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FIXED POINTS FOR SUM OR PRODUCT OF OPERATORS IN WEAK TOPOLOGY WITH APPLICATIONS (A SURVEY)

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Abstract. The existence of fixed points for sum or product of two operators has been studied by several authors, for example Burton [14], Barroso [12], Bin [38], Cichoń[19] and Dhage [21].
Here, a review on fixed points for sum or product of two operators in weak topology will be given and some of their application will be presented in this survey paper.

1. Introduction

The existence of fixed points for the sum of two mappings has been focused of interest for several years and their applications are often in nonlinear analysis to solve problems in natural sciences, mathematical physics, mechanics and population dynamics. A useful tool to deal with such problems is the celebrated fixed point theorem due to Krasnosel'skii in 1958 (see for example Theorem 11.Bin [38]) which asserts that, if Ω is a nonempty closed and convex subset of a Banach space E; A and B are two mappings from Ω into E such that (i)A is compact, (ii) B is a contraction and (iii) $A\Omega+B\Omega \subset \Omega$, then $A+B$ has at least one fixed point in Ω. Since then, a wide class of problems, for instance in integral equations and stability theory have been contemplated by the Krasnosel'skii fixed point principle. However, in some applications, the verification of the hypothesis (iii) is quite hard. As a tentative approach to grapple with such a difficulty, many attempts have appeared in the literature in the direction of weakening the hypothesis (iii). For example, in [14] Burton improved the Krasnosel'skii principle by requiring, instead of (iii), the more general condition $[x = Bx + Ay, y \in \Omega] \Rightarrow x \in \Omega$. The new variant of Krasnosel'skii's fixed point theorem developed by Burton was applied in stability theory and integral equations, covering cases where the result of Krasnosel'skii can not be applied. Subsequently, the new asymptotic requirement was introduced by Barroso [12].

In recent years, many interesting works have been made in order to establish the

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analogue of Krasnosel'skii principle for the weak topology. Also, many authors have focused on the resolution of the equation

$$x = Ax\ Bx + Cx \tag{1}$$

and obtained a lot of valuable results . These studies were mainly based on the convexity of the bounded domain, the celebrate Schauder fixed point theorem [35] and properties of operators A, B and C (cf. completely continuous, k-set contractive, condensing and the potential tool of the axiomatic measures of non-compactness). Because the weak topology is the practice setting and natural to investigate the problems of existence of solutions of different types of nonlinear integral equations and nonlinear differential equations in Banach algebras, it turns out that the results mentioned above cannot be easily applied.
The equation (1) arises frequently in biology, engineering, physics, mechanics and economics [38] and [15].

This paper is devoted to a survey of the current state of the theory of fixed point of sum and product of operators in weak topology.

2. Solvability in Dunford-Pettis Space

In 2005, *A. Ben Amar, A. Jeribi, M. Mnif* [9] gave a generalization of the Schauder and Krasnosel'skii fixed point theorems in Dunford-Pettis spaces. Both of these theorems can be used to resolve some open problems posed by Jeribi (Nonlinear Anal.: Real World Appl. 2002; 3:85-105); and Latrach (J. Math. Phys. 1996; 37:1336-1348). Further, they applied their work to prove some existence results for a source problem with general boundary conditions in L_1 spaces. More precisely, let X be a Dunford Pettis space, M be a closed bounded convex subset of X and A a weakly compact linear operator. If A leaves M invariant then A has a fixed point in M.

The central purpose of this paper is to give some existence results for the stationary model on L_1 spaces:

$$\upsilon_3 \frac{\partial \psi}{\partial x}(x,\upsilon) + \sigma(x,\upsilon)\psi(x,\upsilon) - \lambda\psi(x,\upsilon) = \int_K r(x,\upsilon,\upsilon',\psi(x,\upsilon'))d\upsilon' \text{ in D}$$

where $D = (0,1) \times K$, K is the unit sphere of $\mathbb{R}^3$, $x \in (0,1)$,
$\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3) \in K$, $r(.,.,.,.)$ is a non-linear function of ψ, $\sigma(.,.)$ is a positive bounded function and λ is a complex number.
The main point in this equation is the non-linear dependence of the functions $r(x,\upsilon,\upsilon',(x,\upsilon'))$ on ψ.

Theorem [9]
Let X be a Dunford-Pettis space, M be a non-empty closed bounded convex subset of X and A a weakly compact linear operator on X . If A leaves M invariant then A has a fixed point in M:

In addition, if B is a contraction map of M into X, $Ax + By \in M$ for $x; y$ in M and $(I - B)^{-1}A$ is a weakly compact operator, then $A + B$ has a fixed point in M. consider now the sum of a weakly compact operator and a contraction mapping.

Theorem[9]
Let X be a Dunford-Pettis space, and let M a non-empty closed bounded convex subset of X . Suppose that A and B map M into X such that:

(i) $A \in \mathcal{L}(X)$and weakly compact;
(ii) B is a contraction mapping;
(iii) $Ax + By \in M$ for all $x; y$ in M;
(iv) $(I - B)^{-1}A(M)$ is a weakly compact set.
Then there exist y in M such that $Ay + By = y$

A. Ben Amar et al.. [9] discussed briefly the existence of positives solutions. Let X_1 and X_2 be two Banach lattice spaces, with positive *cônes* X_1^+ and X_2^+, respectively. An operator T from X_1 into X_2 is said to be positive, if it carries the positive cône X_1^+ into $X_2^+(i.e.T(X_1^+) \subset X_2^+)$.

Theorem [9]
Let X be a Dunford-Pettis space, M be a non-empty closed bounded convex subset of X such that and A is a positive weakly compact linear operator on X . If A leaves M invariant then A has at least a positive fixed point in M.

In 2010, *Afif Ben Amar* [4] presented some new variants of Leray Schauder type fixed point theorems and eigenvalue results for decomposable single-valued nonlinear weakly compact operators in Dunford-Pettis spaces.

The main results of [4] is to give new alternatives of Leray- Schauder type for some nonlinear weakly compact composite operators $F = GT$ in Dunford-Pettis spaces, where G and T verify some sequential conditions H_1 and H_2,
where

$$(H_1) \begin{cases} If\ (x_n)_{n\in N} \text{ is a weakly convergent sequence in } E, \text{ then} \\ (Fx_n)_{n\in N} \text{ has a strongly convergent subsequence in } E, \end{cases}$$

$$(H_2) \begin{cases} If\ (x_n)_{n\in N} \text{ is a weakly convergent sequence in } E, \text{ then} \\ (Fx_n)_{n\in N} \text{ has a weakly convergent subsequence in } E. \end{cases}$$

Remark :

(i) A strongly continuous operator F verifies (H_1).
(ii) An operator F verifies (H_2) if and only if it maps relatively weakly compact sets into relatively weakly compact ones.

As an application, *Afif Ben Amar* proved the existence of positive eigenvalues for such decomposable operators. Finally, he established some new nonlinear alternatives of Leray-Schauder type for decomposable nonlinear positive, weakly compact operators in the positive cône of Dunford-Pettis lattices. One important feature of this approach is that it allows the relaxation of boundedness and convexity of domains.
First, we introduce the definition and the proposition.

Definition Let E be a Banach space. E is said to have the Dunford- Pettis property (for short property DP) if for each Banach space E_1 every weakly compact linear operator $F : E \to E_1$ takes weakly compact sets in E into norm compact sets of E_1.

Proposition Let E be a Dunford-Pettis space and T a weakly compact linear operator on E. Then T is strongly continuous.

Afif Ben Amar obtained a nonlinear alternative of Leray Schauder type for decomposable nonlinear weakly compact operators in Dunford-Pettis spaces.

Theorem Let E be a Dunford-Pettis space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $z \in U$. If $G : E \to E$ and $T : U \to E$ are operators satisfying:

(1) G is a bounded linear weakly compact operator.
(2) T is a nonlinear continuous operator satisfying (H_1).
(3) $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$.

Then, either

(A_1) GT has a fixed point in U , or
(A_2) there is a point $x \in \partial_\Omega U$ (the boundary of U in Ω) and $\lambda \in (0, 1)$ with $x = (1 - \lambda)z + \lambda GTx$.

3. Solvability in Banach algebra

In 2011, *Afif Ben Amar , Soufiene Chouayekh and Aref Jeribi*[6] introduced a class of Banach algebras satisfying certain sequential condition

$$(\mathcal{P}) \begin{cases} \text{For any sequences } (x_n)_{n\in N} \text{ and } (x_n)_{n\in N} \text{ of } \varepsilon, \text{ such that } x_n \rightharpoonup x \\ \\ \text{and } y_n \rightharpoonup y, \text{ then } x_n y_n \rightharpoonup xy; \text{ where } \varepsilon \text{ is a Banach algebra} \end{cases}$$

and proved fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators.
Also, they illustrated the applicability of their results by considering the following examples of nonlinear functional integral equations.

Example 1:
Let $(X, ||.||)$ be a Banach algebra satisfying condition $(\mathcal{P})$. Let $J = [0, 1]$ the closed and bounded interval in R, the set of all real numbers. Let $E = C(J, X)$ the Banach algebra of all continuous functions from $[0, 1]$ to X, endowed with the sup-norm $||.||_\infty$, defined by $||f||_\infty = \sup ||f(t)||;\ t \in [0, 1]$, for each $f \in C(J, X)$. The nonlinear functional integral equation (in short, FIE):

$$x(t) = a(t) + (T_1x)(t)\left[\left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda\ s))\ ds\right).u\right], 0 < \lambda < 1,$$

for all $t \in J$, where $u \neq 0$ is a fixed vector of X and the functions a, q, σ, p, T_1 are given, while $x = x(t)$ is an unknown function, is considered under the following conditions:

- (H_1) $a : J \to X$ is a continuous function.
- (H_2) $\sigma : J \to J$ is a continuous and nondecreasing function.
- (H_3) $q : J \to R$ is a continuous function.
- (H_4) The operator $T_1 : C(J, X) \to C(J, X)$ is such that
  - (a) T_1 is Lipschitzian with a Lipschitzian constant α,
  - (b) T_1 is regular on $C(J, X)$,
  - (c) T_1 is sequentially weakly continuous on $C(J, X)$,
  - (d) T_1 is weakly compact.
- (H_5) The function $p : J \times J \times X \times X \to R$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \to p(t, s, x, y)$ is continuous uniformly for $(s, x, y) \in J \times X \times X$.
- (H_6) There exists $r_0 > 0$ such that
  - (a) $|p(t, s, x, y)| \leq r_0 - ||q||_\infty$ for each $t, s \in J;\ x, y \in X$ such that $||x|| \leq r_0$ and $||y|| \leq r_0$,
  - (b) $||T_1 x||_\infty \leq (1 - \frac{||a||_\infty}{r_0}) \frac{1}{||u||}$ for each $x \in C(J, X)$,
  - (c) $\alpha\ r_0\ ||u|| < 1$.

Also, the nonlinear functional integral equation (in short, FIE) in C(J,X) is considered

$$x(t) = a(t) + (T_2 x)(t) \left[\left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda\ s))\ ds \right) .u \right], 0 < \lambda < 1,$$

for all $t \in J$, where $u \neq 0$ is a fixed vector of X and the functions a, q, σ, p, T_2 are given, while $x \in C(J, X)$ is an unknown function. Under the following conditions:

- (H_1) $a : J \to X$ is a continuous function with $||a||_\infty < 1$.
- (H_2) $\sigma : J \to J$ is a continuous and nondecreasing function.
- (H_3) $q : J \to R$ is a continuous function.
- (H_4) The operator $T_2 : C(J, X) \to C(J, X)$ is such that
  - (a) T_2 is Lipschitzian with a Lipschitzian constant α,
  - (b) T_2 is regular on $C(J, X)$,
  - (c) $(\frac{I}{T_2})^{-1}$ is well defined on $C(J, X)$,
  - (d) $(\frac{I}{T_2})^{-1}$ is sequentially weakly continuous on $C(J, X)$.
- (H_5) The function $p : J \times J \times X \times X \to R$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \mapsto p(t, s, x, y)$ is continuous uniformly for $(s, x, y) \in J \times X \times X$.
- (H_6) There exists $r_0 > 0$ such that
  - (a) $|p(t, s, x, y)| \leq r_0 - ||q||_\infty$ for each $t, s \in J;\ x, y \in X$ such that $||x|| \leq r_0$ and $||y|| \leq r_0$,
  - (b) $||T_2 x||_\infty \leq (1 - \frac{||a||_\infty}{r_0}) \frac{1}{||u||}$ for each $x \in C(J, X)$,
  - (c) $\alpha\ r_0\ ||u|| < 1$.

2014 *Jozef Banas and Mohamed-Aziz Taoudi* [13] proved some fixed point theorems for operators acting in Banach algebras and satisfying conditions expressed mainly with help of weak topology and measures of weak noncompactness. The existence of solutions to some class of operator equations in Banach algebras is also discussed. Some examples are presented to illustrate their results.

In [13] *Jozef Banas et al.* also discussed the solvability of some operator equations in Banach algebras. their analysis used the concept of measures of noncompactness.

In the sequel the concept defined below will play a crucial role in the following considerations.

Definition Let X be a Banach algebra. We say that X is a WC-Banach algebra if the product KK' of arbitrary weakly compact subsets $K,\ K'$ of X is weakly compact.

The following result concerning the existence of fixed points for operators acting in a WC-Banach algebra and satisfying some conditions expressed in terms of weak sequential continuity and the measure of weak noncompactness w.

Theorem [13]
Assume that Ω is a nonempty, closed and convex subset of a WC-Banach algebra X. Further, assume that P and T are operators acting weakly sequentially continuously from Ω into X in such a way that $P\Omega$ and $T\Omega$ are bounded. Apart from this we require that the operator $S = PT$ (the product of P and T) transforms Ω into itself and is weakly sequentially continuous. If the operators P and T satisfy the Darbo condition with respect to the De- Blasi measure of weak noncompactness w, with constants k_1 and k_2, respectively, then the operator S satisfies on Ω the Darbo condition (with respect to w) with the constant $k_1||T\Omega|| + k_2||P\Omega|| + k_1k_2w(\Omega)$. Particularly, if $k_1||T\Omega|| + k_2||P\Omega|| + k_1k_2w(\Omega) < 1$, then S is a contraction with respect to w and has at least one fixed point in the set Ω.

Now consider the condition

$$(\mathfrak{P})\ \begin{cases} \text{For any sequences } (x_n)_{n\in N} \text{ and } (x_n)_{n\in N} \text{ of } \chi, \text{ such that } x_n \rightharpoonup x \\ \text{and } y_n \rightharpoonup y, \text{ then } x_ny_n \rightharpoonup xy; \text{ where } \chi \text{ is a Banach algebra} \end{cases}$$

The condition $(\mathfrak{P})$ defined above implies the WC-Banach algebra structure. Indeed, the following is established:

Lemma [13]
If X is a Banach algebra satisfying condition $(\mathfrak{P})$ then X is a WC-Banach algebra.

Lemma [13]
Let Ω be a nonempty bounded closed subset of a Banach algebra X and let $A, C : X \to X$ be D-Lipschitzian mappings with D- functions φ_A and φ_C respectively. Assume that for each $r > 0$ we have $||\Omega||\phi_A(r) + \varphi_C(r) < r$. Then $(\frac{I-C}{A})^{-1} : \Omega \to X$ exists and is continuous.

Theorem [13]
Let X be a Banach algebra and let Ψ be a measure of weak noncompactness on X. Let Ω be a nonempty closed and convex subset of X and let $A, C : X \to X$ and $B : \Omega \to X$ be weakly sequentially continuous operators satisfying the following conditions:

(i) The operators A and C are D-Lipschitzian mappings with D- functions φ_A and φ_C respectively.
(ii) The set $B(\Omega)$ is bounded and the operator $(\frac{I-C}{A})^{-1}B$ is Ψ-condensing on Ω.
(iii) The equality $(x = AxBy + Cx)$ with $y \in \Omega$ implies $x \in \Omega$. Then the operator equation $x = AxBx + Cx$ has a solution in the set Ω provided that $Q\varphi_A(r) + \varphi_C(r) < r$ for $r > 0$, where $Q = ||B(\Omega)||$.

Taking $C \equiv 0$ in the above theorem we obtain the following result.

Theorem [13] Let X be a Banach algebra and let Ψ be a measure of weak noncompactness on X. Let Ω be a nonempty closed and convex subset of X and let $A : X \to X$ and $B : \Omega \to X$ be weakly sequentially continuous operators satisfying the following conditions:

(i) The operator A is a D-Lipschitzian mapping with D- functions ϕ_A.
(ii) The set $B(\Omega)$ is bounded and the operator $(\frac{I}{A})^{-1}B$ is Ψ-condensing on Ω.
(iii) The equality $(x = Ax\ By)$ with $y \in \Omega$ implies $x \in \Omega$. Then the operator equation $x = Ax\ Bx$ has a solution in the set Ω provided $Q\varphi_A(r) < r$ for $r > 0$, where $Q = ||B(\Omega)||$.

Theorem [13]
Let Ω be a nonempty, closed and convex subset of a WC- Banach algebra X and let $A, C : X \to X$ and $B : \Omega \to X$ be weakly sequentially continuous operators satisfying the following conditions:

(i) The operators A and C are D-Lipschitzian mappings with $D-$functions φ_A and φ_C respectively.
(ii) The set $B(\Omega)$ is relatively weakly compact.
(iii) The equality $(x = Ax\ By + Cx)$ with $y \in \Omega$ implies $x \in \Omega$. Then the operator equation $x = Ax\ Bx + Cx$ has a solution in the set Ω provided $Q\varphi_A(r) + \varphi_C(r) < r$ for $r > 0$, where $Q = ||B(\Omega)||$.

As an application of these results, *Jozef Banas and Mohamed-Aziz Taoudi* [13] discussed the solvability of the quadratic integral equation

$$x(t) = u(x(t)) + f(x(t)) \int_0^1 k(t, s)g(s, x(s))ds,\ t \in [0, 1].$$

The integral in the above equation is understood to be the Pettis integral and its solutions will be sought in $E := C([0, 1], X)$, where X is a (real) reflexive Banach algebra and E is endowed with its standard norm $||x|| = sup_{t\in[0,1]}||x(t)||$.
This equation is a general form of many integral equations, such as the Chandrasekhar integral equation arising in radiative transfer [16] and the Hammerstein integral equation [17].

In 2015 *Aref Jeribi , Bilel Krichen , Bilel Mefteh* [32] proved some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators acting on a WC-Banach algebra. their results improve some recent results and extend some several earlier works using the condition $(\mathfrak{P})$. Jeribi established a weak variant of Theorem 1.1 [20] (which is established by B. Dhage [20]) in Banach algebras, satisfying the sequential condition $(\mathfrak{P})$.

It is interesting to notice that their result requires both the weak sequential continuity and the weak compactness of the operators A, B and C. Their proof is based on the Arino, Gautier and Penot's fixed point theorem [2] and also on the weak sequential continuity of $(\frac{1-C}{A})^{-1}\ B$.

In the paper [7], *Ben Amar, Chouayekh and Jeribi* have established some fixed point theorems in Banach algebras satisfying the condition $(\mathfrak{P})$ under the weak topology. Their results were based on the class of weakly condensing, weak sequential continuity and weakly compact.

Let $A : \mathcal{D}(A) \subseteq \chi \to \chi$ be an operator. In what follows, we will use the following conditions:

$$(\mathcal{H}_1)\ \begin{cases} If\ (x_n)_{n\in N} \subseteq\ \mathcal{D}(A) \text{ is a weakly convergent sequence in } \chi \text{ , then} \\ \\ (Ax_n)_{n\in N} \text{ has a strongly convergent subsequence in } \chi \text{ ,} \end{cases}$$

$$(\mathcal{H}_2)\ \begin{cases} If\ (x_n)_{n\in N} \subseteq\ \mathcal{D}(A) \text{ is a weakly convergent sequence in } \chi \text{ , then} \\ \\ (Ax_n)_{n\in N} \text{ has a weakly convergent subsequence in } \chi \text{ .} \end{cases}$$

Aref Jeribi , Bilel Krichen , Bilel Mefteh [32] proved the following theorem:

Theorem
Let S be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra χ and let $A, C : \chi \to X$ and $B : S \to \chi$ be three weakly sequentially continuous operators, satisfying the following conditions:

(i) $(\frac{I-C}{A})^{-1}$ exists on $B(S)$,
(ii) A satisfies $(\mathcal{H}_1)$, and $A(S)$ is relatively weakly compact,
(iii) B is an $\omega - \beta$-contraction,
(iv) C is an $\omega - \alpha-$ contraction, and
(v) $(x = Ax\ By + Cx, y \in S) \Rightarrow\ x \in S$.

Then, the operator equation $x = AxBx + Cx$ has, at least, a solution in S, whenever $\frac{\gamma^{\beta}}{1-\alpha} < 1$, where $\gamma = ||A(S)||$.
by the help of the following:

Theorem [3]
Let S be a nonempty, bounded, closed, and convex subset of a Banach space Y and let $A : S \to S$ be a weakly sequentially continuous mapping. If A is ω-condensing, then it has, at least, a fixed point in S.

Lemma [13]
Let M and M' be two bounded subsets of a WC-Banach algebra X. Then, we have the following inequality $\omega(MM') \leq\ ||M'||\ \omega(M) + ||M||\ \omega(M') + \omega(M)\ \omega(M')$.

4. Solvability in Banach spaces

In 2010, *Smaïl Djebali , Zahira Sahnoun* [23] established new variants of some nonlinear alternatives of Leray-Schauder and Krasnosel'skii type involving the weak topology of Banach spaces. The De Blasi measure of weak noncompactness is used. An application to solving a nonlinear Hammerstein integral equation in L_1 spaces is given.

In 1958, M. A. Krasnosel'skii [33] and [35] proved a fixed point theorem which is an important supplement to both the Schauder fixed point theorem and the Banach contraction principle.

In 2010, *Mohamed Aziz Taoudi* [37] proved the following Krasnosel'skii type fixed point theorem under weak topology features.

Theorem [36] Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that $A : M \to X$ and $B : X \to X$ are two weakly sequentially continuous mappings satisfying:

(i) AM is relatively weakly compact,
(ii) B is a strict contraction,
(iii) $(x = Bx + Ay, y \in M) \Rightarrow x \in M$. Then $A + B$ has at least one fixed point in M.

If we take $B = 0$ in the previous theorem, then we obtain the following version of the Arino Gautier Penot fixed point theorem in Banach spaces [2].

Corollary Let M be a nonempty bounded closed convex subset of a Banach space X. Assume that $A : M \to M$ is a weakly sequentially continuous mapping. If AM is relatively weakly compact, then A has at least one fixed point in M.

Definition Let X be a Banach space. A mapping $B : X \to X$ is said to be nonexpansive (or sometimes called a contraction) if

$$||Bx - By|| \leq ||x - y|| \text{ for all } x, y \in X.$$

Mohamed Aziz Taoudi [37] proved the following fixed point theorem for the sum of a weakly compact and a nonexpansive mapping.

Theorem Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that $A : M \to X$ and $B : X \to X$ are weakly sequentially continuous mappings satisfying:

(i) AM is weakly compact,
(ii) B is nonexpensive,
(iii) if x_n is a sequence of elements of M such that $(I - B)x_n \rightharpoonup y$, then x_n has a weakly convergent subsequence,
(iv) If $\lambda \in (0, 1)$ and $x = \lambda Bx + Ay$ for some $y \in M$, then $x \in M$. Then there is $x \in M$ such that $Ax + Bx = x$.

In 2011, *Afif Ben Amar* [5], examined the set of weakly continuous solutions for a Volterra integral equation in Henstock-Kurzweil-Pettis integrability settings. his result extends those obtained in several kinds of integrability settings. Besides, he proved some new fixed point theorems for function spaces relative to the weak topology which are basic in his considerations and comprise the theory of differential and integral equations in Banach spaces.

Afif Ben Amar [5] proved the following theorem:

Theorem Let E be a Banach space with Q a non-empty subset of $C(I,E)$. Assume also that Q is a closed convex subset of $C_w(I,E), F : Q \to Q$ is continuous with respect to the weak uniform convergence topology, $F(Q)$ is bounded and F is β-condensing (i.e., $\beta(F(X)) < \beta(X)$ for all bounded subsets $X \subset Q$ such that $\beta(X) \neq 0$). In addition, suppose the family $F(Q)$ is weakly equi-continuous. Then the set of fixed points of F is non-empty and compact in $C_w(I,E)$.

This fixed point is motivated by the weak sequential compactness of weakly compact subsets of a Banach space.

Theorem [3]
Let Q be a non-empty, convex closed set in a Banach space E. Assume $F : Q \to Q$ is a weakly sequentially continuous map which is also β- condensing (i.e., $\beta(F(X)) < \beta(X)$ for all bounded subsets $X \subset Q$ such that $\beta(X) \neq 0$). In addition, suppose that $F(Q)$ is bounded. Then F has a fixed point.

Also, *Afif Ben Amar* [5] proved a fixed point theorem for weakly sequentially continuous mappings.

Theorem Let E a Banach space and Q be a non-empty, convex closed set in E. Assume $F : Q \to Q$ is a weakly sequentially continuous map and the family $F(Q)$ is bounded and strongly equi-continuous. In addition, suppose that for each $t \in I, F(Q)(t)$ is relatively weakly compact in E. Then F has a fixed point.
He introduced the concept of Henstock-Kurzweil-Pettis integrability as:

Definition
A function $f : I \to E$ is said to be Henstock-Kurzweil integrable, or simply HK-integrable on I, if there exists $w \in E$ with the following property : for $\epsilon > 0$ there exists a gauge δ on I such that $||\sigma(g,P)w|| < \epsilon$ for each δ-fine Perron partition P of I. We set $w = (HK)\int_0^T f(s)ds$.

Remark This definition includes the generalized Riemann integral (see [25]). In a special case, when δ is a constant function, we get the Riemann integral.

The following result states that the HK-integrability for real functions is preserved under multiplication by functions of bounded variation.

Lemma ([[26], Theorem 12.21]). Let $f : I \to R$ be an HK-integrable function and let $g : I \to R$ be of bounded variation. Then fg is HK-integrable.

The following integration by parts result inspired from the previous lemma and [[26], Theorem 12.8]:

Lemma $f : [a,b] \to R$ be HK-integrable function and let $g : I \to R$ be of bounded variation. Then, for every $t \in [a,b]$

$$(HK)\int_a^t f(s)g(s)ds = g(t)(HK)\int_a^t f(s)ds \int_a^t ((HK)\int_a^s f(\tau)d\tau)dg(s),$$

the last integral being of Riemann-Stieltjes type.
The generalization of the Pettis integral obtained by replacing the Lebesgue integrability of the functions by the Henstock-Kurzweil integrability produces the Henstock-Kurzweil-Pettis integral (for the definition of Pettis integral see [22]).

Definition [18]
A function $f : I \to E$ is said to be Henstock-KurzweilPettis integrable, or simply HKP-integrable, on I if there exists a function $g : I \to E$ with the following properties:

(i) $\forall x^* \in E^*,\ x^* f$ is Henstock-Kurzweil integrable on I;
(ii) $\forall t \in I, \forall x^* \in E^*,\ x^* g(t) = (HK)\int_0^t x^* f(s)ds$. This function g will be called a primitive of f and by $g(T) = \int_0^t f(t)dt$ we will denote the Henstock-Kurzweil-Pettis integral of f on the interval I.

Afif Ben Amar [5] dealt with the existence of weak solution of the Volterra integral equation

$$x(t) = h(t) + \int_0^t K(t,s)f(s,x(s))\ ds \text{ on } I,$$

here "$\int$" denotes the HKP-integral.

Theorem Let $f : I \times E \to E,\ h : I \to E$ and $K : I \times I \to R$ satisfy the following conditions:

(1) h is weakly continuous on I.
(2) For each $t \in I,\ K(t,.)$ continuous, $K(t,.) \in BV(I,R)$ and the mapping $t \mapsto K(t,.)$ is $||.||_{BV}$-continuous. (Here $BV(I,R)$ represents the space of real bounded variation functions with its classical norm $||.||_{BV}$.)
(3) $f : I \times E \to E$ is a weakly-weakly continuous function such that for all $x \in C_w(I,E)$, for all $t \in I$, $f(.,x(.))$ and $K(t,.)f(.,x(.))$are HKP-integrable on I.
(4) For all $r > 0$ and $\epsilon > 0$, there exists $\delta_{\epsilon,r} > 0$ such that

$$||\int_\tau^t f(s,x(s))\ ds|| < \epsilon,\ \forall\ |t-\tau| < \delta_{\epsilon,r},\ \forall\ x \in C_w(I,E),\ ||x|| \le r.$$

(5) There exists a nonnegative function $L(.,.)$ such that:
   (a) for each closed subinterval J of I and bounded subset X of E,

$$\beta(f[J \times X]) \le supL(t,\beta(X)),\ t \in J;$$

   (b) the function $s \mapsto L(s,r)$ is continuous for each $r \in [0,+\infty[$, and

$$sup_{t\in I}\left\{(HK)\int_0^t |K(t,s)|L(s,r)ds\right\} < r \text{ for all} r > 0.$$

Then there exists an interval $J = [0, a]$ such that the set of weakly continuous solutions of the Volterra-type integral equation

$$x(t) = h(t) + \int_0^t K(t, s)f(s, x(s))ds,$$

defined on J is non-empty and compact in the space $C_w(J, E)$.

In 2014, *Gang Cai and Shangquan Bu* [31] study fixed point theorems and new variants of some nonlinear alternatives of Krasnosel'skii type in Banach spaces by using measures of weak noncompactness. Then they give an application to solve a nonlinear Hammerstein integral equation in L_1 spaces.

Definition [30] A mapping $T : C \subseteq X \to X$ is said to be a $w - k$-contraction for some $k \in [0, 1)$, if T is continuous and satisfies $w(T(A)) \leq kw(A)$ for every bounded set $A \subseteq M$ with $w(A) > 0.T$ is said to be w-condensing if T is continuous and $w(T(A)) < w(A)$ for every bounded set $A \subseteq C$ with $w(A) > 0$.

By the help of the following:
Definition [29] A mapping $T : C \subseteq X \to X$ is said to be a ϕ-contraction if there exists a continuous nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ such that $\phi(0) = 0,\ \phi(r) < r$ for any $r > 0$ and $||Tx - Ty|| \leq \phi(||x - y||)$ for all $x, y \in C$.

Definition[30] Let $T : D(T) \subseteq X \to X$ be a mapping. The map T is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for every $x, y \in D(T)$. T is said to be pseudocontractive, if for every $x, y \in D(T)$ and for all $r > 0$, there holds the following inequality

$$||x - y|| \leq ||(1 + r)(x - y) + r(Ty - Tx)||.$$

We recall that a mapping $A : D(A) \to X$ is said to be accretive if the inequality $||x - y + \geq (Ax - Ay)|| \geq ||x - y||$ holds for all $\lambda \geq 0,\ x, y \in D(A)$. It is well known that T is pseudo-contractive if and only if $I - T$ is accretive, where I is the identity operator.

Definition [28] A mapping $T : D(T) \subseteq X \to X$ is said to be ψ-expansive if there exists a function $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(1) $\psi(0) = 0$;
(2) $\psi(r) > 0 for\ r > 0$;
(3) either it is continuous or nondecreasing; such that, for every $x, y \in D(T)$, the inequality $||Tx - Ty|| \geq \psi(||x - y||)$ holds.

such that, for every $x, y \in D(T)$, the inequality $||Tx - Ty|| = \psi(||x - y||)$ holds.

Gang Cai and Shangquan Bu [31] proved the following Krasnosel'skii-type fixed point theorem which improved the result of Garcia-Falset et al. [29].

Theorem Let M be a nonempty closed convex and bounded subset of a Banach space X and let $A : M \to X,\ B : X \to X$ be two continuous mappings. If $A,\ B$ satisfy the following conditions,

(i) A satisfies

$$(\mathcal{A}_1)\ \text{If}\ (x_n)_{n\in\mathcal{N}} \subseteq D(T)\ \text{is a weakly convergent sequence in}\ X,\ \text{then}\ (Tx_n)_{n\in\mathcal{N}}\ \text{has a strongly convergent subsequence in}\ X,$$

(ii) B is pseudocontractive and $I-B$ is ψ-expansive,
(iii) $w(A(S)+B(S)) < w(S)$ for all $S \subseteq M$ such that $w(S) > 0$,
(iv) $[x = Bx + Ay,\ y \in M] \Rightarrow x \in M$,

then the equation $x = A(x) + B(x)$ has a solution.
As an easy consequence of this theorem we may obtain Theorem 3.2 of [29].

Theorem ([29], Theorem 3.2.) Let X be a Banach space. Let M be a nonempty closed convex and bounded subset of X and let $A,\ B : M \to X$ be two continuous mappings. If $A,\ B$ satisfy the following conditions,

(i) $A(M)$ is relatively weakly compact,
(ii) A satisfies $(\mathcal{A}_1)$,
(iii) B is nonexpansive and w-condensing,
(iv) $I-B$ is ψ-expansive,
(v) $A(M)+B(M) \in M$,

then the equation $x = A(x) + B(x)$ has a solution.
The following theorem extends and improves Theorem 3.4 of Garcia-Falset et al. [29].

Theorem [31] Let$X$ be a Banach space and let $A,\ B : X \to X$ be two continuous mappings. If $A,\ B$satisfy the following conditions,

(i) A maps bounded sets into relatively weakly compact ones,
(ii) A satisfies $(\mathcal{A}_1)$,
(iii) B is pseudocontractive and w-condensing,
(iv) $I-B$ is ψ-expansive where ψ is either strictly increasing or $\lim_{r\to infty}\psi(r) = \infty$,

then, either

(a) the equation $x = B(x) + A(x)$ has a solution, or
(b) the set $\{x \in X : x = \lambda B(\frac{x}{\lambda}) + \lambda A(x)\}$ is unbounded for some $\lambda \in (0,1)$.

As an application of their results *Gang Cai and Shangquan Bu* [31] studied the existence of solutions for the following generalized Hammerstein integral equation:

$$y(t) = g(t, y(t)) + \lambda \int_{\Omega} k(t,s) f(s, y(s)) ds,\ t \in \Omega,$$

in $L_1(\Omega, X)$, the space of Lebesgue integrable functions on a measurable domain Ω of R^n with values in a finite dimensional Banach space X.

In 2014, *A. B. Amar, I. Feki and A. Jeribi* [8] established some new variants of Leray-Schauder type fixed point theorems for the sum of two weakly sequentially continuous mappings, A and B defined on a closed convex subset Ω of a Banach space E, where A satisfies some conditions and B is a separate contraction (resp. nonlinear contraction) mapping. Note here that Ω need not to be bounded. Moreover, they gave Leray-Schauder and Furi-Pera fixed point theorems for a larger class of weakly sequentially continuous mappings under weaker assumptions and they explored this kind of generalization by looking for the multivalued mapping

$(I-B)^{-1}A$, when $I-B$ may not be injective. An illustrative application to a source problem in L_1 setting with general boundary conditions is presented.

The existence of fixed points for the sum of two mappings has been focus of interest for several years and their applications are often in nonlinear analysis to solve problems in natural sciences, mathematical physics, mechanics and population dynamics. A useful tool to deal with such problems is the celebrated fixed point theorem due to Krasnosel'skii in 1958 (see for example Theorem 11.B in [38]) which asserts that, if Ω is a nonempty closed and convex subset of a Banach space E; A and B are two mappings from Ω into E such that (i) A is compact, (ii) B is a contraction and (iii) $A\Omega + B\Omega \subset \Omega$, then $A+B$ has at least one fixed point in Ω. Since then, a wide class of problems, for instance in integral equations and stability theory have been contemplated by the Krasnosel'skii fixed point principle. However, in some applications, the verification of the hypothesis (iii) is quite hard. As a tentative approach to grapple with such a difficulty, many attempts have appeared in the literature in the direction of weakening the hypothesis (iii). For example, in [14] Burton improved the Krasnosel'skii principle by requiring, instead of (iii), the more general condition $[x = Bx + Ay, y \in \Omega] \Rightarrow x \in \Omega$. The new variant of Krasnosel'skii's fixed point theorem developed by Burton was applied in stability theory and integral equations, covering cases where the result of Krasnosel'skii can not be applied. Subsequently, the following new asymptotic requirement was introduced by Barroso [12].

$$\text{If } \lambda \in (0,1) \text{ and } x = Bx + Ay \text{ for some } y \in \Omega, \text{ then } x \in \Omega.$$

Theorem [8]
Let Ω be a nonempty closed and convex subset of a Banach space E. In addition, let U be a weakly open subset of Ω with $\theta \in U, A : \overline{U^w} \to E$ weakly sequentially continuous and $B : E \to E$ satisfying:

(i) $A(\overline{U^w})$ is relatively weakly compact.
(ii) B is linear, bounded and there exists $p \in N^*$ such that B^p is a separate contraction.
(iii) $[x = Bx + Ay, y \in \overline{U^w}] \Rightarrow x \in \Omega$. Then, either $A+B$ has a fixed point or there is a point $x \in \partial_\Omega U$ (the weak boundary of U in Ω) and a $\lambda \in (0,1)$ with $x = Bx + \lambda\ Ax$.

The above theorem remains true if we suppose that there exists $p \in N^*$ such that B^p is a nonlinear contraction.

Theorem [8]
Let Ω be a nonempty closed and convex subset of a Banach space E. In addition, let U be a weakly open subset of Ω with $\theta \in U, A : \overline{U^w} \to E$ weakly sequentially continuous and $B : E \to E$ satisfying:

(i) $A(\overline{U^w})$ is relatively weakly compact.
(ii) B is linear, bounded and there exists $p \in N^*$ such that B^p is a nonlinear contraction.
(iii) $[x = Bx + Ay, y \in \overline{U^w}] \Rightarrow x \in \Omega$. Then, either $A+B$ has a fixed point or there is a point $u \in \partial_\Omega U$ (the weak boundary of U in Ω) and a $\lambda \in (0,1)$ with $u = Bu + \lambda Au$.

Theorem [8]
Let Ω be a nonempty closed and convex subset of a Banach space E. In addition, let U be a weakly open subset of Ω with $\theta \in U, A : \overline{U^w} \to E$ weakly sequentially continuous and $B : E \to E$ satisfying:

(i) A is a weakly compact.
(ii) B is a nonlinear contraction.
(iii) $[x = Bx + Ay, y \in \overline{U^w}] \Rightarrow x \in \Omega$.
(iv) $(I - B)^{-1}A(\overline{U^w})$ is a bounded subset of E. Then, either $A + B$ has a fixed point or there is a point $x \in \partial_\Omega U$ (the weak boundary of U in Ω) and a $\lambda \in (0, 1)$ with $x = \lambda\ B(\frac{x}{\lambda}) + \lambda Ax$.

As application *A. B. Amar et al.*. [8] considered the following problem

$$v_3 \frac{\partial \Psi}{\partial x}(x, v) + \mathfrak{B}(x, v, \Psi(x, v)) - \lambda\ \Psi(x, v) = \int_K r(x, v, v', \Psi(x, v'))\ dv' \quad \text{in } D$$

with a general boundary condition, where $D = (0, 1) \times K$ with K is the unit sphere of $R^3, x \in (0, 1), v = (v_1, v_2, v_3) \in K$. $r(., ., .)$ is a nonlinear function of $\Psi, \mathfrak{B}(., ., ., .)$ is a function on $[0, 1] \times K \times C$ and λ is a complex number. This equation describes the transport solution $\Psi(x, v)$ in the vertical direction and characterizes the possible leakage of energy at the boundary of the slab.

In 2015, *A. B. Amar*[10] investigated the set of solutions for nonlinear Volterra type integral equations in Banach spaces in the weak sense and under Henstock-Kurzweil-Pettis integrability. Moreover, a fixed point result was presented for weakly sequentially continuous mappings defined on the function space $C(K, X)$, where K is compact Hausdorff and X is a Banach space. The main condition is expressed in terms of axiomatic measure of weak noncompactness.
A. B. Amar focused on the existence of solutions in the weak sense for the nonlinear Volterra type integral equation in Banach spaces

$$x(t) = h(t) + \int_0^t G(t, s)\ f(s, x(s),\ \int_0^s\ k(s, \tau)\ x(\tau)\ d\tau)\ ds,$$

involving the Henstock-Kurzweil-Pettis integral. The main tools used in our study are associated with the techniques of measure of weak noncompactness, properties and convergence theorems mainly of Vitali type for Henstock-Kurzweil-Pettis integrals based on the notion of equi-integrability.

In 2015, *K. Ezzinbi and M. Taoudi*[24] introduced the concept of a convex-power condensing mapping T with respect to another mapping S as a generalization of condensing and convex-power condensing mappings. Some fixed point theorems for the sum $T + S$ with S is a strict contraction and T is convex-power condensing with respect to S are established. The cases where S is nonexpansive or expansive are also considered. Our fixed point results encompass the well known Sadovskii's fixed point theorem and a number of its generalizations. To show the usefulness and the applicability of their fixed point results they investigate the existence of mild solutions to a broad class of neutral differential equations.

Theorem [24]
Let X be a Banach space and ψ a measure of noncompactness on X. Let M be a

nonempty bounded closed convex subset of X. Suppose that $T,\ S : M \to X$ are two continuous mappings satisfying:

(i) there are an integer n and a vector $x_0 \in M$ such that T is S-convex-power condensing about x_0 and n_0 w.r.t. ψ,
(ii) S is a strict contraction,
(iii) $Tx + Sy \in M$ for all $x, y \in M$.

Then $T + S$ has at least one fixed point in M.

If we take S = 0 in Theorem 3.2 we obtain the following sharpening of the well-known Sadovskii fixed point theorem, which was proved in [34].

Corollary [24]
Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that $T : M \to M$ is continuous and there exist an integer n and a vector $x_0 \in M$ such that T is convex-power condensing about x_0 and n. Then T has at least one fixed point in M.
Definition[24]
A mapping $T : D(T) \subseteq X \to X$ is called nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in D(T)$.

Definition [24]
We say that $T : D(T) \subseteq X \to X$ is demiclosed if for any sequence x_n weakly convergent to an element $x^* \in D(T)$ with $\{Tx_n\}$ norm-convergent to an element y, then $Tx^* = y$.

Theorem [[35], Theorem 5.1.2.] Let$M$ be a bounded closed convex subset of a Banach space X and T a nonexpansive mapping of M into M. Then for each $\epsilon > 0$, there exists $x_\epsilon \in M$ such that $||Tx_\epsilon - x_\epsilon|| < \epsilon$.

Theorem [24]
Let X be a Banach space and ψ a measure of noncompactness on X. Let M be a nonempty bounded closed convex subset of X and $T,\ S : M \to X$ two continuous mapping satisfying:

(i) there are an integer n and a vector $x_0 \in M$ such that T is S-convex-power condensing about x_0 and n_0 w.r.t. ψ,
(ii) S is a nonexpansive mapping,
(iii) if (x_n) is a sequence of M such that $((I - S)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence,
(iv) $I - S$ is injective and demiclosed,
(v) $Tx + Sy \in M$ for all $x, y \in M$.

Then $T + S$ has at least one fixed point in M.
An easy consequence of the above theorem is the following.

Corollary[24]
Let M be a nonempty bounded closed convex subset of a reflexive Banach space X. Suppose that $T,\ S : M \to X$ are two continuous mappings satisfying:

(i) there are an integer n and a vector $x_0 \in M$ such that T is S-convex-power condensing about x_0 and n_0 w.r.t. ψ,
(ii) S is nonexpansive,
(iii) $I - S$ is injective and demi-closed,
(iv) $Tx + Sy \in M,$ for all $x, y \in M$.

Then $S + T$ has at least one fixed point in M.
Another easy consequence of that theorem is the following sharpening of [[1], Theorem 2.13].

Corollary [24]
Let X be a Banach space and ψ a measure of noncompactness on X. Let M be a nonempty bounded closed convex subset of X and $T,\ S : M \to X$ two continuous mapping satisfying:

(i) there are an integer n_0 and a vector $x_0 \in M$ such that $\mathcal{F}^{(n_0,x_0)}(T, S, M)$ is relatively compact,
(ii) S is a nonexpansive mapping,
(iii) if (x_n) is a sequence of M such that $((I - S)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence,
(iv) $I - S$ is injective and demiclosed,
(v) $Tx + Sy \in M$ for all $x,\ y \in M$.

Then $T + S$ has at least one fixed point in M.
Also, some results for expansive mapping are proved [24].

As an application of the fixed point results obtained in this paper [24], the existence of mild solutions to the following neutral differential equation

$$\frac{d}{dx}(x(t) - g(x(t))) = A\ (x(t) - g(x(t))) + f(t, x(t)),\ t \in I,\ x(0) = x_0 \qquad (2)$$

is discussed. Where $I = [0, a],\ a > 0,\ A$ is the generator of a strongly continuous semigroup $(U(t))_{t\geq 0}$ of linear operators, not necessarily compact, defined on a Banach space $E,\ f : [0, a] \times E \to E$ and $g : E \to E$ are suitably defined functions satisfying certain conditions:

(a) The semigroup $(U(t))_{t\geq 0}$ is equicontinuous.
(b) The function g maps E into itself and there is a constant $k \in [0, 1)$ such that for all $u,\ v \in E$ we have $||g(u) - g(v)|| \leq k||u - v||$.
(c) (i) The map $f : [0, a] \times E \to E$ is such that for almost every $t \in [0, a]$, the function $f(t, .) : E \to E$ is continuous and for all $x \in E$, the function $f(., x) : [0; a] \to E$ is measurable.
(ii) There exist a function $m \in L_1((0, a); R+)$ and a nondecreasing continuous function $\Omega : R_+ \to (0; +\infty)$ such that

$$||f(t, x)|| \leq m(t)\ \Omega(||x||),$$

for all $x \in E$ and all $t \in [0, a]$.
(iii) There exists a positive constant c such that for every bounded set $D \subseteq E$ we have $\alpha(f(t, D)) \leq c.\alpha(D)$ for almost every $t \in [0, a]$.

The definition of the mild solution of the problem (2) may defined as [24].

Definition

A continuous function $x : [0, a] \to E$ is said to be a mild solution to the neutral differential Eq. (2) if:

(i) $x(0) = x_0$,
(ii) $x(t) = U(t)(x_0 - g(x_0)) + g(x(t)) + \int_0^t U(t-s) f(s, x(s)) ds$ for $t \in [0, a]$.

If we take $g = 0$ we recapture the following result which was proved in Sun and Zhang [34].

Corollary
Assume that the conditions (a) and (c) are satisfied. Then the following equation

$$\frac{d}{dx} x(t) = Ax(t) + f(t, x(t)), t \in I, \ x(0) = x_0$$

has at least one mild solution on $[0, a]$.

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