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SOME NEW OSCILLATION CRITERIA FOR FOURTH-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAY

CEMIL TUNÇ AND OMAR BAZIGHIFAN

Abstract. In this paper, we study the oscillation of fourth-order neutral differentiale quations with continuously distributed delay

$$\left[r(t)\left[[z(t)]'''\right]^{\alpha}\right]' + \int_c^d q(t,\ \xi) f(x(g(t,\ \xi)))\, d(\xi) = 0$$

where

$$z(t) = x(t) + \int_a^b p(t,\ \sigma)\, x(\tau(t,\ \sigma))\, d\sigma$$

under a condition

$$\int_{t_\circ}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)} dt = \infty.$$

New oscillation criteria are obtained by employing a refinement of the generalized Riccati transformations. An illustrative example is given for illustrations.

1. Introduction

In this paper, we consider a fourth order neutral differential equation with a continuously distributed delay of the form

$$\left[r(t)\left[\left[x(t) + \int_a^b p(t,\ \sigma)\, x(\tau(t,\ \sigma))\, d\sigma\right]'''\right]^{\alpha}\right]' + \int_c^d q(t,\ \xi) f(x(g(t,\ \xi)))\, d(\xi) = 0. \tag{1}$$

We assume the following conditions hold:

- (H_1) α is a quotient of odd positive integers.
- (H_2) $r(t) \in C([t_0,\ \infty), \mathbb{R}), r'(t) \geq 0$.
- (H_3) $p(t,\ \sigma),\ \tau(t,\ \sigma) \in C([t_0,\ \infty) \times [a,b], \mathbb{R}),\ 0 \leq p(t) \equiv \int_a^b p(t,\ \sigma)\, d\sigma \leq P < 1,\ \tau'(t) > 0,\ \tau(t,\ \sigma)$ is a nondecreasing function in $\sigma,\ \tau(t,\ \sigma) \leq t, \lim_{t\to\infty} \tau(t,\ \sigma) = \infty$.

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(H_4) $q(t,\ \xi),\ q(t,\ \xi) \in C\left([t_0,\ \infty) \times [c,d], \mathbb{R}\right),\ q(t,\ \xi)$ is positive, $g(t,\ \xi)$ is a nondecreasing function in ξ, $g(t,\xi) \leq t\ \lim_{t\to\infty} g(t,\xi) = \infty$.

(H_5) There exists a constant $k > 0$ such that $f(u)/u^{\gamma} \geq k$, for $u \neq 0$.

We define the corresponding function $z(t)$ of a solution $x(t)$ of (1) by

$$z(t) = x(t) + \int_a^b p(t,\ \sigma)\, x(\tau(t,\ \sigma))\, d\sigma.$$

We mean a non-trivial real function $x(t) \in C([t_x, \infty))$, $t_x \geq t_0$, satisfying (1) on $[t_x, \infty)$ and moreover, having the properties: $z(t)$, $z'(t)$, $z''(t)$ and $r(t)\,[z'''(t)]^{\alpha}$ are continuously differentiable for all $t \in [t_x, \infty)$.
We consider only those solutions $x(t)$ of (1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$, for any $T \geq t_x$.
A solution of (1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

The problem of the oscillation of higher and fourth order differential equations have been widely studied by many authors, who have provided many techniques for obtaining oscillatory criteria for higher and fourth order differential equations. We refer the reader to the related books (see [2, 6, 18], [13]-[15]) and to the papers (see [1], [3]-[12], [16]-[23]).
In what follows, we present some relevant results that have provided the background and the motivation for the present work.

Li et al. [16] studied the oscillatory behavior of the fourth-order nonlinear differential equation

$$[r(t)\, z(t)]^{(4)} + q(t)\, x(\tau(t,\ \sigma)) = 0 \qquad t \geq t_0.$$

Elabbasy et al. [9, 10], Moaaz et al. [18] and Zhang et al. [23] examined the oscillation of the fourth-order nonlinear delay differential equation

$$\left[r(t)\,(x'''(t))^{\alpha}\right]' + q(t)\, x^{\alpha}(t) = 0 \qquad t \geq t_0.$$

Parhi and Tripathy [19] have considered the fourth-order neutral differential equations of the form

$$\left[r(t)\,(y(t) + p(t)\, y(t-\tau))''\right]'' + q(t)\, G(y(t-\sigma)) = 0$$

and

$$\left[r(t)\,(y(t) + p(t)\, y(t-\tau))''\right]'' + q(t)\, G(y(t-\sigma)) = f(t)$$

and they established the oscillation and asymptotic behavior of the above equations, under the conditions

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)} d\dot{t} < \infty$$

and

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)} dt = \infty \tag{2}$$

respectively.

Our aim, in the present paper, is to use the Riccati method to establish new conditions for the oscillation of all solutions of (1) under condition (2).

The proof of our main result is essentially based on the following lemmas.

Lemma 1. [21] Let $\beta \geq 1$ be a ratio of two numbers, where U and V are constants. Then

$$Uy - Vy^{\frac{\beta+1}{\beta}} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^{\beta}},\ V > 0.$$

Lemma 2. [14] If the function z satisfies $z^{(i)} > 0$, $i = 0, 1, ..., n$, and $z^{(n+1)} < 0$, then

$$\frac{z(t)}{t^{n}/n!} \geq \frac{z'(t)}{t^{n-1}/(n-1)!}. \tag{3}$$

Lemma 3. [2] Let $h \in C^{n}([t_0, \infty), (0, \infty))$. Assume that $h^{(n)}(t)$ is of a fixed sign and not identically zero on $[t_0, \infty)$ and that there exists a $t_1 \geq t_0$ such that $h^{(n-1)}(t) h^{(n)}(t) \leq 0$ for all $t \geq t_1$. If, $\lim_{t\to\infty} h(t) \neq 0$ then for every $\lambda \in (0, 1)$ there exists $t_\lambda \geq t_0$ such that

$$h(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| h^{(n-1)}(t) \right| \text{ for } t \geq t_\lambda. \tag{4}$$

2. Main result

In this section, we establish some oscillation criteria for equation (1).

For convenience, we denote

$$\eta(t) = \int_t^\infty \frac{1}{r^{\frac{1}{\alpha}}(s)} ds,\ \rho'_+(t) := \max\{0,\ \rho'(t)\},\text{ and } Q(t) = \int_a^b q(t, \xi)\, d(\xi).$$

Theorem. Let $(H_1) - (H_5)$ hold. If there exists a positive function $\rho \in C([t_0, \infty))$ such that

$$\int_{t_0}^\infty \left(k\rho(s)(1-P)^{\alpha} (g(s,\ a) \backslash s)^{3\alpha} Q(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{r(t)\left(\rho'_+(t)\right)^{\alpha+1}}{\mu^{\alpha} t^{2\alpha} \rho^{\alpha}(t)} \right) ds = \infty \tag{5}$$

for some $\mu \in (0, 1)$ and

$$\int_{t_\circ}^\infty Q(s)(g(s,\ a) \backslash s)^{\alpha} ds = \infty, \tag{6}$$

then all solutions of (1) are oscillatory.

Proof. Let x be a nonoscillatory solution of equation (1) defined in the interval $[t_0, \infty)$. Without loss of generality, we can assume that $x(t)$ is eventually positive. It follows from (1) that there are two possible cases, for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large:

$$(C_1) \quad z'(t) > 0, z''(t) > 0, z'''(t) > 0, \left(r(t)(z'''(t))^{\alpha}\right) < 0,$$
$$(C_2) \quad z'(t) > 0, z''(t) < 0, z'''(t) > 0, \left(r(t)(z'''(t))^{\alpha}\right) < 0.$$

Assume that case (C_1) holds. Since $\tau(t,\ \sigma) \leq t$ and $z'(t) > 0,\ z(t) > x(t)$, we get

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t,\ \sigma)\, x(\tau(t,\ \sigma))\, d\sigma \\ &\geq z(t) - \int_a^b p(t,\ \sigma)\, z(\tau(t,\ \sigma))\, d\sigma \\ &\geq z(t) - z(\tau(t,\ b)) \int_a^b p(t,\ \sigma)\, d\sigma \\ &\geq z(t)\left(1 - \int_a^b p(t,\ \sigma)\, d\sigma\right) = (1-P)z(t). \end{aligned}$$

From equation (1), we see that

$$\left(r(t)\left(\left(x(t) + \int_a^b p(t,\ \sigma)\, x(\tau(t,\ \sigma))\, d\sigma\right)'''(t)\right)^{\alpha}\right)' = -\int_c^d q(t,\xi)\, f(x(g(t,\xi)))\, d(\xi)$$

so that

$$\begin{aligned} \left(r(t)(z'''(t))^{\alpha}\right)' &\leq -(1-P)^{\alpha} \int_c^d kq(t,\xi)\, x^{\alpha}(g(t,\xi))\, d(\xi) \\ &\leq -(1-P)^{\alpha} kz^{\alpha}(g(t,\ c)) \int_c^d q(t,\xi)\, d(\xi), \end{aligned}$$

$$\left(r(t)(z'''(t))^{\alpha}\right)' \leq -kQ(t)(1-P)^{\alpha} z^{\alpha}(g(t,\ c)). \tag{7}$$

Now, we define a generalized Riccati substitution by

$$\omega(t) := \rho(t) \frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha}(t)}.$$

Then $\omega(t) > 0$. Differentiating $\omega(t)$ and using the inequality (7), we obtain

$$\begin{aligned} \omega'(t) \leq\ & \frac{\rho'(t)}{\rho(t)}\omega(t) - k\rho(t)Q(t)(1-P)^{\alpha}\frac{z^{\alpha}(g(t,\ a))}{z^{\alpha}(t)} \\ & -\alpha\rho(t)\frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha+1}(t)} z'(t). \end{aligned} \tag{8}$$

From Lemma 3, we have that $z(t) \geq \frac{t}{3} z'(t)$, and hence

$$\frac{z(g(t,\ a))}{z(t)} \geq \frac{g^3(t,\ a)}{t^3}. \tag{9}$$

Since $r'(t) > 0$ and $\left(r(t)(z'''(t))^{\alpha}\right)' \leq 0$, then we get $z^{(4)}(t) < 0$. It follows from Lemma 4 that

$$z'(t) \geq \frac{\mu}{2} t^2 z'''(t), \tag{10}$$

for all $\mu \in (0,1)$ and every sufficiently large t. Thus, by (8), (9) and (10), we get

$$\begin{aligned} \omega'(t) \leq\ & \frac{\rho'_+(t)}{\rho(t)}\omega(t) - k\rho(t)Q(t)(1-P)^{\alpha}\frac{g^{3\alpha}(t,\ a)}{t^{3\alpha}} \\ & -\alpha\mu\frac{t^2}{2r^{1/\alpha}(t)\,\rho^{1/\alpha}(t)}\omega^{\frac{\alpha+1}{\alpha}}(t). \end{aligned}$$

Using Lemma 1 with $U=\frac{\rho'_+(t)}{\rho(t)}, V=\frac{\alpha\mu t^2}{2r^{1/\alpha}(t)\rho^{1/\alpha}(t)}$ and $y=\omega$, we get

$$\omega'(t)\le -k\rho(t)(1-P)^\alpha\frac{g^{3\alpha}(t,\ a)}{t^{3\alpha}}Q(t)+\frac{2^\alpha}{(\alpha+1)^{\alpha+1}}\frac{r(t)\left(\rho'_+(t)\right)^{\alpha+1}}{\mu^\alpha t^{2\alpha}\rho^\alpha(t)}.$$

This implies that

$$\int_{t_1}^{t}\left(k\rho(s)(1-P)^\alpha\frac{g^{3\alpha}(s,\ a)}{s^{3\alpha}}Q(s)-\frac{2^\alpha}{(\alpha+1)^{\alpha+1}}\frac{r(t)\left(\rho'_+(t)\right)^{\alpha+1}}{\mu^\alpha t^{2\alpha}\rho^\alpha(t)}\right)ds\le\omega(t_1),$$

for some $\mu\in(0,1)$ which contradicts with (5).

Assume that case (C_2) holds. Integrating (1) from t_1 to t, we obtain

$$-r(t_1)\left(z'''(t_1)\right)^\alpha\le-\int_{t_1}^{t}kQ(t)x^\alpha(g(s,\ \xi))ds. \tag{11}$$

From $z'(t)>0,\ x(t)\ge(1-P)z(t)$ and $g(s,\ \xi)\le t$, it follows that

$$\int_{t_1}^{t}kQ(s)(1-P)^\alpha z^\alpha(g(s,\ a))ds\le r(t_1)\left(z'''(t_1)\right)^\alpha.$$

From (9), we get

$$\int_{t_1}^{t}kQ(s)(1-P)^\alpha(g(s,\ a)\backslash s)^{3\alpha}ds\le r(t_1)\left(\frac{z'''(t_1)}{z(t_1)}\right)^\alpha,$$

which contradicts with (5).

The proof of the theorem is complete. □

Let $\rho(t)=t^3$. As a consequence of Theorem 2, we obtain the following oscillation criterion.

Corollary. Let $(H_1)-(H_5)$ and (2) hold, for some constant $\lambda_0\in(0,\ 1)$

$$\int_{t_0}^{\infty}\left(kt^3Q(t)(1-P)^\alpha(g(t,\ a)\backslash t)^{3\alpha}-\left(\frac{3}{\alpha+1}\right)^{\alpha+1}\left(\frac{2}{\lambda_0}\right)^\alpha t^{2-3\alpha}r(t)\right)ds=\infty,$$

and (6) are satisfied. Then all solution of (1) are oscillatory.

3. **Example**

In this section, we give the following example to illustrate our main results.

Eexample. Consider the following differential equation of fourth order

$$\left(\left[x(t)+\int_0^1\frac{1}{2}x\left(\frac{t-\sigma}{3}\right)d\sigma\right]'''\right)'+\int_0^1\left(\nu\backslash t^4\right)\xi x\left(\frac{t-\xi}{2}\right)d\xi=0, \tag{12}$$

where $\upsilon>0$ is a constant. Let

$\alpha=1,\ r(t)=1,\ p(t,\ \sigma)=\dfrac{1}{2},\ \tau(t,\ a)=\dfrac{t}{3},\ g(t,\ a)=\dfrac{t}{2},\ q(t,\ \xi)=\left(\nu\backslash t^4\right)\xi,\ f(x)=x.$

Then, we get

$$Q(t)=\int_0^1 q(t,\ \xi)d\xi=\frac{\nu}{2t^4}.$$

If we now set $k = 1$, then

$$\int_{t_0}^{\infty} \left(kt^3 Q(t) (1-P)^{\alpha} (g(t,\ a) \setminus t)^{3\alpha} - \left(\frac{3}{\alpha+1}\right)^{\alpha+1} \left(\frac{2}{\lambda_0}\right)^{\alpha} t^{2-3\alpha} r(t) \right) ds,$$

$$= \left(\frac{\nu}{32} - \frac{9}{2\lambda_0}\right) \int_{t_0}^{\infty} \frac{1}{t} dt = \infty, \text{ if } \nu > \frac{144}{\lambda_0} \text{ for some constant } \lambda_0 \in (0,\ 1).$$

Thus, by Corollary 2, every solution of equation (12) is oscillatory, provided $\nu > \frac{144}{\lambda_0}$.

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Cemil Tunç, Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University,65080, Campus-VAN-TURKEY.
E-mail address: cemtunc@yahoo.com

Omar Bazighifan, Department of Mathematics, Hadhramout University, Yemen.
E-mail address: o.bazighifan@gmail.com