agarwal [11] associated with the right hand part of Hermite-Hadamard inequality.

Lemma 1.([11]) Let $\Phi : \mathtt{I}^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathtt{I}^\circ$ and $a, b \in \mathtt{I}^\circ$ with $a < b$. If $\Phi' \in \mathtt{L}^1[a, b]$, then the following identity holds:

$$\frac{\Phi(a) + \Phi(b)}{2} - \frac{1}{b-a} \int_a^b \Phi(s) ds = \frac{b-a}{2} \int_0^1 (1 - 2t)\Phi'(ta + (1 - t)b) dt. \tag{11}$$

Theorem 1.([11]) Let $\Phi : \mathtt{I}^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathtt{I}^\circ$ and $a, b \in \mathtt{I}^\circ$ with $a < b$. If $\Phi' \in \mathtt{L}^1[a, b]$ and $|\Phi'|$ is convex on $[a, b]$, then we have the following inequality:

$$\left|\frac{\Phi(a) + \Phi(b)}{2} - \frac{1}{b-a} \int_a^b \Phi(s) ds\right| \leq \frac{(b-a)}{4} \left(\frac{|\Phi'(a)| + |\Phi'(b)|}{2}\right). \tag{12}$$

In [26], U. S. Kirmaci gave the following results.
Lemma 2.([26]) Let $f : \mathtt{I}^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathtt{I}^\circ$, $a, b \in \mathtt{I}^\circ$ with $a < b$. If $\Phi' \in \mathtt{L}^1[a, b]$, then the following equality holds:

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \Phi(s)ds - \Phi\left(\frac{a+b}{2}\right) \\ = \;& (b-a)\left[\int_0^{\frac{1}{2}} t\Phi'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)\Phi'(ta+(1-t)b)dt\right]. \quad (13) \end{aligned}$$

Theorem 2.([26]) Let $\Phi : \mathtt{I}^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathtt{I}^\circ$ and $a, b \in \mathtt{I}^\circ$ with $a < b$. If $\Phi' \in \mathtt{L}^1[a, b]$ and $|\Phi'|$ is convex on $[a, b]$, then we have the following inequality:

$$\left|\frac{1}{b-a}\int_a^b \Phi(s)ds - \Phi\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)(|\Phi'(a)| + |\Phi'(b)|)}{8}. \quad (14)$$

In one of the recent paper, Anderson [4] investigated the following conformable integral version of Hermite-Hadamard inequality:
Theorem 3. ([4]) Let $\alpha \in (0, 1]$ and $\Phi : [a, b] \to \mathbb{R}$ be an α-differentiable function with $0 < a < b$, such that $\mathtt{D}_\alpha\Phi$ is increasing, then we have the following inequality

$$\frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \Phi(s)d_\alpha s \le \frac{\Phi(a) + \Phi(b)}{2}. \quad (15)$$

Moreover, if the function Φ is decreasing on $[a, b]$, then we have

$$\Phi\left(\frac{a+b}{2}\right) \le \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \Phi(s)d_\alpha s. \quad (16)$$

Remark 2. It is obvious that, if we choose $\alpha = 1$, then the inequalities (15) and (16) reduce to inequality (8).

In 2017, Chu et al. [9] established the following identity and obtained some interesting results for it.
Lemma 3. Let $a, b \in \mathbb{R}^+$ and $\Phi : [a, b] \to \mathbb{R}$ be an α-differentiable function on (a, b) for $\alpha \in (0, 1]$. If $\mathtt{D}_\alpha(\Phi) \in \mathtt{L}^1_\alpha([a, b])$, then the following identity holds:

$$\begin{aligned} &\Phi\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \Phi(s)d_\alpha s \\ = \;& \frac{b-a}{b^\alpha - a^\alpha}\left[\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{2\alpha-1} - a^\alpha((1-t)a+tb)^{\alpha-1}\right)\right. \\ \times \;& \mathtt{D}_\alpha(\Phi)((1-t)a+tb)t^{1-\alpha}d_\alpha t + \int_{\frac{1}{2}}^1 \left(((1-t)a+tb)^{2\alpha-1} - b^\alpha((1-t)a+tb)^{\alpha-1}\right) \\ \times \;& \left.\mathtt{D}_\alpha(\Phi)((1-t)a+tb)t^{1-\alpha}d_\alpha t\right]. \quad (17) \end{aligned}$$

Very recently in 2017, Khan et al. [21] proved the following identity and presented certain useful results by making use of it.
Lemma 4. Let $a, b \in \mathbb{R}^+$ and $\Phi : [a, b] \to \mathbb{R}$ be an α-fractional differentiable function on (a, b) for $\alpha \in (0, 1]$. If $\mathtt{D}_\alpha(\Phi) \in \mathtt{L}^1_\alpha([a, b])$, then the following identity

holds:

$$\begin{aligned}
&\frac{\Phi(a)+\Phi(b)}{2}-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b \Phi(s)d_\alpha s \\
= \ & \frac{b-a}{2(b^\alpha-a^\alpha)}\Bigg[\int_0^1 \big((ta+(1-t)b)^{2\alpha-1}-b^\alpha(ta+(1-t)b)^{\alpha-1}\big) \\
\times \ & \mathtt{D}_\alpha(\Phi)(ta+(1-t)b)t^{1-\alpha}d_\alpha t+\int_0^1 \big((ta+(1-t)b)^{2\alpha-1}-a^\alpha(ta+(1-t)b)^{\alpha-1}\big) \\
\times \ & \mathtt{D}_\alpha(\Phi)(ta+(1-t)b)t^{1-\alpha}d_\alpha t\Bigg]. \qquad (18)
\end{aligned}$$

Remark 3. By setting $\alpha = 1$, the identity in (18) reduces to (13).
For further study on the topic and details treatment, also see [44, 7, 8, 23, 41, 39, 42, 40, 25, 24] and the articles cited there in.

In the present paper, firstly, we consider the identities obtained by Khan et al. [21] and Chu et al. [9] for the class of classical convex functions. Secondly, using these results for quasi convex functions, we establish some new Hermite-Hadamard type inequalities for conformable fractional integrals. Finally, applications of the main results to certain means of positive real numbers, the midpoint, and trapezoidal formulas are provided.

2. Main Results

We begin this section with our first main result based on Lemma 4.
Theorem 4. Let $a, b \in \mathbb{R}^+$ and $\Phi : [a, b] \to \mathbb{R}$ be an α-differentiable function on (a, b) for $\alpha \in (0, 1]$. If $\mathtt{D}_\alpha(\Phi) \in \mathtt{L}^1_\alpha([a, b])$ and $|\Phi'|$ is quasi convex on $[a, b]$, then we have the following inequality:

$$\begin{aligned}
&\left|\frac{\Phi(a)+\Phi(b)}{2}-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b \Phi(s)d_\alpha s\right| \\
\leq \ & \frac{b-a}{2(b^\alpha-a^\alpha)}\max\{|\Phi'(a)|, |\Phi'(b)|\}\left[\frac{5b^\alpha-7a^\alpha+b^{\alpha-1}a+a^{\alpha-1}b}{6}\right] \qquad (19)
\end{aligned}$$

Proof. It follow from Lemma 4 and the convexities of the functions $s \to s^{\alpha-1}$ and $s \to -s^\alpha$ on $(0, \infty)$ together with quasi convexity of $|\Phi'|$ on $[a, b]$ that

$$\begin{aligned}
&\left|\frac{\Phi(a)+\Phi(b)}{2}-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b \Phi(s)d_\alpha s\right| \\
\leq \ & \frac{b-a}{2(b^\alpha-a^\alpha)}\Bigg[\int_0^1 \left((ta+(1-t)b)^\alpha-a^\alpha\right)|\Phi'(ta+(1-t)b)|\,dt \\
+ \ & \int_0^1 \left(b^\alpha-(ta+(1-t)b)^\alpha\right)|\Phi'(ta+(1-t)b)|\,dt\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{b-a}{2(b^\alpha-a^\alpha)}\left[\int_0^1 \left((ta+(1-t)b)^{\alpha-1}(ta+(1-t)b)-a^\alpha\right)|\Phi'(ta+(1-t)b)|\,dt\right.\\
&+ \left.\int_0^1 (b^\alpha-(ta^\alpha+(1-t)b^\alpha))\,|\Phi'(ta+(1-t)b)|\,dt\right]\\
&\leq \frac{b-a}{2(b^\alpha-a^\alpha)}\left[\int_0^1 \left((ta^{\alpha-1}+(1-t)b^{\alpha-1})(ta+(1-t)b)-a^\alpha\right)|\Phi'(ta+(1-t)b)|\,dt\right.\\
&+ \left.\int_0^1 (b^\alpha-(ta^\alpha+(1-t)b^\alpha))\,|\Phi'(ta+(1-t)b|\,dt\right]\\
&\leq \frac{b-a}{2(b^\alpha-a^\alpha)}\left[\int_0^1 \left((ta^{\alpha-1}+(1-t)b^{\alpha-1})(ta+(1-t)b)-a^\alpha\right)\max\{|\Phi'(a)|,|\Phi'(b)|\}\,dt\right.\\
&+ \left.\int_0^1 (b^\alpha-(ta^\alpha+(1-t)b^\alpha))\max\{|\Phi'(a)|,|\Phi'(b)|\}\,dt\right]\\
&\leq \frac{b-a}{2(b^\alpha-a^\alpha)}\left[\max\{|\Phi'(a)|,|\Phi'(b)|\}\int_0^1 \left((ta^{\alpha-1}+(1-t)b^{\alpha-1})(ta+(1-t)b)-a^\alpha\right)dt\right.\\
&+ \left.\max\{|\Phi'(a)|,|\Phi'(b)|\}\int_0^1 (b^\alpha-(ta^\alpha+(1-t)b^\alpha))\,dt\right]\\
&= \frac{b-a}{2(b^\alpha-a^\alpha)}\max\{|\Phi'(a)|,|\Phi'(b)|\}\left[\int_0^1 \left((ta^{\alpha-1}+(1-t)b^{\alpha-1})(ta+(1-t)b)-a^\alpha\right)dt\right.\\
&+ \left.\int_0^1 (b^\alpha-(ta^\alpha+(1-t)b^\alpha))\,dt\right]\\
&= \frac{b-a}{2(b^\alpha-a^\alpha)}\max\{|\Phi'(a)|,|\Phi'(b)|\}\left[\frac{1}{3}a^\alpha+\frac{1}{6}ab^{\alpha-1}+\frac{1}{6}a^{\alpha-1}b+\frac{1}{3}b^\alpha-a^\alpha-\frac{1}{2}a^\alpha-\frac{1}{2}b^\alpha+b^\alpha\right]\\
&= \frac{b-a}{2(b^\alpha-a^\alpha)}\max\{|\Phi'(a)|,|\Phi'(b)|\}\left[\frac{5b^\alpha-7a^\alpha+b^{\alpha-1}a+a^{\alpha-1}b}{6}\right],
\end{aligned}
$$

which completes the desired proof. □

Remark 4. If we set $\alpha=1$, then the inequality (19) becomes

$$
\left|\frac{\Phi(a)+\Phi(b)}{2}-\frac{1}{b-a}\int_a^b \Phi(s)ds\right| \leq \frac{b-a}{2}\max\{|\Phi'(a)|,|\Phi'(b)|\}
$$

Theorem 5. Let $\alpha \in (0,1]$ with $a,b \in \mathbb{R}^+$ and $\Phi : [a,b] \to \mathbb{R}$ be an α-differentiable function, then the inequality

$$\begin{aligned}
&\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \Phi(s) d_\alpha s\right| \\
\leq\ & \frac{b-a}{2(b^\alpha - a^\alpha)}\Bigg[\mathtt{A}_1(\alpha)\Big(\max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\}\Big)^{\frac{1}{q}} \\
+\ & \mathtt{A}_2(\alpha)\Big(\max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\}\Big)^{\frac{1}{q}}\Bigg] \qquad (20)
\end{aligned}$$

is valid if $\mathtt{D}_\alpha(\Phi) \in \mathtt{L}^1_\alpha([a,b])$ and for $q > 1$ $|\Phi'|^q$ is quasi-convex on $[a,b]$, where

$$\mathtt{A}_1(\alpha) = \left[\frac{a^{\alpha+1} - b^{\alpha+1}}{(a-b)(\alpha+1)}\right] - a^\alpha$$

$$\mathtt{A}_2(\alpha) = b^\alpha - \left[\frac{a^{\alpha+1} - b^{\alpha+1}}{(a-b)(\alpha+1)}\right]$$

Proof. It follows from Lemma 4 that

$$\begin{aligned}
&\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \Phi(s) d_\alpha s\right| \\
\leq\ & \frac{b-a}{2(b^\alpha - a^\alpha)}\left[\int_0^1 \left((ta+(1-t)b)^\alpha - a^\alpha\right)|\Phi'(ta+(1-t)b)|\, dt\right. \\
+\ & \left.\int_0^1 \left(b^\alpha - (ta+(1-t)b)^\alpha\right)|\Phi'(ta+(1-t)b)|\, dt\right]. \qquad (21)
\end{aligned}$$

By power-mean inequality

$$\begin{aligned}
&\int_0^1 \left((ta+(1-t)b)^\alpha - a^\alpha\right)|\Phi'(ta+(1-t)b)|\, dt \\
\leq\ & \left(\int_0^1 \left((ta+(1-t)b)^\alpha - a^\alpha\right) dt\right)^{1-\frac{1}{q}} \\
\times\ & \left(\int_0^1 \left((ta+(1-t)b)^\alpha - a^\alpha\right)|\Phi'(ta+(1-t)b)|^q\, dt\right)^{\frac{1}{q}} \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left(b^\alpha - (ta+(1-t)b)^\alpha\right)|\Phi'(ta+(1-t)b)|\, dt \\
\leq\ & \left(\int_0^1 \left(b^\alpha - (ta+(1-t)b)^\alpha\right) dt\right)^{1-\frac{1}{q}} \\
\times\ & \left(\int_0^1 \left(b^\alpha - (ta+(1-t)b)^\alpha\right)|\Phi'(ta+(1-t)b)|^q\, dt\right)^{\frac{1}{q}}. \qquad (23)
\end{aligned}$$

Furthermore, the quasi-convexity of $|\Phi'|$ implies that

$$\begin{aligned}
& \int_0^1 \left((ta+(1-t)b)^\alpha - a^\alpha\right) |\Phi'(ta+(1-t)b)|^q \, dt \\
\leq \quad & \int_0^1 \left((ta+(1-t)b)^\alpha - a^\alpha\right) \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} dt \\
\leq \quad & \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} \int_0^1 \left((ta+(1-t)b)^\alpha - a^\alpha\right) dt \\
= \quad & \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} \left[\frac{a^{\alpha+1}-b^{\alpha+1}}{(a-b)(\alpha+1)} - a^\alpha\right]. \qquad (24)
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
& \int_0^1 \left(b^\alpha - (ta+(1-t)b)^\alpha\right) |\Phi'(ta+(1-t)b)|^q \, dt \\
\leq \quad & \int_0^1 \left(b^\alpha - (ta+(1-t)b)^\alpha\right) \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} dt \\
\leq \quad & \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} \int_0^1 \left(b^\alpha - (ta+(1-t)b)^\alpha\right) dt \\
= \quad & \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} \left[b^\alpha - \frac{a^{\alpha+1}-b^{\alpha+1}}{(a-b)(\alpha+1)}\right]. \qquad (25)
\end{aligned}$$

Therefore, the desired inequality (20) follows easily from (21)-(25). □

Remark 5. If we set $\alpha = 1$, then the inequality (20) becomes

$$\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(s)ds\right| \leq \frac{b-a}{2}\left[\left(\max\{|\Phi'(a)|^q, |\Phi'(b)|^q\}\right)^{\frac{1}{q}}\right],$$

where

$$\mathtt{A}_1(1) = \frac{b-a}{2} = \mathtt{A}_2(1).$$

In the next theorem, we present another important result associated with Lemma 4.

Theorem 6. Let $\alpha \in (0,1]$ and $a, b \in \mathbb{R}^+$. If $\Phi : [a,b] \to \mathbb{R}$ be an α-differentiable function on (a,b), then the inequality

$$\begin{aligned}
& \left|\Phi\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \Phi(s) d_\alpha s\right| \\
\leq \quad & \frac{b-a}{b^\alpha - a^\alpha} \max\{|\Phi'(a)|, |\Phi'(b)|\} \left[\frac{2b^\alpha - 4a^\alpha + b^{\alpha-1}a + a^{\alpha-1}b}{12}\right] \qquad (26)
\end{aligned}$$

holds, if $\mathtt{D}_\alpha(\Phi) \in \mathtt{L}^1_\alpha([a,b])$ and $|\Phi'|$ is quasi-convex on $[a,b]$.

Proof. It follow from Lemma 3 and the convexities of the functions $s \to s^{\alpha-1}$ and $s \to -s^{\alpha}$ on $(0,\infty)$ together with quasi convexity of $|\Phi'|$ on $[a,b]$ that

$$\begin{aligned}
&\left|\Phi\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \Phi(s)d_\alpha s\right| \\
\leq\ & \frac{b-a}{b^\alpha - a^\alpha}\left[\int_0^{\frac{1}{2}} \big(((1-t)a+tb)^\alpha - a^\alpha\big)\left|\Phi'((1-t)a+tb)\right| dt\right. \\
+\ & \left.\int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a+tb)^\alpha\big)\left|\Phi'((1-t)a+tb)\right| dt\right] \\
\leq\ & \frac{b-a}{b^\alpha - a^\alpha}\left[\int_0^{\frac{1}{2}} \big(((1-t)a+tb)^{\alpha-1}((1-t)a+tb) - a^\alpha\big)\left|\Phi'((1-t)a+tb)\right| dt\right. \\
+\ & \left.\int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a^\alpha+tb^\alpha)\big)\left|\Phi'((1-t)a+tb)\right| dt\right] \\
\leq\ & \frac{b-a}{b^\alpha - a^\alpha}\left[\int_0^{\frac{1}{2}} \big(((1-t)a^{\alpha-1}+tb^{\alpha-1})((1-t)a+tb) - a^\alpha\big)\left|\Phi'((1-t)a+tb)\right| dt\right. \\
+\ & \left.\int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a^\alpha+tb^\alpha)\big)\left|\Phi'((1-t)a+tb)\right| dt\right] \\
\leq\ & \frac{b-a}{b^\alpha - a^\alpha}\left[\int_0^{\frac{1}{2}} \big(((1-t)a^{\alpha-1}+tb^{\alpha-1})((1-t)a+tb) - a^\alpha\big)\max\left\{\left|\Phi'(a)\right|,\left|\Phi'(b)\right|\right\}\right. \\
+\ & \left.\int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a^\alpha+tb^\alpha)\big)\max\left\{\left|\Phi'(a)\right|,\left|\Phi'(b)\right|\right\}\right] \\
\leq\ & \frac{b-a}{b^\alpha - a^\alpha}\left[\max\left\{\left|\Phi'(a)\right|,\left|\Phi'(b)\right|\right\}\int_0^{\frac{1}{2}} \big(((1-t)a^{\alpha-1}+tb^{\alpha-1})((1-t)a+tb) - a^\alpha\big)dt\right. \\
+\ & \left.\max\left\{\left|\Phi'(a)\right|,\left|\Phi'(b)\right|\right\}\int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a^\alpha+tb^\alpha)\big)dt\right] \\
=\ & \frac{b-a}{b^\alpha - a^\alpha}\max\left\{\left|\Phi'(a)\right|,\left|\Phi'(b)\right|\right\}\left[\int_0^{\frac{1}{2}} \big(((1-t)a^{\alpha-1}+tb^{\alpha-1})((1-t)a+tb) - a^\alpha\big)dt\right. \\
+\ & \left.\int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a^\alpha+tb^\alpha)\big)dt\right] \\
=\ & \frac{b-a}{b^\alpha - a^\alpha}\max\left\{\left|\Phi'(a)\right|,\left|\Phi'(b)\right|\right\}\left[\frac{2b^\alpha - 4a^\alpha + b^{\alpha-1}a + a^{\alpha-1}b}{12}\right],
\end{aligned}$$

which completes the desired proof.

□

Remark 6. If $\alpha = 1$, then the inequality (26) becomes

$$\left|\Phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \Phi(s)ds\right| \leq \frac{b-a}{4}\max\left\{\left|\Phi'(a)\right|,\left|\Phi'(b)\right|\right\}$$

Theorem 7. Let $\alpha \in [a,b]$ with $a,b \in \mathbb{R}^+$ and $\Phi : [a,b] \to \mathbb{R}$ be an α-differentiable function, then the inequality

$$\begin{aligned}
&\left|\Phi\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \Phi(s)d_\alpha s\right| \\
\leq\ & \frac{b-a}{b^\alpha - a^\alpha}\left[\mathtt{A}_1(\alpha)\Big(\max\{|\Phi'(a)|^q, |\Phi'(b)|^q\}\Big)^{\frac{1}{q}}\right. \\
+\ & \left.\mathtt{A}_2(\alpha)\Big(\max\{|\Phi'(a)|^q, |\Phi'(b)|^q\}\Big)^{\frac{1}{q}}\right] \qquad (27)
\end{aligned}$$

is valid if $\mathtt{D}_\alpha(\Phi) \in L^1_\alpha([a,b])$ and for $q > 1$ $|\Phi'|^q$ is quasi-convex on $[a,b]$, where

$$\mathtt{A}_1(\alpha) = \left[\frac{(a+b)^{\alpha+1} - (2a)^{\alpha+1}}{(2)^{\alpha+1}(b-a)(\alpha+1)}\right] - \frac{a^\alpha}{2}$$

$$\mathtt{A}_2(\alpha) = \frac{b^\alpha}{2} - \left[\frac{(2b)^{\alpha+1} - (a+b)^{\alpha+1}}{(2)^{\alpha+1}(b-a)(\alpha+1)}\right]$$

Proof. It follows from Lemma 3 that

$$\begin{aligned}
&\left|\Phi\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \Phi(s)d_\alpha s\right| \\
\leq\ & \frac{b-a}{(b^\alpha - a^\alpha)}\left[\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^\alpha - a^\alpha\right)|\Phi'((1-t)a+tb)|\,dt\right. \\
+\ & \left.\int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^\alpha\right)|\Phi'((1-t)a+tb)|\,dt\right]. \qquad (28)
\end{aligned}$$

By power-mean inequality

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^\alpha - a^\alpha\right)|\Phi'((1-t)a+tb)|\,dt \\
\leq\ & \left(\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^\alpha - a^\alpha\right)dt\right)^{1-\frac{1}{q}} \\
\times\ & \left(\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^\alpha - a^\alpha\right)|\Phi'((1-t)a+tb)|^q\,dt\right)^{\frac{1}{q}} \qquad (29)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^\alpha\right)|\Phi'((1-t)a+tb)|\,dt \\
\leq\ & \left(\int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^\alpha\right)dt\right)^{1-\frac{1}{q}} \\
\times\ & \left(\int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^\alpha\right)|\Phi'((1-t)a+tb)|^q\,dt\right)^{\frac{1}{q}}. \qquad (30)
\end{aligned}$$

Furthermore, the quasi-convexity of $|\Phi'|$ implies that

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^\alpha - a^\alpha\right) |\Phi'((1-t)a+tb)|^q \, dt \\
\leq\ & \int_0^{\frac{1}{2}} \left(((1-t)a+tb)^\alpha - a^\alpha\right) \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} dt \\
\leq\ & \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} \int_0^{\frac{1}{2}} \left(((1-t)a+tb)^\alpha - a^\alpha\right) dt \\
=\ & \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} \left[\left[\frac{(a+b)^{\alpha+1} - (2a)^{\alpha+1}}{(2)^{\alpha+1}(b-a)(\alpha+1)}\right] - \frac{a^\alpha}{2}\right]. \qquad (31)
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^\alpha\right) |\Phi'((1-t)a+tb)|^q \, dt \\
\leq\ & \int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^\alpha\right) \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} dt \\
\leq\ & \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} \int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^\alpha\right) dt \\
=\ & \max\left\{|\Phi'(a)|^q, |\Phi'(b)|^q\right\} \left[\frac{b^\alpha}{2} - \left[\frac{(2b)^{\alpha+1} - (a+b)^{\alpha+1}}{(2)^{\alpha+1}(b-a)(\alpha+1)}\right]\right]. \qquad (32)
\end{aligned}$$

Therefore, the desired inequality (27) follows easily from (28)-(32). □

Remark 7. If we set $\alpha = 1$, then the inequality (27) becomes

$$\begin{aligned}
&\left|\Phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \Phi(s)ds\right| \\
\leq\ & \frac{b-a}{4}\left[\left(\max\{|\Phi'(a)|^q, |\Phi'(b)|^q\}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\mathtt{A}_1(1) = \frac{b-a}{8} = \mathtt{A}_2(1).$$

3. Application to special means

We begin, this section by considering some particular means for arbitrary positive real numbers $a, b \in \mathbb{R}^+$ such that $a \neq b$. So, for this purpose, we recall the following well-known definition in the literature:

(1) The arithmetic mean:

$$\mathtt{A} = \mathtt{A}(a,b) := \frac{a+b}{2}, \quad a, b \in \mathbb{R}^+,$$

(2) The logarithmic mean:

$$\mathtt{L} = \mathtt{L}(a,b) := \frac{b-a}{\ln b - \ln a}, \quad a \neq b, a, b \in \mathbb{R}^+,$$

(3) The generalized logarithmic (α, r)-th mean:

$$\mathtt{L}_{(\alpha,r)} = \mathtt{L}_{(\alpha,r)}(a,b) = \left[\frac{\alpha(b^{r+\alpha} - a^{r+\alpha})}{(b^{\alpha} - a^{\alpha})(r+\alpha)}\right], \quad a \neq b, r \neq 0, -\alpha, \ \alpha \in (0,1], \ r \in \mathbb{R}.$$

Now, by making use of the result obtained in Section 2, we give some applications to special means of real numbers.

Proposition 1. Let $a, b \in \mathbb{R}^{+}$ with$0 < a < b$ $r > 1$ and $\alpha \in (0,1]$, then the following holds:

$$\begin{aligned} & \left|\mathtt{A}(a^r, b^r) - \mathtt{L}^{r}_{(\alpha,r)}(a,b)\right| \\ \leq \ & \frac{r(b-a)}{b^{\alpha} - a^{\alpha}} \max\left\{|a|^{r-1}, |b|^{r-1}\right\} \left[\frac{5b^{\alpha} - 7a^{\alpha} + b^{\alpha-1}a + a^{\alpha-1}b}{6}\right] \end{aligned}$$

Proof. The result follows from Theorem 2 for the convex function $\Phi(x) = x^r$, $x > 0$. ☐

Proposition 2. Let $a, b \in \mathbb{R}^{+}$ with $r > 1$ and $\alpha \in (0,1]$, then we have the following inequality:

$$\begin{aligned} & \left|\mathtt{A}(a^r, b^r) - \mathtt{L}^{r}_{(\alpha,r)}(a,b)\right| \\ \leq \ & \frac{r(b-a)}{b^{\alpha} - a^{\alpha}} \Bigg[\mathtt{A}_1(\alpha)\Big(\max\left\{|a|^{(r-1)q}, |b|^{(r-1)q}\right\}\Big)^{\frac{1}{q}} \\ + \ & \mathtt{A}_2(\alpha)\Big(\max\left\{|a|^{(r-1)q}, |b|^{(r-1)q}\right\}\Big)^{\frac{1}{q}}\Bigg] \end{aligned}$$

Proof. The result follows from Theorem 2 for the convex function $\Phi(x) = x^r$, $x > 0$. ☐

Proposition 3. Let $a, b \in \mathbb{R}^{+}$ with $r > 1$ and $\alpha \in (0,1]$, then the following holds:

$$\begin{aligned} & \left|\mathtt{A}(a^{-1}, b^{-1}) - \mathtt{L}^{-1}_{(\alpha,-1)}(a,b)\right| \\ \leq \ & \frac{b-a}{b^{\alpha} - a^{\alpha}} \max\left\{|a|^{-2}, |b|^{-2}\right\} \left[\frac{5b^{\alpha} - 7a^{\alpha} + b^{\alpha-1}a + a^{\alpha-1}b}{6}\right] \end{aligned}$$

Proof. The result follows from Theorem 2 for the convex function $\Phi(x) = x^r$, $x > 0$. ☐

Proposition 4. Let $a, b \in \mathbb{R}^{+}$ with $r > 1$ and $\alpha \in (0,1]$, then we have the following inequality:

$$\begin{aligned} & \left|\mathtt{A}(a^{-1}, b^{-1}) - \mathtt{L}^{-1}_{(\alpha,-1)}(a,b)\right| \\ \leq \ & \frac{b-a}{b^{\alpha} - a^{\alpha}} \Bigg[\mathtt{A}_1(\alpha)\Big(\max\left\{|a|^{-2q}, |b|^{-2q}\right\}\Big)^{\frac{1}{q}} \\ + \ & \mathtt{A}_2(\alpha)\Big(\max\left\{|a|^{-2q}, |b|^{-2q}\right\}\Big)^{\frac{1}{q}}\Bigg] \end{aligned}$$

Proof. The result follows from Theorem 2 for the convex function $\Phi(x) = x^r$, $x > 0$. ☐

Proposition 5. Let $a,b \in \mathbb{R}^+$ with $r > 1$ and $\alpha \in (0,1]$, then the following holds:

$$\begin{aligned} &\left|\mathtt{A}^r(a,b) - \mathtt{L}^r_{(\alpha,r)}(a,b)\right| \\ \leq\ & \frac{(r-1)(b-a)}{b^\alpha - a^\alpha} \max\left\{|a|^{r-1}, |b|^{r-1}\right\} \left[\frac{2b^\alpha - 4a^\alpha + b^{\alpha-1}a + a^{\alpha-1}b}{12}\right] \end{aligned}$$

Proof. The result follows from Theorem 2 for the convex function $\Phi(x) = x^r$, $x > 0$. □

Proposition 6. Let $a,b \in \mathbb{R}^+$ with $r > 1$ and $\alpha \in (0,1]$, then we have the following inequality:

$$\begin{aligned} &\left|\mathtt{A}^r(a,b) - \mathtt{L}^r_{(\alpha,r)}(a,b)\right| \\ \leq\ & \frac{(r-1)(b-a)}{b^\alpha - a^\alpha}\left[\mathtt{A}_1(\alpha)\Big(\max\left\{|a|^{(r-1)q}, |b|^{(r-1)q}\right\}\Big)^{\frac{1}{q}}\right. \\ +\ & \left.\mathtt{A}_2(\alpha)\Big(\max\left\{|a|^{(r-1)q}, |b|^{(r-1)q}\right\}\Big)^{\frac{1}{q}}\right] \end{aligned}$$

Proof. The result follows from Theorem 2 for the convex function $\Phi(x) = x^r$, $x > 0$. □

Proposition 7. Let $a,b \in \mathbb{R}^+$ with $r > 1$ and $\alpha \in (0,1]$, then the following inequality holds:

$$\begin{aligned} &\left|\mathtt{A}^{-1}(a,b) - \mathtt{L}^{-1}_{(\alpha,-1)}(a,b)\right| \\ \leq\ & \frac{b-a}{b^\alpha - a^\alpha} \max\left\{|a|^{-2}, |b|^{-2}\right\} \left[\frac{2b^\alpha - 4a^\alpha + b^{\alpha-1}a + a^{\alpha-1}b}{12}\right] \end{aligned}$$

Proof. The result follows from Theorem 2 for the convex function $\Phi(x) = x^r$, $x > 0$. □

Proposition 8. Let $a,b \in \mathbb{R}^+$ with $r > 1$ and $\alpha \in (0,1]$, then we have the following inequality:

$$\begin{aligned} &\left|\mathtt{A}^{-1}(a,b) - \mathtt{L}^{-1}_{(\alpha,-1)}(a,b)\right| \\ \leq\ & \frac{b-a}{b^\alpha - a^\alpha}\left[\mathtt{A}_1(\alpha)\Big(\max\left\{|a|^{-2q}, |b|^{-2q}\right\}\Big)^{\frac{1}{q}}\right. \\ +\ & \left.\mathtt{A}_2(\alpha)\Big(\max\left\{|a|^{-2q}, |b|^{-2q}\right\}\Big)^{\frac{1}{q}}\right] \end{aligned}$$

Proof. The result follows from Theorem 2 for the convex function $\Phi(x) = x^r$, $x > 0$. □

4. Application to the trapezoidal and mid point formulas

Let $\mathtt{P}$ be a division $a = x_o < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b \Phi(x) d_\alpha x = \mathtt{M}_\alpha(\Phi, \mathtt{P}) + \mathtt{E}_\alpha^{\mathtt{M}}(\Phi, \mathtt{P}), \tag{33}$$

where

$$\mathtt{M}_\alpha(\Phi, \mathtt{P}) = \sum_{i=0}^{n-1} \Phi\left(\frac{x_i + x_{i+1}}{2}\right) \frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} \tag{34}$$

is the midpoint version and $\mathtt{E}_\alpha^{\mathtt{M}}(\Phi, \mathtt{P})$ denotes the associated approximation error, and

$$\int_a^b \Phi(\mu) d_\alpha \mu = \mathtt{T}_\alpha(\Phi, \mathtt{P}) + \mathtt{E}_\alpha^{\mathtt{T}}(\Phi, \mathtt{P}), \tag{35}$$

where

$$\mathtt{T}_\alpha(\Phi, \mathtt{P}) = \sum_{i=0}^{n-1} \frac{\Phi(x_i) + \Phi(x_{i+1})}{2} \frac{\left(x_{i+1}^\alpha - x_i^\alpha\right)}{\alpha}. \tag{36}$$

for the trapezoidal version, and $\mathtt{E}_\alpha^{\mathtt{T}}(\Phi, \mathtt{P})$ denotes the associated approximation error. Here, we are going to derive some new error estimates for the midpoint and trapezoidal formulas.

Proposition 9. Let $\alpha \in (0, 1]$ and $x_i, x_{i+1} \in \mathbb{R}^+$. If $\Phi : [x_i, x_{i+1}] \to \mathbb{R}$ be an α-differentiable function on (a, b), then the inequality

$$\begin{aligned} \left|\mathtt{E}_\alpha^{\mathtt{M}}(\Phi, \mathtt{P})\right| &\leq \sum_{i=0}^{n-1} \frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} \max\{|\Phi'(x_i)|, |\Phi'(x_{i+1})|\} \\ \times & \left[\frac{2x_{i+1}^\alpha - 4x_i^\alpha + x_{i+1}^{\alpha-1} x_i + x_i^{\alpha-1} x_{i+1}}{12}\right] \end{aligned} \tag{37}$$

holds, if $\mathtt{D}_\alpha(\Phi) \in \mathtt{L}_\alpha^1([x_i, x_{i+1}])$ and $|\Phi'|$ is quasi-convex on $[x_i, x_{i+1}]$.

Proof. since

$$\begin{aligned} & \left|\Phi\left(\frac{x_i + x_{i+1}}{2}\right) \frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} - \int_{x_i}^{x_{i+1}} \Phi(x) d_\alpha x\right| \\ \leq \quad & \frac{(x_{i+1} - x_i)}{2} \max\{|\Phi'(x_i)|, |\Phi'(x_{i+1})|\} \left[\frac{2x_{i+1}^\alpha - 4x_i^\alpha + x_{i+1}^{\alpha-1} x_i + x_i^{\alpha-1} x_{i+1}}{12}\right] \end{aligned}$$

As we know that

$$\begin{aligned}
&\left|\int_{x_i}^{x_{i+1}} \Phi(x)d_\alpha x - \mathtt{M}_\alpha(\Phi,\mathtt{P})\right| \\
= &\left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} \Phi(x)d_\alpha x - \Phi\left(\frac{x_i+x_{i+1}}{2}\right)\frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha}\right\}\right| \\
\le &\sum_{i=0}^{n-1}\left|\left\{\int_{x_i}^{x_{i+1}} \Phi(x)d_\alpha x - \Phi\left(\frac{x_i+x_{i+1}}{2}\right)\frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha}\right\}\right| \\
\le &\sum_{i=0}^{n-1}\frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha}\max\left\{\left|\Phi'(x_i)\right|,\left|\Phi'(x_{i+1})\right|\right\}\left[\frac{2x_{i+1}^\alpha - 4x_i^\alpha + x_{i+1}^{\alpha-1}x_i + x_i^{\alpha-1}x_{i+1}}{12}\right].
\end{aligned}$$

□

Proposition 10. Let $\alpha \in [x_i, x_{i+1}]$, $q > 1$ with $x_i, x_{i+1} \in \mathbb{R}^+$ and $\Phi : [x_i, x_{i+1}] \to \mathbb{R}$ be an α-differentiable function, then the inequality

$$\begin{aligned}
\left|\mathtt{E}_\alpha^{\mathtt{M}}(\Phi,\mathtt{P})\right| \le &\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)}{\alpha}\Bigg[\mathtt{A}_1(\alpha)\Big(\max\left\{|\Phi'(x_i)|^q, |\Phi'(x_{i+1})|^q\right\}\Big)^{\frac{1}{q}} \\
+ \quad &\mathtt{A}_2(\alpha)\Big(\max\left\{|\Phi'(x_i)|^q, |\Phi'(x_{i+1})|^q\right\}\Big)^{\frac{1}{q}}\Bigg]. \qquad (38)
\end{aligned}$$

is valid if $\mathtt{D}_\alpha(\Phi) \in \mathtt{L}_\alpha^1([x_i, x_{i+1}])$ and $|\Phi'|^q$ is quasi-convex on $[x_i, x_{i+1}]$, where

$$\mathtt{A}_1(\alpha) = \left[\frac{(x_i + x_{i+1})^{\alpha+1} - (2x_i)^{\alpha+1}}{(2)^{\alpha+1}(x_{i+1}-x_i)(\alpha+1)}\right] - \frac{x_i^\alpha}{2}$$

$$\mathtt{A}_2(\alpha) = \frac{x_{i+1}^\alpha}{2} - \left[\frac{(2x_{i+1})^{\alpha+1} - (x_i + x_{i+1})^{\alpha+1}}{(2)^{\alpha+1}(x_{i+1}-x_i)(\alpha+1)}\right]$$

Proposition 11. Let $\alpha \in (0,1]$ and $x_i, x_{i+1} \in \mathbb{R}^+$. If $\Phi : [x_i, x_{i+1}] \to \mathbb{R}$ be an α-differentiable function on (a,b), then the inequality

$$\begin{aligned}
&\left|\mathtt{E}_\alpha^{\mathtt{T}}(\Phi,\mathtt{P})\right| \\
\le &\sum_{i=1}^{n-1}\frac{x_{i+1}-x_i}{\alpha}\max\left\{\left|\Phi'(x_i)\right|,\left|\Phi'(x_{i+1})\right|\right\}\left[\frac{5x_{i+1}^\alpha - 7x_i^\alpha + x_{i+1}^{\alpha-1}x_i + x_i^{\alpha-1}x_{i+1}}{6}\right] \qquad (39)
\end{aligned}$$

holds, if $\mathtt{D}_\alpha(\Phi) \in \mathtt{L}_\alpha^1([x_i, x_{i+1}])$ and $|\Phi'|$ is quasi-convex on $[x_i, x_{i+1}]$.

Proof. since

$$\begin{aligned}
&\left|\frac{\Phi(x_i)+\Phi(x_{i+1})}{2}\frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} - \int_{x_i}^{x_{i+1}} \Phi(x)d_\alpha x\right| \\
\le &\frac{(x_{i+1}-x_i)}{2}\max\left\{\left|\Phi'(x_i)\right|,\left|\Phi'(x_{i+1})\right|\right\}\left[\frac{5x_{i+1}^\alpha - 7x_i^\alpha + x_{i+1}^{\alpha-1}x_i + x_i^{\alpha-1}x_{i+1}}{6}\right].
\end{aligned}$$

As we know that

$$
\begin{aligned}
&\left|\int_a^b \Phi(x)d_\alpha x - \mathtt{T}_\alpha(\Phi,\mathtt{P})\right| \\
= & \left|\sum_{i=0}^{n-1}\left[\frac{\Phi(x_i)+\Phi(x_{i+1})}{2}\frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} - \int_{x_i}^{x_{i+1}}\Phi(x)d_\alpha x\right]\right| \\
\leq & \sum_{i=0}^{n-1}\left|\left[\frac{\Phi(x_i)+\Phi(x_{i+1})}{2}\frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} - \int_{x_i}^{x_{i+1}}\Phi(x)d_\alpha x\right]\right| \\
\leq & \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)}{\alpha}\max\left\{\left|\Phi'(x_i)\right|,\left|\Phi'(x_{i+1})\right|\right\}\left[\frac{5x_{i+1}^\alpha - 7x_i^\alpha + x_{i+1}^{\alpha-1}x_i + x_i^{\alpha-1}x_{i+1}}{6}\right].
\end{aligned}
$$

□

Proposition 12. Let $\alpha \in [x_i, x_{i+1}]$, $q > 1$ with $x_i, x_{i+1} \in \mathbb{R}^+$. If $\Phi : [x_i, x_{i+1}] \to \mathbb{R}$ be an α-differentiable function, then the inequality

$$
\begin{aligned}
\left|\mathtt{E}_\alpha^{\mathtt{T}}(\Phi,\mathtt{P})\right| \leq & \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)}{2\alpha}\Bigg[\mathtt{A}_1(\alpha)\Big(\max\left\{|\Phi'(x_i)|^q, |\Phi'(x_{i+1})|^q\right\}\Big)^{\frac{1}{q}} \\
+ & \mathtt{A}_2(\alpha)\Big(\max\left\{|\Phi'(x_i)|^q, |\Phi'(x_{i+1})|^q\right\}\Big)^{\frac{1}{q}}\Bigg] \qquad (40)
\end{aligned}
$$

is valid if $\mathtt{D}_\alpha(\Phi) \in \mathtt{L}_\alpha^1([x_i, x_{i+1}])$ and $|\Phi'|^q$ is quasi-convex on $[x_i, x_{i+1}]$, where

$$
\mathtt{A}_1(\alpha) = \left[\frac{x_i^{\alpha+1} - x_{i+1}^{\alpha+1}}{(x_i - x_{i+1})(\alpha+1)}\right] - x_i^\alpha
$$

$$
\mathtt{A}_2(\alpha) = x_{i+1}^\alpha - \left[\frac{x_i^{\alpha+1} - x_{i+1}^{\alpha+1}}{(x_i - x_{i+1})(\alpha+1)}\right]
$$

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