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ON SOME SEMICONSERVATIVE FK SPACES

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ABSTRACT. Snyder and Wilansky give the definition of semiconservative FK space and investigate the properties of this space in [5]. In their article, an FK space X containing ϕ is called semiconservative space if $X^f \subset cs$ holds. In this paper, we study a new type of semiconservative space. Also, we give some characterizations of these spaces.

1. INTRODUCTION

Let ω denote the space of all real or complex valued sequences. An FK space is a locally convex vector subspaces of ω which is also a Fréchet space (complete linear metric) with continuous coordinates. A BK space is normed a normed FK space, [1]. The basic properties of such spaces can be found in [6], [7], [8] and [9].

By m, c_0 we denote the space of all bounded sequences, null sequences, respectively. These are FK space under $\|x\| = \sup_n |x_n|$. By ℓ we shall denote the space of all absolutely summable sequences. The sequences space

$$\begin{aligned} cs &= \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} x_n \text{ convergent} \right\}, \\ bs &= \left\{ x = (x_n) \in \omega : \sup_k \left| \sum_{n=1}^{k} x_n \right| < \infty \right\}, \\ \chi &= \left\{ x = (x_n) \in \omega : \forall n \in \mathbb{N}^0, x_n \in \{0,1\} \right\}, \\ \rho &= \left\{ \alpha \in \omega : \sum_n a_{nk} \text{ convergent and } \sup_m \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{nk-1}) \right| < \infty \right\}, \end{aligned}$$

in which $A = (a_{nk}) = R.diag(\alpha_1, \alpha_2, \dots).R^{-1} = R.diag(\alpha).R^{-1}$ and R is Riesz matrix.

In this paper, let $X = (x_{nk})$ be a matrix;

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$$(x_{nk}) = \begin{cases} \frac{k.(q_k - q_{k+1})}{Q_n} & , \ k < n \\ \frac{k.q_k}{Q_k} & , \ k = n \\ 0 & , \ k \geq n \end{cases}$$

and we suppose that $\sum_n x_{nk}$ convergent and

$$\sup_m \sum_k \left| \sum_{n=0}^{m} (x_{nk} - x_{nk-1}) \right| < \infty.$$

$$\begin{aligned} (cs)_R = rs &= \left\{ x \in \omega : \lim_n \left| \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j \right| \text{ exists} \right\}, \\ (bs)_R = rb &= \left\{ x \in \omega : \sup_n \left| \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j \right| < \infty \right\}, \\ (c_0)_R = r_0 &= \left\{ x \in \omega : \lim_n \left| \frac{1}{Q_n} \sum_{j=1}^{n} q_j x_j \right| = 0 \right\} \end{aligned}$$

are FK spaces with the norms

$$\|x\|_{rb} = \sup_n \left| \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j \right|,$$

$$\|x\|_{r_o} = \sup_n \frac{1}{Q_n} \left| \sum_{j=1}^{n} q_j x_j \right|$$

respectively.

Let $\phi = span\{\delta^k : k \in N\}$ and $\phi_1 = \phi \cup \{\delta\}$. The topological dual of X is denoted by $X^{'}$. Let (X, τ) be a K space with $\phi \subset X$ and dual space $X^{'}$, and let $x = (x_k) \in X$ be arbitrarily given. Define the n^{th} section of x to be sequence $x^{[n]} = \sum_{k=1}^{n} x_k \delta^k = (x_1, x_2, \ldots, x_n, 0, \ldots)$, where δ^k denotes the sequence having 1 in the j-th position and 0's elsewhere. Also, $r^{[n]}x = \frac{1}{Q_n} \sum_{k=1}^{n} q_k x_k \delta^k$ is called the n^{th} Riesz section of x . This here r is the set $\{r^{[n]} : n \in \mathbb{N}\}$. We define the following properties:

x has $AK(sectional\ convergence)$ if $x^{[n]} \to x$ in (X, τ),
x has $SAK(weak\ sectional\ convergence)$ if $x^{[n]} \to x$ in $(X, \sigma(X, X^{'}))$,
x has $FAK(functional\ sectional\ convergence)$ if $\sum_k x_k f(e^k)$ converges for all $f \in X^{'}$,
x has $AB(sectional\ boundedness)$ if $\{x^{[n]} : n \in \mathbb{N}\}$ is bounded in (X, τ).

In addition an FK space is said to have rK space (or rK) if $X \supset \phi$ and for each $x \in X$,

$$\frac{1}{Q_n} \sum_{k=1}^{n} q_k x^{[k]} \to x \ , n \to \infty.$$

Every AK space is a rK space. For example ω, c_0 are AK spaces and so rK space [2], [3].

Then

$$X^f = \left\{ \{f(\delta^k)\} : f \in X^{'} \right\}.$$

In addition

$$\begin{aligned} X^Y &= \{x : yx = (y_k x_k) \in Y \text{ for every } y \in X\} = (X \to Y), \\ X^\beta &= \{x : yx = (y_k x_k) \in cs \text{ for every } y \in X\} \\ &= \left\{ x : \sum_{k=1}^{\infty} x_k y_k \text{ exists for every } y \in X \right\}, \\ X^r &= \{x : yx = (y_k x_k) \in rs \text{ for every } y \in X\} \\ &= \left\{ x : \lim_n \frac{1}{Q_n} \left| \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j y_j \right| \text{ exists for every } y \in X \right\}, \\ X^{rb} &= \left\{ x : \sup_n \frac{1}{Q_n} \left| \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j y_j \right| < \infty \text{ for every } y \in X \right\}. \end{aligned}$$

For example, it is claim $(rs)^r = \rho$:

$$\begin{aligned} (rs)^r &= ((cs)_R)^r \\ &= \{\alpha \in \omega : (\alpha x) \in (cs)_R, \forall x \in (cs)_R\} \\ &= \{\alpha \in \omega : A = R.diag(\alpha).R^{-1} \in (cs, cs)\} \\ &= \{\alpha \in \omega : A \in (cs, cs)\} \\ &= \{\alpha \in \omega : \sum_n a_{nk} \text{ convergent and } \sup_m \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{nk-1}) \right| < \infty\} \\ &= \rho \end{aligned}$$

By taking advantage of [1] , we can easily see the following lemma:

Lemma 1.1 Let X, X_1 be sets of sequences. Then for $k = f, \beta, r, rb$
(1) $X \subset X^{kk}$,
(2) $X^{kkk} = X^k$,
(3) if $X \subset X_1$ *then* $X_1^k \subset X^k$ holds.

Lemma 1.2 Let X be an FK space containing ϕ and $\lim_{n\to\infty} \frac{n}{Q_n} = 1$. Then
(1)$X^\beta \subset X^r \subset X^{rb} \subset X^f$,
(2)If X is rK space then $X^f = X^r$,
(3)If X is an AD space then $X^r = X^{rb}$.

Proof. The proof is quite clear from [10].

Let $A = (a_{ij})$ be an infinite matrix. The matrix A may be considered as a linear transformation of sequence by the formula $y = Ax$, where $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$. For an FK space (λ, u) we consider the summability domain $\lambda_A = \{x \in \omega \ : Ax \in \lambda\}$. Then λ_A is an FK space under the semi-norms $p_i = |x_i|$, $(1, 2, \dots)$. A conservative matrix A, and the corresponding matrix method, is called coregular if $\chi(A) \neq 0$ and conull if $\chi(A) = 0$, where $\chi(A) = \lim_A \delta - \sum_k \lim_A \delta^k$, [1].

Recall that, given a matrix A with $\ell_A \supset \phi$ is called ℓ-replaceable if there is a matrix $B = (b_{nk})$ with $\ell_B = \ell_A$ and $\sum_{k=1}^{\infty} b_{nk} = 1$ for all $k \in \mathbb{N}$, [4].

In addition an FK space X is called semiconservative if $X^f \subset cs$, this means that $X \supset \phi$ and $\sum_{j=1}^{\infty} f(\delta^j)$ is convergent for each $f \in X^{'}$, [5].

2. Riesz semiconservative FK Space

Firstly, we have defined the notations of Riesz semiconservative FK space and Riesz conull in this section. Then, we investigate the properties of these spaces and we also give the relationship between ℓ- reblaceable and Riesz semiconservative FK space. Furthermore, in this paper it is accepted $Q_n \to \infty, (n \to \infty)$.

Definition 2.1 An FK space X is called Riesz semiconservative or shortly (Rsc) if $X^f \subset rs$. It is obvious that $X^f \subset rs$ if and only if δ^k is weakly Riesz Cauchy i. e. $\left\{\frac{1}{Q_n}\sum_{k=1}^{n} q_k f(\delta^k)\right\}$ is convergent for each $f \in X'$ equivalently

$$\lim_n \left\{ \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_k f(\delta^k) \right\}$$

exists.

Definition 2.2 An FK space containing ϕ_1 is called Riesz conull if

$$f(\delta) = \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j f(\delta^j),$$

for all, $f \in X^{'}$.
For example c_0 is Riesz semiconservative FK space. Every semiconservative FK space is an Riesz semiconservative FK space.

Theorem 2.1 If a matrix A is ℓ-replaceable then ℓ_A is not Riesz semiconservative FK space.

Proof. If A is ℓ-replaceable then there is $f \in \ell_A^{'}$ such that $f(\delta^j) = 1$ for all $j \in \mathbb{N}$,[4]. Hence

$$\lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j f(\delta^j)$$

does not exist, so ℓ_A is not Riesz semiconservative space.

Theorem 2.2 If X_A is Riesz conull FK space then it is Riesz semiconservative space.

Proof. Suppose that X_A is Riesz conull. Then

$$f(\delta) = \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j f(\delta^j),$$

for all $f \in X_A^{'}$. Hence $X_A^f \subset rs$.

Theorem 2.3 (1) A closed subspace, containing ϕ, of a Riesz semiconservative space is a Riesz semiconservative space.
(2) An FK space that contains a Riesz semiconservative space must be a Riesz semiconservative space.
(3) A countable intersection of Riesz semiconservative space is a Riesz semiconservative space.

Proof. (1) Let $\phi \subset X$. If X is closed in Y then $X^f = Y^f$ [6, Theorem 7.2.6] Y is Rsc space $\Leftrightarrow Y^f \subset rs$. By $Y^f = X^f \subset rs$, $X^f \subset rs \Leftrightarrow X$ Rsc. Hence (1) is true.
(2) Let X Rsc space and $X \subset Y$. $X^f \subset rs$ since X Rsc space. Since $X \subset Y \Rightarrow$

$Y^f \subset X^f$, then $Y^f \subset X^f \subset rs$. So, $Y^f \subset rs \Rightarrow Y$ Rsc space.
(3) First the intersection $X = \cap X_n$ is an FK space by [6, Theorem 4.2.15]. Every $f \in X^{'}$ can be written $f = \sum_{k=1}^{m} g_k$ where each $g_k \in X_n^{'}$ for some n by [6, 4.0.3,4.0.8].

Theorem 2.4 If z^r is a Riesz semiconservative space then $z \in rs$.

Proof. Let z^r be a Riesz semiconservative space. Then $z^{rf} \subset rs$. Since z^r is a rK space, we have $z^{rf} = z^{rr}$. So since $\{z\} \subset z^{rr} \subset rs$, we get $z \in rs$.

Definition 2.3 An FK space is called bounded convex Riesz semiconservative if it is Riesz semiconservative space and includes δ.

The definition of Riesz conull FK space X which $X \supset \phi$, can be given as follows by using Riesz semiconservative; A Riesz semiconservative space X is called Riesz conull, if

$$f(\delta) = \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j f(\delta^j),$$

for all $f \in X^{'}$. A Riesz semiconservative space need not contain δ but must contain δ , if it is Riesz conull. A Riesz conull space is automatically bounded convex Riesz semiconservative space.

3. Distinguished Subspaces of Riesz Semiconservative FK Spaces

In this section we give the relation between the distinguished subspaces which are rF^+, rF, rB^+, rB, Riesz semiconservative and bounded convex Riesz semiconservative FK spaces. Additionally, we proved some theorems on these spaces. Let X be an FK space containing ϕ. Then we define

$$\begin{aligned} rF^+(X) &= \left\{ x \in \omega : \left\{ \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} q_k x_k \right\} \textit{ is weakly Cauchy in } X \right\} \\ &= \left\{ x \in \omega : \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j f(\delta^j) \textit{ exists for all } f \in X^{'} \right\} \\ &= \left\{ x \in \omega : \{x_n f(\delta^n)\} \in rs \textit{ for all } f \in X^{'} \right\} = (X^f)^r. \end{aligned}$$

$$\begin{aligned} rB^+(X) &= \left\{ x \in \omega : \left\{ \frac{1}{Q_n} \sum_{k=1}^{n} q_k x^{(k)} \right\} \textit{ is bounded in } X \right\} \\ &= \left\{ x \in \omega : \{x_n f(\delta^n)\} \in rb \textit{ for all } f \in X^{'} \right\} \\ &= (X^f)^{rb} \end{aligned}$$

Also $rF = rF^+ \cap X$ and $rB = rB^+ \cap X$.

Theorem 3.1 Let X be an FK space containing ϕ and $z \in \omega$. Then $z \in rF^+$ if and only if $Y = z^{-1} \cdot X = \{x : zx \in X\}$ is Riesz semiconservative FK space, where, $zx = \{z_n x_n\}$ in particular $\delta \in rF^+$ if and only if X is Riesz semiconservative FK space.

Proof. Let $(z^{-1} \cdot X)$ be Rsc, so $(z^{-1} \cdot X)^f \subset rs$. Hence $f \in (z^{-1} \cdot X)^{'}$. Then $f(x) = \alpha x + g(z \cdot x)$, $\alpha \in \phi, g \in Y^{'}$, by [6, Theorem 4.4.10] and $f(\delta^n) = \alpha_n + g(z \cdot$

$\delta^n) = \alpha_n + g(z_n\delta^n) = \alpha_n + z_n g(\delta^n)$. Hence, since $\alpha \in \phi \subset rs$ then $\{f(\delta^n)\} \in rs$ if and only if $\{z_n g(\delta^n)\} \in rs$, i. e. $z \in rF^+$.

Theorem 3.2 Let X be an FK space containing ϕ and $z \in \omega$. Then $z \in rF$ if and only if $z^{-1} \cdot X$ is bounded convex Riesz semiconservative FK space in particular $\delta \in rF$ if and only if X convex Riesz semiconservative FK space.

Proof. Let $z \in rF$. Since $rF = rF^+ \cap X$ then $z \in X$ so $\delta \in z^{-1} \cdot X$ and since $z \in rF^+$, $z^{-1} \cdot X$ is Riesz semiconservative FK space by Theorem 3.1. Thus $z^{-1} \cdot X \in X$ is bounded convex Riesz Semiconservative FK space. Conversely, let $z^{-1} \cdot X$ be a bounded convex Riesz semiconservative FK space. So, since $z \in rF^+$ by Theorem 3.1 and $z \in X$ by Definition 2.3, then $z \in rF$.

Theorem 3.3 Let Y be a Riesz semiconservative FK space and Z an AD space. Suppose that for an FK space X, $X \supset Y.Z$. Then $X \supset Z$,where $Y \cdot Z = \{y.z : y \in Y, z \in Z\}$.

Proof. Let $z \in Z$. Then, since $X \supset Y \cdot Z$, $z^{-1} \cdot X \supset Y$. Thus, since Y is Riesz semiconservative space then $z^{-1} \cdot X$ is Riesz semiconservative space by Theorem 2.3 and so $z \in rF^+$ by Theorem 3.1. Hence $Z \subset rF^+ = X^{fr}$. Thus $X^f \subset X^{frr} \subset Z^r \subset Z^f$ and so $Z \subset X$ by [6, Theorem 8.6.1].

Authors' contributions

This work was carried out in collaboration between all authors. All authors declare that there is no conflict of interest regarding the publication of this paper.

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