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WELL POSEDNESS AND ASYMPTOTIC BEHAVIOR FOR COUPLED QUASILINEAR PARABOLIC SYSTEM WITH SOURCE TERM

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Abstract. In this paper we are interested in the study of a coupled quasilinear parabolic system of the form:

$$\begin{cases} u_t - \Delta u = f_1(u, v), \\ \\ v_t - \Delta v = f_2(u, v) \end{cases}$$

in a bounded domain, we prove global existence of the solutions by combining the energy method with the Faedo-Galerkin's procedure. Furthermore we study the asymptotic stability in using Nakao's technique, we show also blow up of the solution in finite time when the initial energy is negative.

1. Introduction

We omit the space variable x of $u(x,t),\ v(x,t),\ u_t(x,t),\ v_t(x,t)$ and for simplicity reason denote $u(x,t) = u,\ v(x,t) = v$ and $u_t(x,t) = u,\ v_t(x,t) = v$, when no confusion arises also the functions considered are all real valued, here $u_t = du(t)/dt$, $v_t = dv(t)/dt$
Our main interest lies in the following system

$$\begin{cases} u_t - \Delta u = f_1(u, v), & on \quad \Omega \times (0, \infty) \\ v_t - \Delta v = f_2(u, v), & on \quad \Omega \times (0, \infty) \\ u = v = 0 & on \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = u^0(x),\ v(x, 0) = v^0(x) & in \quad \Omega. \end{cases} \tag{1}$$

Where Ω is a bounded domain of R^n, $n \geq 1$ with a smooth boundary $\partial\Omega$. $f_i(.,.) : R^2 \to R$ $i = 1, 2$, are given functions which will be specified later.
To motivate our work, let us recall some results regarding heat equations. The single heat equation of the form

$$\frac{\partial u}{\partial t} - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0, \tag{2}$$

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where $u(x, y, z, t)$ is not velocity. It is an arbitrary function being considered; often it is temperature. The heat equation is More generally in any coordinate system:

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0, \tag{3}$$

where α is a positive constant, ∇^2 denotes the Laplace operator. In the physical problem of temperature variation, $u(x, y, z, t)$ is the temperature and α is the thermal diffusivity. For the mathematical treatment it is sufficient to consider the case $\alpha = 1$.
The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In probability theory, the heat equation is connected with the study of Brownian motion via the Fokker-Planck equation. In financial mathematics it is used to solve the Black-Scholes partial differential equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes, suppose one has a function u that describes the temperature at a given location (x, y, z). This function will change over time as heat spreads throughout space. The heat equation is used to determine the change in the function u over time.

The heat equation is used in probability and describes random walks. It is also applied in financial mathematics for this reason, the control of PDEs has become an active area of research, see for instance [7, 8, 9, 10, 11] and the references therein. For example, Nakao [7] studied the system

$$\begin{cases} u_t - \Delta\beta(u) + div(G(u)) + h(u) = 0 \ in \ \Omega \times (0, +\infty), \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad on \quad \Gamma \times (0, +\infty), \end{cases} \tag{4}$$

where all the functions β, G and h could be nonlinear. Choosing in that paper, $\beta(u) = |u|^m$, the author showed that global solutions exist for sufficiently small intial data and gave a decay result, from what he derived an exponential decay for $m = 0$ and a polynomial decay for $m > 0$.

Pucci and Serrin [12] studied a parabolic equation with a nonlinearity in the term containing u_t, precisely they discussed the following system:

$$\begin{cases} A(t) \mid u_t \mid^{m-2} u_t = \Delta u - f(x, u) \quad in \ \Omega \times (0, +\infty), \\ u = 0 \qquad\qquad\qquad\qquad\qquad on \quad \Gamma \times (0, +\infty), \end{cases} \tag{5}$$

Where $m \geq 2$ and Ω is a bounded open subset of $R^n (n \geq 1)$. The values of u are taken in R^n, f(x,u) is a source term, generally it is nonlinear term and $A \in C(R^+)$ is a bounded square matrix satisfying

$$(\upsilon, \upsilon) \geq c_0 \mid \upsilon \mid^2, \quad \forall t \in R^+, \upsilon \in R^n.$$

They proved for $m > 1$ that strong solutions tend to the rest state as $t \to \infty$, however no rate of decay has been given, Berrimi and Messaoudi [10] showed that if A satisfies $((A(t)\upsilon, \upsilon) \geq c_0 \mid \upsilon \mid^2 \ \forall t \in R^+, \upsilon \in R^n)$, then the solutions with small initial energy decay exponentially for $m = 2$ and polynomially if $m > 2$. Research of global existence and nonexistence and finite time blow-up of solutions are discussed see the works of Levine. [3] . Levine et al. [11] Messaoudi. [21]. Results concerning global existence, asymptotic behavior have been proved by Nakao. [7],Nakao and

Ohara.[8].
Recently, Maatouk [13] investigated the following system

$$\begin{cases} |u_t|^\rho u_t - \Delta u + \beta(u)|u|^p u = 0 & in\ \Omega \times (0, +\infty) \\ u = 0 & on \quad \Gamma \times (0, +\infty) \\ u(x, 0) = u_0(x) & in\ \Omega \end{cases} \tag{6}$$

where Ω is a bounded domain of R^n, $n \geq 1$, with a smooth boundary Γ and ρ; p are real numbers such that $0 < \rho; p \leq \frac{2}{n-2}$ for $n \geq 3$, and $0 < \rho; p < +\infty$ for $n \in \{1, 2\}$. He obtained global existence when $\rho = 0$ and energy decay result.
This paper is addressed to coupled phenomena of a heat equation who is interested in mathematical aspect(global existence, asymptotic behavior, blow-up of solution). For more research in a coupled wave equation with viecoelastic term we refer the readers to ([22, 23, 24, 25]).
This paper is organized as follows. In Section 2, we present the preliminaries and some lemmas. In Section 3, the global existence is discussed by using standard Faedo-Galerkin's method. In section 4 the decay property are studied. Finally, the blow-up result of (1) is obtained in the case of the initial energy being negative.

2. Preliminary Results

we will use embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q \leq \frac{2n}{n-2}$, if $n \geq 3$ and $q \geq 2$, if $n = 1, 2$; and $L^r(\Omega) \hookrightarrow L^q(\Omega)$, for $q < r$. We will use , in this case, the same embedding constant denoted by c_s

$$\|\nu\|_q \leq c_s \|\nabla \nu\|_2, \quad \|\nu\|_q \leq c_s \|\nu\|_r \quad \text{for} \quad \nu \in H_0^1(\Omega).$$

In this section, we present some material in the proof of our main result. We consider the Hilbert space $L^2(\Omega)$ endowed with the scalar product

$$(\varphi, \phi) = \int_\Omega \varphi(x), \phi(x) dx,$$

and the corresponding norm

$$\| \varphi \| = (\varphi, \varphi)^{\frac{1}{2}}.$$

Generally , the norm of the space $L^p(\Omega,)$ is noted

$$\| \varphi \|_p = \left(\int_\Omega | \varphi(x) |^p \, dx \right)^{\frac{1}{p}},$$

for all $1 \leq p < +\infty$. We consider the space $H_0^1(\Omega)$, which is the closure of C_c^∞in the Sobolev space $H^1(\Omega)$ with respect to its strong topology induced by the scalar product

$$(\varphi, \phi)_{H^1(\Omega)} = (\varphi, \phi) + (\nabla \varphi, \nabla \phi).$$

The space $H_0^1(\Omega)$ endowed with the norm induced by the scalar product

$$(\varphi, \phi)_{H_0^1(\Omega)} = (\nabla \varphi, \nabla \phi),$$

owing to the Poincare's inequality, recalled below, a Hilbert space. We establish the following assumptions

A_1: we take f_1,f_2 as in [6]

$$f_1(u,\upsilon) = a \mid u+\upsilon \mid^{p-1} (u+\upsilon) + b \mid u \mid^{\frac{p-3}{2}} \mid \upsilon \mid^{\frac{p+1}{2}} u, \tag{7}$$

$$f_2(u,\upsilon) = a \mid u+\upsilon \mid^{p-1} (u+\upsilon) + b \mid \upsilon \mid^{\frac{p-3}{2}} \mid u \mid^{\frac{p+1}{2}} \upsilon. \tag{8}$$

With

$$a,b>0,\ p\geq 3 \quad if\ N=1,2 \quad or \quad p=3 \quad if\ N=3. \tag{9}$$

Further, one can easily verify that

$$uf_1(u,\upsilon)+\upsilon f_2(u,\upsilon) = (p+1)F(u,\upsilon), \forall (u,\upsilon)\in R^2.$$

Where

$$F(u,\upsilon) = \frac{1}{(p+1)}(a \mid u+\upsilon \mid^{p+1} + 2b \mid u\upsilon \mid^{\frac{p+1}{2}}). \quad f_1(u,\upsilon) = \frac{\partial F}{\partial u}, \quad f_2(u,\upsilon) = \frac{\partial F}{\partial \upsilon}.$$

And there exists C, such that

$$\left|\frac{\partial f_i}{\partial u}(u,\upsilon)\right| + \left|\frac{\partial f_i}{\partial \upsilon}(u,\upsilon)\right| \leq C(|u|^{p-1}+|\upsilon|^{p-1}), \quad i=1,2 \quad where \quad 1\leq p<6.$$

A_2: There exists $c_0, c_1>0$, such that

$$c_0(\mid u\mid^{p+1} + \mid \upsilon \mid^{p+1}) \leq F(u,\upsilon) \leq c_1(\mid u\mid^{p+1} + \mid \upsilon \mid^{p+1}), \forall (u,\upsilon)\in R^2.$$

We define the energy related with problem (1) by

$$E(t) = \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|\nabla \upsilon(t)\|_2^2 - \int_\Omega F(u,\upsilon)dx, \tag{10}$$

we define also,

$$I(t) = \|\nabla u(t)\|_2^2 + \|\nabla \upsilon(t)\|_2^2 - (p+1)\int_\Omega F(u,\upsilon)dx. \tag{11}$$

Since $2\leq p\leq \frac{2N}{N-2}$, if $N\geq 3$ or $1\leq p<\infty$, if $N=\{1,2\}$. According to Sobolev's embedding, we have

$$H_0^1(\Omega) \hookrightarrow L^{2(p+1)}(\Omega) \hookrightarrow L^{p+1}(\Omega). \tag{12}$$

lemma 1. ([6]). Suppose that (9) holds.Then there exists $\eta>0$ such that for any $(u,\upsilon)\in H_0^1(\Omega)\times H_0^1(\Omega)$. We have

$$\|u+\upsilon\|_{p+1}^{p+1} + 2\|u\upsilon\|_{\frac{p+1}{2}}^{\frac{p+1}{2}} \leq \eta(\|\nabla u\|_2^2 + \|\nabla \upsilon\|_2^2)^{\frac{p+1}{2}}$$

lemma 2([19]). Let $\phi(t)$ be a nonincreasing and nonnegative function on $[0,T]$, $T>1$, such that

$$\phi(t)^{1+r} \leq \omega_0(\phi(t)-\phi(t+1)), \quad \text{on } [0,T],$$

where $\omega_0>1$ and $r\geq 0$. Then we have, for all $t\in[0,T]$

(i) if $r=0$, then

$$\phi(t)\leq \phi(0)e^{-\omega_1[t-1]^+}$$

where $\omega_1 = \ln\left(\frac{\omega_0}{\omega_0-1}\right)$ and $[t-1]^+ = max(t-1,0)$.

Definition 1 Under the assumption $(A_1)-(A_2)$, a pair function (u,υ) defined on $[0,T]$ is called a weak solution of (1) if $u,\upsilon \in C([0,T];H_0^1(\Omega)), u', \upsilon' \in C([0,T];L^2(\Omega))$, $(u(x,0),\upsilon(x,0)) = (u^0(x),\upsilon^0(x)) \in H_0^1(\Omega)\times H_0^1(\Omega)$ and $(u(t),\upsilon(t))$ satisfies

$$\langle u(t),\phi\rangle - \langle u^0,\phi\rangle + \int_0^t \langle \nabla u(s),\nabla\phi\rangle ds = \int_0^t \langle f_1(u(s),\upsilon(s))\phi\rangle ds, \tag{13}$$

and

$$\langle \upsilon(t),\psi\rangle - \langle \upsilon^0,\psi\rangle + \int_0^t \langle \nabla \upsilon(s),\nabla\psi\rangle ds = \int_0^t \langle f_2(u(s),\upsilon(s))\psi\rangle ds. \tag{14}$$

For all $t\in[0,T]$, $\phi,\psi \in H_0^1(\Omega)$.

Remark 1 By avoiding the complexity of the matter, we take $a=b=1$ in (7) − (8)
.

the above inequalities will be used later, for some constant $C>0$, and all $\alpha,\beta\in R$, we have

$$\left||\alpha|^k - |\beta|^k\right| \le C|\alpha-\beta|\left(|\alpha|^{k-1}+|\beta|^{k-1}\right), \tag{15}$$

for some constant $C>0$, all $k\ge 1$, and all $\alpha,\beta\in R$. all $p\ge 0$. Also

$$||\alpha|^p\alpha - |\beta|^p\beta| \le C|\alpha-\beta|\left(|\alpha|^p+|\beta|^p\right). \tag{16}$$

3. Global existence

In this section, we shall prove the global existence results of the solution to the problem in question.

Lemma 3. Let (u,υ) be a global solution to the problem (1) on $[0,\infty)$. Then we have

$$E'(t) = -\|u'(t)\|_2^2 - \|\upsilon'(t)\|_2^2 \le 0. \tag{17}$$

Proof. Multiplying the equation (1) by u_t, and the second equation in (1) by υ_t, and integrating over $(0,t)\times\Omega$, we get

$$E(t) + \int_0^t \|u'(s)\|_2^2 ds + \int_0^t \|\upsilon'(s)\|_2^2 ds = E(0), \tag{18}$$

after deriving (18) we get the desired results.

Theorem 1 Let $(u^0(x),\upsilon^0(x)) \in H_0^1(\Omega)\times H_0^1(\Omega)$. Assume $(A_1)-(A_2)$, (15) − (16) hold. Then the problem (1) admits a global strong solution $u(x,t)$ defined on $[0,+\infty)$ satisfying

$$u(x,t) \in C([0,+\infty);H_0^1(\Omega)) \cap C([0,+\infty),L^2(\Omega)) \cap H_0^1([0,+\infty);L^2(\Omega)).$$

Hence, we obtain the following decay property:

$$E(t) \le E(0)e^{-\tau t}, \quad \forall\ t\ge 0, \quad \tau = \ln\left(\frac{c_9}{c_9-1}\right). \tag{19}$$

Proof. We use the standard Faedo-Galerkin's method to construct approximate solution. Let $\{w_j\}_{j=1}^{\infty}$ be the eigenfunctions of the operator $A=-\Delta$ with zero Dirichlet boundary condition and $D(A) = H^2(\Omega)\bigcap H_0^1(\Omega)$. It is known that

$\{w_j\}_{j=1}^{\infty}$ forms an orthonormal basis for $L^2(\Omega)$ as well as for $H_0^1(\Omega)$. Moreover. The linear span of $\{w_j\}_{j=1}^{\infty}$ is dense in $L^q(\Omega)$ for any $1 \leq q < \infty$. V_k the linear span of $\{w_1.....w_k\}$, $k \geq 1$. Let $u_k(t) = \sum_{j=1}^{k} u_{k,j}(t)w_j$,$\upsilon_k(t) = \sum_{j=1}^{k} \upsilon_{k,j}(t)w_j$ be the approximate solution to (1) in V_k, then $u_k(t), \upsilon_{k,j}(t)$ verify the following system of ODEs:

$$\langle u_k'(t), w_j\rangle + \langle \nabla u_k(t), \nabla w_j\rangle = \langle f_1(u_k(t), \upsilon_k(t)), w_j\rangle, \tag{20}$$

$$\langle \upsilon_k'(t), w_j\rangle + \langle \nabla \upsilon_k(t), \nabla w_j\rangle = \langle f_2(u_k(t), \upsilon_k(t)), w_j\rangle, \tag{21}$$

for $j = 1.....k$. More specifically

$$u_k(0) = \sum_{j=1}^{k} u_{k,j}(0)w_j, \upsilon_k(0) = \sum_{j=1}^{k} \upsilon_{k,j}(0)w_j, u_k'(0) = \sum_{j=1}^{k} u_{k,j}'(0)w_j, \upsilon_k'(0) = \sum_{j=1}^{k} \upsilon_{k,j}'(0)w_j, \tag{22}$$

where

$$u_k(0) = \langle u^0, w_j\rangle, \upsilon_k(0) = \langle \upsilon^0, w_j\rangle, u_k'(0) = \langle u^1, w_j\rangle, \upsilon_k'(0) = \langle \upsilon^1, w_j\rangle,$$

$j = 1, ..., k$. Obviously, $u_k(0) \to u^0$, $\upsilon_k(0) \to \upsilon^0$ strongly in $H_0^1(\Omega)$, $u_k'(0) \to u^1$, $\upsilon_k'(0) \to \upsilon^1$strongly in $L^2(\Omega)$ as $k \to \infty$.

We shall prove that the problem (20) – (22) admits a local solution in $[0, t_m)$, $0 < t_m < T$, for an arbitrary $T > 0$. The extension of the solution to the whole interval $[0, T]$ is a consequence of the estimates below .

Now we try to get the a priori estimate for the approximate solutions $(u_k(t), \upsilon_k(t))$.

Lemma 4. There exists a constant $T > 0$ such that the approximate solutions $(u_k(t), \upsilon_k(t))$ satisfy for all $k \geq 1$

$$\begin{cases} u_k, \upsilon_k \text{ are bounded in } & L^\infty(0, T; H_0^1(\Omega)), \\ u_k', \upsilon_k' \text{ are bounded in } & L^\infty(0, T; H_0^1(\Omega)). \end{cases} \tag{23}$$

Proof.

First estimate.

Multiplying (20) by $u_{k,j}^{'}(t)$, (21) by $\upsilon_{k,j}^{'}(t)$, and summing with respect to j from 1 to k, respectively, we have

$$\|u_k^{'}(t)\|_2^2 + \frac{d}{dt}\|\nabla u_k(t)\|_2^2 = \int_\Omega f_1(u_k(t), \upsilon_k(t))u_k^{'}(t)dx, \tag{24}$$

$$\|\upsilon_k^{'}(t)\|_2^2 + \frac{d}{dt}\|\nabla \upsilon_k(t)\|_2^2 = \int_\Omega f_2(u_k(t), \upsilon_k(t))\upsilon_k^{'}(t)dx. \tag{25}$$

By summing (24), (25) and integrating over $(0, t)$, $0 < t < T_k$, we get

$$\begin{aligned} \int_0^t \|u_k^{'}(s)\|_2^2 ds + \int_0^t \|\upsilon_k^{'}(s)\|_2^2 ds + \|\nabla \upsilon_k(t)\|_2^2 + \|\nabla u_k(t)\|_2^2 \\ = \int_0^t \int_\Omega [f_1(u_k(s), \upsilon_k(s))u_k^{'}(s) + f_2(u_k(s), \upsilon_k(s))\upsilon_k^{'}(s)]dxds \\ \leq C_0 + \int_0^t \int_\Omega [f_1(u_k(s), \upsilon_k(s))u_k^{'}(s) + f_2(u_k(s), \upsilon_k(s))\upsilon_k^{'}(s)]dxds, \end{aligned} \tag{26}$$

where

$$C_0 = C\left(\|\nabla u^0\|_2, \|\nabla \upsilon^0\|_2\right)$$

is a positive constant, we just need to estimate the right hand terms of (26). Applying , Holder's and Young's inequalities, Sobolev's embedding theorem , for ϵ saficiently small we obtain

$$\begin{aligned}
&\int_0^t \left|\int_\Omega f_1(u_k(s), \upsilon_k(s))u_k^{'}(s)dxds\right| \\
&\leq C\int_0^t \int_\Omega \left(|u_k(s)|^p + |\upsilon_k(s)|^p + |u_k(s)|^{\frac{p-1}{2}} |\upsilon_k(s)|^{\frac{p+1}{2}}\right)|u_k^{'}(s)|dxds, \\
&\leq C\int_0^t \left(\|u_k(s)\|_{2p}^p + \|\upsilon_k(s)\|_{2p}^p + \|u_k(s)\|_{3(p-1)}^{\frac{p-1}{2}} \|u_k(s)\|_{\frac{3(p+1)}{2}}^{\frac{p+1}{2}}\right)\|u_k^{'}(s)\|_2 ds, \\
&\leq C\int_0^t \left(\|\nabla u_k(s)\|_2^p + \|\nabla \upsilon_k(s)\|_2^p + \|\nabla u_k(s)\|_2^{\frac{p-1}{2}} \|\nabla \upsilon_k(s)\|_2^{\frac{p+1}{2}}\right)\|u_k^{'}(s)\|_2 ds, \\
&\leq C\int_0^t \left(\|u_k^{'}(s)\|_2^2 + \|\nabla u_k(s)\|_2^{2p} + \|\nabla \upsilon_k(s)\|_2^{2p} + \|\nabla u_k(s)\|_2^{p-1}\|\nabla \upsilon_k(s)\|_2^{p+1}\right)ds.
\end{aligned} \tag{27}$$

Likewise, we obtain

$$\begin{aligned}
&\int_0^t \left|\int_\Omega f_2(u_k(s), \upsilon_k(s))\upsilon_k^{'}(s)dxds\right| \\
&\leq C\int_0^t \left(\|\upsilon_k^{'}(s)\|_2^2 + \|\nabla u_k(s)\|_2^{2p} + \|\nabla \upsilon_k(s)\|_2^{2p} + \|\nabla u_k(s)\|_2^{p+1}\|\nabla u_k(s)\|_2^{p-1}\right)ds.
\end{aligned} \tag{28}$$

Let

$$y_k(t) = \|\nabla u_k(t)\|_2^2 + \|\nabla \upsilon_k(t)\|_2^2. \tag{29}$$

Then, we infer from (26) – (29) and for a best constant C sufficiently small, we can obtain

$$y_k(t) + (1-C)\int_0^t \|u_k^{'}(s)\|_2^2 ds + (1-C)\int_0^t \|\upsilon_k^{'}(s)\|_2^2 ds \leq C_0 + C\int_0^t (y_k(s))^p ds. \tag{30}$$

Particularly, we have

$$y_k(t) \leq C_0 + C\int_0^t (y_k(s))^p ds.$$

Using a Gronwall's type inequality, we can get

$$y_k(t) \leq [C_0 - (p-1)Ct]^{-1/(p-1)}.$$

Finally , for T arbitrary constant, we obtain

$$y_k(t) \leq C_0 e^{cT}. \tag{31}$$

Thus, we deduce from (31) that there exists a time $T > 0$ such that

$$y_k(t) \leq C_1, \quad \forall t \in [0, T]. \tag{32}$$

Where C_1 is a positive constant independent of k.

Second estimate.

In order to calculate the Second estimate we take the derivatives of (20) – (21)

$$\langle u_k''(t), w_j\rangle + \langle \nabla u_k'(t), \nabla w_j\rangle = \langle Df_1(u_k(t), \upsilon_k(t)), w_j\rangle, \tag{33}$$

$$\langle \upsilon_k''(t), w_j\rangle + \langle \nabla \upsilon_k'(t), \nabla w_j\rangle = \langle Df_2(u_k(t), \upsilon_k(t)), w_j\rangle. \tag{34}$$

Multiplying (33) by $u_k'(t)$, (34) by $\upsilon_k'(t)$, summing with respect to j from 1 to k, respectively, and integrating over Ω, then we obtain

$$\frac{d}{dt}\|u_k'(t)\|_2^2 + \|\nabla u_k'(t)\|_2^2 = \int_\Omega Df_1(u_k(t), \upsilon_k(t))u_k'(t)dx, \tag{35}$$

$$\frac{d}{dt}\|\upsilon_k'(t)\|_2^2 + \|\nabla \upsilon_k'(t)\|_2^2 = \int_\Omega Df_2(u_k(t), \upsilon_k(t))\upsilon_k'(t)dx. \tag{36}$$

By summing (35), (36), and integrating over $(0,t)$, $0 < t \le T$. We get

$$\begin{aligned}&\|u_k'(t)\|_2^2 + \|\upsilon_k'(t)\|_2^2 + \int_0^t \|\nabla u_k'(s)\|_2^2 ds + \int_0^t \|\nabla \upsilon_k'(s)\|_2^2 ds\\ &\le c_0 + c\int_0^t\int_\Omega Df_1(u_k(s), \upsilon_k(s))u_k'(s) + Df_2(u_k(s), \upsilon_k(s))\upsilon_k'(s)dsdx\end{aligned} \tag{37}$$

we just need to estimate the right-hand terms of (37). Applying , Holder and Young inequalities, Sobolev embedding theorem, we obtain

$$\begin{aligned}\left|\int_\Omega Df_1(u_k(t), \upsilon_k(t))u_k'(t)dx\right| &\le C\left[(\|u_k\|^{p-1} + \|\upsilon_k\|^{p-1})\|u_k'\|_2\right]\\ &\quad + \left[\|u_k\|^{p-1} + \|\upsilon_k\|^{p-1})\|\upsilon_k'\|_2\right]\|u_k'\|_2\\ &\le C[(\|u_k'\|_2 + \|\upsilon_k'\|_2)(\|u_k\|^{p-1} + \|\upsilon_k\|^{p-1})]\|u_k'\|_2\\ &\le \|u_k'\|_2^2 + \|\upsilon_k'\|_2^2 + c\end{aligned} \tag{38}$$

likewise, we get

$$\left|\int_\Omega Df_2(u_k(t), \upsilon_k(t))\upsilon_k'(t)dx\right| \le \|u_k'\|_2^2 + \|\upsilon_k'\|_2^2 + c \tag{39}$$

From (37) we have

$$\|u_k'(t)\|_2^2 + \|\upsilon_k'(t)\|_2^2 \le c_0 + c\int_0^t \|\upsilon_k'(s)\|_2^2 + \|u_k'(s)\|_2^2 ds \tag{40}$$

Particulary from the first estimate, we have

$$\|u_k'(t)\|_2^2 + \|\upsilon_k'(t)\|_2^2 \le c_2 + c\int_0^t (\|\upsilon_k'(s)\|_2^2 + \|u_k'(s)\|_2^2)ds. \tag{41}$$

Let

$$y_k(t) = \|u_k'(t)\|_2^2 + \|\upsilon_k'(t)\|_2^2,$$

then

$$y_k(t) \le c_2 + c\int_0^t y_k(s)ds. \tag{42}$$

Using Gronwall's inequality, we get

$$y_k(t) \leq c_2 e^{cT}. \tag{43}$$

Thus we deduce from (43) that there exists a time $T > 0$ such that

$$y_k(t) \leq C_3, \tag{44}$$

where C_3 is a positive constant of k. The proof of Lemma 4 is completed

Lemma 5. The sequences of approximate solutions $\{(u_k, \upsilon_k)\}$ satisfy the following

$$\begin{cases} u_k \textit{ are Cauchy sequences in } L^\infty(0,T;H_0^1(\Omega)). \\ \upsilon_k \textit{ are Cauchy sequences in } L^\infty(0,T;H_0^1(\Omega)), \end{cases} \tag{45}$$

Proof. As in [5]. We consider a family of approximate solutions (u_k, υ_k) and (u_l, υ_l). Without loss of generality, we assume $k > l$. Setting:

$$u_{kl}(t) = u_k(t) - u_l(t) = \sum_{j=1}^{k.j}(u_{k.j}(t) - u_{l.j}(t))w_j, \upsilon_{kl}(t) = \upsilon_k(t) - \upsilon_l(t) = \sum_{j=1}^{k.j}(\upsilon_{k.j}(t) - \upsilon_{l.j}(t))w_j,$$

we put $u_{l.j} = \upsilon_{l.j} = 0$ as $j > l$. Then, $u_k(t), \upsilon_k(t)$ verify

$$\langle u'_{kl}(t), w_j\rangle + \langle \nabla u_{kl}(t), \nabla w_j\rangle = \langle f_1(u_k(t), u_k(t)) - f_1(u_l(t), u_l(t)), w_j\rangle. \tag{46}$$

$$\langle \upsilon'_{kl}(t), w_j\rangle + \langle \nabla \upsilon_{kl}(t), \nabla w_j\rangle = \langle f_2(u_k(t), \upsilon_k(t) - f_2(u_l(t), \upsilon_l(t), w_j\rangle. \tag{47}$$

$$\begin{aligned} u_{kl}(0) &= u_k^0 - u_l^0, \ u'_{kl}(0) = u_k^1 - u_l^1 \\ \upsilon_{kl}(0) &= \upsilon_k^0 - \upsilon_l^0, \ \upsilon'_{kl}(0) = \upsilon_k^1 - \upsilon_l^1 \end{aligned} \tag{48}$$

for $j = 1, ..., k$. Multiplying (46) by $u_{kj}(t) - u_{lj}(t)$ and (47) $\upsilon_{kj}(t) - \upsilon_{lj}(t)$ and summing over j from 1 to k, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{kl}(t)\|_2^2 + \frac{1}{2}\|\nabla u_{kl}(t)\|_2^2 = \int_\Omega [f_1(u_k(t), \upsilon_k(t)) - f_1(u_k(t), \upsilon_k(t)]u_{kl}(t)dx \tag{49}$$

$$\frac{1}{2}\frac{d}{dt}\|\upsilon_{kl}(t)\|_2^2 + \frac{1}{2}\|\nabla \upsilon_{kl}(t)\|_2^2 = \int_\Omega [f_2(u_k(t), \upsilon_k(t)) - f_2(u_k(t), \upsilon_k(t))]\upsilon_{kl}(t)dx. \tag{50}$$

Summing (49) , (50) and integrating over $(0, t)$, we have

$$\begin{aligned} &\frac{1}{2}\|u_{kl}(t)\|_2^2 + \frac{1}{2}\|\upsilon_{kl}(t)\|_2^2 + \int_0^t \|\nabla u_{kl}(s)\|_2^2 ds + \int_0^t \|\nabla \upsilon_{kl}(s)\|_2^2 ds \\ &= \frac{1}{2}\|u_{kl}(0)\|_2^2 + \frac{1}{2}\|\upsilon_{kl}(0)\|_2^2 \\ &+ \int_0^t \int_\Omega [f_1(u_k(s), \upsilon_k(s)) - f_1(u_k(s), \upsilon_k(s)]u_{kl}(s)dxds \\ &+ \int_0^t \int_\Omega [f_2(u_k(s), \upsilon_k(s)) - f_2(u_k(s), \upsilon_k(s)]\upsilon_{kl}(s)dxds. \end{aligned} \tag{51}$$

Now we deal with the right hand terms of (51) by (A_1) and a direct computation

$$\begin{aligned}
&\left|\int_\Omega [f_1(u_k(s),\upsilon_k(s)) - f_1(u_l(s),\upsilon_l(s))]u_{kl}(s)dx\right| \\
&\le c\int_\Omega (|u_{kl}| + |\upsilon_{kl}|)(|u_k|^{p-1} + |\upsilon_k|^{p-1} + |u_l|^{p-1} + |\upsilon_l|^{p-1})|u_{kl}|dx \\
&+ c\int_\Omega |u_k|^{\frac{p-1}{2}}|\upsilon_{kl}|(|\upsilon_k|^{\frac{p-1}{2}} + |\upsilon_l|^{\frac{p-1}{2}})|u_{kl}|dx \\
&+ c\int_\Omega |\upsilon_k|^{\frac{p+1}{2}}|u_{kl}|(|u_k|^{\frac{p-3}{2}} + |u_l|^{\frac{p-3}{2}})|u_{kl}|dx, \\
&\le (I_1 + I_2 + I_3).
\end{aligned} \tag{52}$$

estimating I_1, I_2, I_3 in (52). Employing Holder and Young inequalities, Sobolev's embedding theorem, and Lemma 5 we estimate a typical term in I_1 as

$$\begin{aligned}
\left|\int_\Omega |u_{kl}||u_k|^{p-1}|u_{kl}|dx\right| &\le c\|u_{kl}\|_6\|u_k\|^{p-1}_{3(p-1)}\|u_{kl}\|_2, \\
&\le c\|\nabla u_{kl}\|_2\|\nabla u_k\|_2^{p-1}\|u_{kl}\|_2, \\
&\le c\|\nabla u_{kl}\|_2\|u_{kl}\|_2, \\
&\le c(\|\nabla u_{kl}\|_2^2 + \|u_{kl}\|_2^2).
\end{aligned} \tag{53}$$

Thus, we have

$$I_1 \le c(\|u_{kl}\|_2^2 + \|\nabla u_{kl}\|_2^2 + \|\nabla \upsilon_{kl}\|_2^2). \tag{54}$$

Similarly, a typical term in I_2 can be estimated as

$$\begin{aligned}
\int_\Omega |u_k|^{\frac{p-1}{2}}|\upsilon_{kl}|(|\upsilon_k|^{\frac{p-1}{2}}|u_{kl}|_2 dx &\le c\|u_k\|^{\frac{p-1}{2}}_{3(p-1)}\|\upsilon_{kl}\|_6\|\upsilon_{kl}\|^{\frac{p-1}{2}}_{3(p-1)}\|u_{kl}\|_2, \\
&\le c\|\nabla u_k\|_2^{\frac{p-1}{2}}\|\nabla \upsilon_{kl}\|_2\|\nabla \upsilon_{kl}\|_2^{\frac{p-1}{2}}\|u_{kl}\|_2, \\
&\le c\|u_{kl}\|_2\|\nabla \upsilon_{kl}\|_2, \\
&\le c(\|u_{kl}\|_2^2 + \|\nabla \upsilon_{kl}\|_2^2).
\end{aligned} \tag{55}$$

Then

$$I_2 \le c(\|u_{kl}\|_2^2 + \|\nabla \upsilon_{kl}\|_2^2). \tag{56}$$

Noting that $p = 3$ as $n = 3$, we have for $n = 3$

$$\begin{aligned}
I_3 &= \int_\Omega |\upsilon_l|^{\frac{p+1}{2}}|u_{kl}|^2|u_{kl}|^{\frac{p-3}{2}}dx, \\
&\le c\|\upsilon_l\|^{\frac{p+1}{2}}_{\frac{3(p+1)}{2}}\|u_{kl}\|_6^2\||u_k\|^{3(p-3)}_{\frac{p-3}{2}}, \\
&\le c\|\nabla \upsilon_l\|_2^{\frac{p+1}{2}}\|\nabla u_{kl}\|_2^2.
\end{aligned} \tag{57}$$

as $n = 1, 2$, a typical term in I_3 can be estimated as

$$\begin{aligned}
\int_\Omega |\upsilon_l|^{\frac{p+1}{2}}|u_{kl}|^2|u_k|^{\frac{p-3}{2}}dx, &\le \|\upsilon_l\|^{\frac{p+1}{2}}_{3(p+1)}\|u_{kl}\|_6^2\|u_k\|^{\frac{p-3}{2}}_{3(p-3)}, \\
&\le \|\nabla \upsilon_l\|_2^{\frac{p+1}{2}}\|\nabla u_k\|^{\frac{p-3}{2}}_{3(p-3)}\|\nabla u_{kl}\|_2^2, \\
&\le \|\nabla \upsilon_l\|_2^{\frac{p+1}{2}}\|\nabla u_{kl}\|_2^2.
\end{aligned} \tag{58}$$

Combining (51) – (58), we get

$$\begin{aligned}&\left|\int_\Omega [f_1(u_k(s),\upsilon_k(s))-f_1(u_l(s),\upsilon_l(s))]u_{kl}(s)dx\right|\\&\le c\left(\|\nabla u_{kl}\|_2^2+\|\nabla \upsilon_{kl}\|_2^2+\|u_{kl}\|_2^2+\|\nabla u_{kl}\|_2^2\|\nabla \upsilon_l\|_2^{\frac{p+1}{2}}\right).\end{aligned}\tag{59}$$

Likewise, we have

$$\begin{aligned}&\left|\int_\Omega [f_2(u_k(s),\upsilon_k(s))-f_2(u_l(s),\upsilon_l(s))]\upsilon_{kl}(s)dx\right|\\&\le c\left(\|\nabla u_{kl}\|_2^2+\|\nabla \upsilon_{kl}\|_2^2+\|\upsilon_{kl}\|_2^2+\|\nabla u_{kl}\|_2^2\|\nabla \upsilon_l\|_2^{\frac{p+1}{2}}\right).\end{aligned}\tag{60}$$

Then it comes out from (59) – (60) that (51) becomes

$$\begin{aligned}&\frac{1}{2}\|u_{kl}(t)\|_2^2+\frac{1}{2}\|\upsilon_{kl}(t)\|_2^2+\int_0^t\|\nabla u_{kl}(s)\|_2^2ds+\int_0^t\|\nabla \upsilon_{kl}(s)\|_2^2ds,\\&\le\frac{1}{2}\|u_{kl}(0)\|_2^2+\|\upsilon_{kl}(0)\|_2^2\\&+c\int_0^t\left(\|\nabla u_{kl}(s)\|_2^2+\|\nabla \upsilon_{kl}(s)\|_2^2+\|\upsilon_{kl}(s)\|_2^2+\|\nabla u_{kl}(s)\|_2^2\|\nabla \upsilon_l(s)\|_2^{\frac{p+1}{2}}\right)ds.\end{aligned}\tag{61}$$

Letting

$$y_{kl}(t)=\|u_{kl}(t)\|_2^2+\|\upsilon_{kl}(t)\|_2^2,$$

then from (61) we have

$$y_{kl}(t)+\int_0^t\|\nabla u_{kl}(s)\|_2^2ds+\int_0^t\|\nabla \upsilon_{kl}(s)\|_2^2ds\le y_{kl}(0)+c\int_0^t y_{kl}(s)ds.\tag{62}$$

where $y_{kl}(0)=\frac{1}{2}(\|u_{kl}(0)\|_2^2+\|\upsilon_{kl}(0)\|_2^2$. By the strong convergence of the initial data, namely, $y_{kl}(0)\to 0$ as $k,l\to\infty$, then it follows from (62) and Gronwall's inequality that

$$y_{kl}(t)\le y_{kl}(0)e^{cT}\to 0\ as\ k,l\to\infty.\tag{63}$$

Corollary 1. The sequences of approximate solutions $\{u_k,\upsilon_k\}$ satisfy as $k\to\infty$

$$\begin{cases}f_1(u_k,\upsilon_k)\to f_1(u,\upsilon)\ strongly\ in\ L^\infty(0,T;L^2(\Omega))\\ f_2(u_k,\upsilon_k)\to f_2(u,\upsilon)\ strongly\ in\ L^\infty(0,T;L^2(\Omega)).\end{cases}\tag{64}$$

Proof. The proof is similar to that of [5] . completed

3.1. **Limiting process.** By Lemma 5 and Corollary 1 we know that

$$\begin{cases}f_1(u_k,\upsilon_k)\to f_1(u,\upsilon)\ strongly\ in\ L^\infty(0,T;L^2(\Omega)),\\ f_2(u_k,\upsilon_k)\to f_2(u,\upsilon)\ strongly\ in\ L^\infty(0,T;L^2(\Omega)),\\ u_k\to u\qquad strongly\ in\ L^\infty(0,T;H_0^1(\Omega)),\\ \upsilon_k\to \upsilon\qquad strongly\ in\ L^\infty(0,T;H_0^1(\Omega)),\\ u_k'\to u'\qquad strongly\ in\ L^\infty(0,T;H_0^1(\Omega)),\\ \upsilon_k'\to \upsilon'\qquad strongly\ in\ L^\infty(0,T;H_0^1(\Omega)).\end{cases}$$

Remark 2 By virtue of the theory of ordinary differential equations, the system (20) – (21) has local solution which is extended to a maximal interval $[0,T_k[$ with $(\ 0<T_k\le+\infty)$.

Now we will prove that the solution obtained above is global and bounded in time, for this purpose, we have the following lemma.

Lemma 6. Let (u, υ) be the solution of problem (1) . Assume further that $I(0) > 0$

$$\alpha = \eta\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} < 1. \tag{65}$$

Then the solution of problem obtained above is global and bounded in time .

proof First, we show that $I(t) > 0$, on [0,T]. Since $I(0) > 0$, then by continuity, there exists a time $t_1 > 0$ such that $I(t) \geq 0$, for $t \in (0, t_1)$.

Let t_0 be given by

$$\{I(t_0) = 0 \; and \; I(t) > 0, \; 0 \leq t < t_0\}.$$

This, together with (10) − (11), implies that, for $t \in [0, t_0]$.

$$E(t) = \frac{p-1}{2(p+1)}(\|\nabla u(t)\|_2^2 + \|\nabla \upsilon(t)\|_2^2) + \frac{1}{p+1}I(t) \geq \frac{p-1}{2(p+1)}(\|\nabla u(t)\|_2^2 + \|\nabla \upsilon(t)\|_2^2), \tag{66}$$

$$\|\nabla \upsilon(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leq \frac{2(p+1)}{p-1}E(t) \leq \frac{2(p+1)}{p-1}E(0). \tag{67}$$

Employing Lemma 1, we obtain

$$\begin{aligned}(p+1)\int_\Omega F(u(t_0), \upsilon(t_0))dx &\leq \eta(\|\nabla u\|_2^2 + \|\nabla \upsilon\|_2^2)^{\frac{p+1}{2}} \\ &\leq \eta\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}}(\|\nabla u(t_0)\|_2^2 + \|\nabla \upsilon(t_0)\|_2^2) \\ &= \alpha(\|\nabla u(t_0)\|_2^2 + \|\nabla \upsilon(t_0)\|_2^2) \\ &< \|\nabla u\|_2^2 + \|\nabla \upsilon\|_2^2).\end{aligned} \tag{68}$$

Thus $I(t) > 0$ on $[0, t_{max}]$, by repeating these steps and using the fact that $\lim_{t \to t_{max}} \eta\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} \leq \alpha < 1.$

This implies that we can take $t_{max} = T$, from the previous procedure, the solution of (1) is global and bounded in time.

4. Asymptotic Behavior

From now and on, our attention is centered on the decay rate of the solutions to problem (1). We will derive the decay rate of the energy function for problem (1) by Nakao's method [19]. Purposefully, we have to show that the energy function defined by(10). Satisfies the hypothesis of Lemma 2. By integrating (17), over $[t, t+1]$, we have

$$E(t) - E(t+1) = D(t)^2, \tag{69}$$

where

$$D(t)^2 = \frac{1}{2}\int_t^{t+1} \|u_t(t)\|_2^2 dt + \frac{1}{2}\int_t^{t+1} \|\upsilon_t(t)\|_2^2 dt. \tag{70}$$

By virtue of (69). and Holder's inequality, we observe that

$$\frac{1}{2}\int_t^{t+1}\int_\Omega |u_t(t)|^2 dxdt + \frac{1}{2}\int_t^{t+1}\int_\Omega |\upsilon_t(t)|^2 dxdt \leq D(t)^2, \tag{71}$$

where $c_1(\Omega) = vol(\Omega)$. Applying the mean value theorem. There exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t)\|_2^2 + \|\upsilon_t(t)\|_2^2 \leq 4c_2(\Omega)D(t)^2, i = 1, 2. \tag{72}$$

Next, multiplying the first eqution in (1) by u, and the second equation in (1). by υ, integrating over $\Omega \times [t_1, t_2]$, using integration by parts, Holder's inequality and adding them together, we obtain

$$\int_{t_1}^{t_2} I(t)dt \leq \int_{t_1}^{t_2} \|\upsilon\|_2 \|\upsilon_t\|_2 dt + \int_{t_1}^{t_2} \|u\|_2 \|u_t\|_2 dt. \tag{73}$$

By using Sobolev's inequality we get

$$\begin{aligned} \int_{t_1}^{t_2} \|u(t)\|_2 \|u_t(t)\|_2 dt &\leq c_* \int_{t_1}^{t_2} \|\nabla u(t)\|_2 \|u_t(t)\|_2 dt \\ &\leq c_* \left(\frac{2(p+1)}{(p-1)}\right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}} \int_{t_1}^{t_2} \|\nabla u_t\|_2 dt \\ &\leq c_* \left(\frac{2(p+1)}{(p-1)}\right)^{\frac{1}{2}} E(t)^{\frac{1}{2}} D(t). \end{aligned} \tag{74}$$

Similarly

$$\begin{aligned} \int_{t_1}^{t_2} \|\upsilon(t)\|_2 \|\upsilon_t(t)\|_2 dt &\leq c_* \int_{t_1}^{t_2} \|\nabla \upsilon(t)\|_2 \|\upsilon_t(t)\|_2 dt \\ &\leq c_* \left(\frac{2(p+1)}{(p-1)}\right)^{\frac{1}{2}} \sup E(s)^{\frac{1}{2}}{}_{t_1 \leq s \leq t_2} \int_{t_1}^{t_2} \|\nabla \upsilon_t\|_2 dt \\ &\leq c_* \left(\frac{2(p+1)}{(p-1)}\right)^{\frac{1}{2}} E(t)^{\frac{1}{2}} D(t). \end{aligned} \tag{75}$$

Noting that

$$\|\nabla u(t)\|_2^2 + \|\nabla \upsilon(t)\|_2^2 \leq I(t), \tag{76}$$

due to $t_2 - t_1 \geq \frac{1}{2}$,
hence (73), takes the form

$$\int_{t_1}^{t_2} I(t)dt \leq 2c_* \left(\frac{2(p+1)}{(p-1)}\right)^{\frac{1}{2}} E(t)^{\frac{1}{2}} D(t), \tag{77}$$

integrating (17). over (t_1, t_2) , utilizing (77). And using Lemma 2, we deduce that

$$\int_{t_1}^{t_2} E(t)dt \leq \left(\frac{p-1}{2(p+1)} + \frac{1}{(p+1)}\right) \int_{t_1}^{t_2} I(t)dt \leq c_* \left(\frac{2(p+1)}{(p-1)}\right)^{\frac{1}{2}} E(t)^{\frac{1}{2}} D(t). \tag{78}$$

Now integrating (17) over (t, t_2), using (78), and the fact that $E(t_2) \leq 2\int_{t_1}^{t_2} E(t)dt$, due to $t_2 - t_1 \geq \frac{1}{2}$, we obtain

$$
\begin{aligned}
E(t) &= E(t_2) + \frac{1}{2}\int_t^{t_2} \|u_t(t)\|_2^2 dt + \frac{1}{2}\int_t^{t_2} \|\upsilon_t(t)\|_2^2 dt \\
&\leq 2\left(\frac{2(p+1)}{(p-1)}\right)^{\frac{1}{2}} E(t)^{\frac{1}{2}} D(t) \\
&+ 2c_* \left(\frac{2(p+1)}{(p-1)}\right)^{\frac{1}{2}} E(t)^{\frac{1}{2}} D(t) + D(t)^2.
\end{aligned} \tag{79}
$$

Consequently, we obtain

$$
E(t) \leq c_4 E(t)^{\frac{1}{2}} D(t) + D(t)^2 \tag{80}
$$

where $c_4 = c_* \left[\left(\frac{2(p+1)}{(p-1)}\right)^{\frac{1}{2}}\right]$. Using Young's inequality we get

$$
E(t) \leq \epsilon c_4 E(t) + c_4 c(\epsilon) D(t)^2 + c_4 D(t)^2. \tag{81}
$$

for ϵ sufficiently small ,finally , we get

$$
\begin{aligned}
E(t) &\leq c_5 D(t)^2 \\
&\leq c_5[E(t) - E(t+1)].
\end{aligned} \tag{82}
$$

where c_5 is a positive constant. Thus by Lemma 2, we obtain

$$
E(t) \leq E(0)e^{-\tau t},
$$

for $t \geq 0$, with $\tau = \ln\left(\frac{c_5}{c_5 - 1}\right)$, which completes the proof.

5. BLOW UP RESULT

In this section we prove the blow up result by setting the following theorem

Theorem 3. Let $(u^0(x), \upsilon^0(x)) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $u^1(x), \upsilon^1(x) \in L^2(\Omega) \times L^2(\Omega)$. Assume that $(A_1) - (A_2)$ holds and $E(0) < 0$, where $E(0)$ is the initial energy given by

$$
E(0) = \frac{1}{2}\|\nabla u(0)\|_2^2 + \frac{1}{2}\|\nabla \upsilon(0)\|_2^2 - \int_\Omega F(u(0), \upsilon(0)) dx.
$$

Then any solution (u, υ) of problem (1) blows up at a finite time T. The lifespan T is estimated by

$$
0 < T \leq C\epsilon^{-1-\tau} L(0)^{\frac{-\alpha}{1-\alpha}}.
$$

Proof. As in [6] we set $G(t) = \int_\Omega F(u, \upsilon) dx$,

$$
H(t) = -E(t), \tag{83}
$$

where E(t) is defined by (10). From (83) and the energy identity, it is easy to see that taking derivative of (83)

$$
H'(t) = -E'(t) = \|u_t(t)\|_2^2 + \|\upsilon_t(t)\|_2^2 \geq 0, \tag{84}
$$

which implies $H(t)$ is increasing on $[0, T]$. Noting the assumption $E(0) < 0$ and (A_1) , then we have H(t) is increasing over $[0, T]$. Noting that $E(0) < 0$, then we have $H(0) > 0$ and

$$
0 < H(0) \leq H(t) \leq G(t) \leq c_1 \left(\|u(t)\|_{p+1}^{p+1} + \|\upsilon(t)\|_{p+1}^{p+1}\right).
$$

From (A_1), we also see that

$$G(t) \geq c_0 \left(\|u(t)\|_{p+1}^{p+1} + \|u(t)\|_{p+1}^{p+1} \right),$$

we define also

$$L(t) = (1-\alpha)H^{1-\alpha}(t) + \epsilon \int_\Omega |u(t)|^2 dx + \epsilon \int_\Omega |\upsilon(t)|^2 dx. \tag{85}$$

Taking the derivative of (85)

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + 2\epsilon \int_\Omega u_t u dx + 2\epsilon \int_\Omega \upsilon_t \upsilon dx. \tag{86}$$

Using (83) – (84) we obtain

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) - 2\epsilon\|\nabla u(t)\|_2^2 - 2\epsilon\|\nabla \upsilon(t)\|_2^2 + \epsilon(p+1)G(t). \tag{87}$$

Now the definition of H(t) in (83) implies

$$-\left(2\epsilon\|\nabla u(t)\|_2^2 + 2\epsilon\|\nabla \upsilon(t)\|_2^2\right) = 2H(t) - 2G(t). \tag{88}$$

Therefore (87), (88) yields

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + 2\epsilon H(t) - 2\epsilon G(t) + \epsilon(p+1)G(t)L'(t) \\ &\geq (1-\alpha)H^{-\alpha}(t)H'(t) + 2\epsilon H(t) + \epsilon(p-1)G(t). \end{aligned} \tag{89}$$

Since $0 < \alpha < \frac{1}{2} \quad 0 < \epsilon \leq 1$ it follows from (89)

$$L'(t) \geq 2\epsilon H(t) + \epsilon(p-1)G(t) \geq \epsilon C(H(t) + G(t)), \tag{90}$$

for $t \in [0, T)$ and where $C > 0$ is a constant . In particular (85) shows that $L(t)$ is increasing on $[0, T)$ with

$$L(t) = H(t)^{1-\alpha} + \epsilon N(t) \geq H(0)^{1-\alpha} + \epsilon N(0), \tag{91}$$

with

$$N(t) = \epsilon \parallel u(t)\|_2^2 + \epsilon\|\upsilon(t)\|_2^2 + \epsilon\|\nabla u(t)\|_2^2 + \epsilon\|\nabla \upsilon(t)\|_2^2. \tag{92}$$

If $N(0) \geq 0$, then no further condition on ϵ is needed. However, if $N(0) < 0$, then we further adjust ϵ so that $0 < \epsilon \leq -\frac{H(0)^{1-\alpha}}{2N(0)}$. In any case, one has

$$L(t) \geq \frac{1}{2}H(0)^{1-\alpha} > 0 \ for \ t \in [0, T), \tag{93}$$

finally, we prove that

$$L'(t) \geq \epsilon^{1+\tau} C L(t)^{\frac{1}{1-\alpha}} \ for \ t \in [0, T). \tag{94}$$

Where

and $C > 0$ is a positive constant. It is important to note that the definition of α implies that $1 < \frac{1}{1-\alpha} < 2$.

We consider two cases

case 1: $N(t) \leq 0$ for some $t \in [0, T)$ and $\tau = 1$ then

$$L(t)^{\frac{1}{1-\alpha}} = N(t)^{\frac{1}{1-\alpha}} \leq H(t), \tag{95}$$

in this case (94), (95) yields

$$L'(t) \geq C\epsilon H(t) \geq C\epsilon^{1+\tau} H(t) \geq C\epsilon^{1+\tau} L(t)^{\frac{1}{1-\alpha}}. \tag{96}$$

Hence (96) holds for all $t \in [0, T)$ for which $N(t) \leq 0$

case 2: $N(t) \leq 0$ (96) is still valid, but with some more work. First we note that

$$L(t)^{\frac{1}{1-\alpha}} \leq 2^{\frac{1}{1-\alpha}-1}[H(t) + \epsilon N(t)^{\frac{1}{1-\alpha}}]. \tag{97}$$

We estimate $N(t)^{\frac{1}{1-\alpha}}$ as follows and noting that $1 < \frac{1}{1-\alpha} < 2$, recalling that $0 < H(0) \leq H(t) \leq G(t)$ and using Sobolev's embedding we have

$$\begin{aligned} N(t)^{\frac{1}{1-\alpha}} &= 2^{\frac{1}{1-\alpha}}[\|u(t)\|_2^2 + \|\upsilon(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\nabla \upsilon(t)\|_2^2]^{\frac{1}{1-\alpha}}, \\ &\leq 2^{\frac{1}{1-\alpha}} C(\|u(t)\|_2^{\frac{2}{1-\alpha}} + \|\upsilon(t)\|_2^{\frac{2}{1-\alpha}} + \|\nabla u(t)\|_2^{\frac{2}{1-\alpha}} + \|\nabla \upsilon(t)\|_2^{\frac{2}{1-\alpha}}). \end{aligned} \tag{98}$$

Then we have from Lemma 1 that yields the following estimations

$$\|u(t)\|_2^{\frac{2}{1-\alpha}} \leq C^{\frac{1}{1-\alpha}} \|\nabla u(t)\|_2^{\frac{2}{1-\alpha}} \leq \frac{C^{\frac{1}{1-\alpha}} H(t)}{H(0)} \leq C^{\frac{1}{1-\alpha}} G(t)\epsilon^{-1}, \tag{99}$$

$$\|\upsilon(t)\|_2^{\frac{2}{1-\alpha}} \leq C^{\frac{1}{1-\alpha}} \|\nabla \upsilon(t)\|_2^{\frac{2}{1-\alpha}} \leq \frac{C^{\frac{1}{1-\alpha}} H(t)}{H(0)} \leq C^{\frac{1}{1-\alpha}} G(t)\epsilon^{-1}, \tag{100}$$

$$\|\nabla \upsilon(t)\|_2^{\frac{2}{1-\alpha}} \leq \frac{C^{\frac{1}{1-\alpha}} H(t)}{H(0)} \leq C^{\frac{1}{1-\alpha}} G(t)\epsilon^{-1}, \tag{101}$$

$$\|\nabla u(t)\|_2^{\frac{2}{1-\alpha}} \leq \frac{C^{\frac{1}{1-\alpha}} H(t)}{H(0)} \leq C^{\frac{1}{1-\alpha}} G(t)\epsilon^{-1}. \tag{102}$$

Combining (99), (102), we have

$$N(t)^{\frac{1}{1-\alpha}} \leq C\epsilon^{-\tau}(H(t) + G(t)). \tag{103}$$

Using (96), finally we get

$$L'(t) \geq C\epsilon^{1+\tau}(L(t))^{\frac{1}{1-\alpha}}, \tag{104}$$

for all values of $t \in [0, T)$ for which $N(t) > 0$ Hence, (95) is valid , simple integration of (96) over $[0, T)$ we get

$$T < C\epsilon^{-1-\tau} L(0)^{\frac{-\alpha}{1-\alpha}}. \tag{105}$$

where α, is a constant defined above

therefore $L(t)$ blows up in finite time T, with $0 < T \leq C\epsilon^{-1-\tau} L(0)^{\frac{-\alpha}{1-\alpha}}$ which completes the proof.

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