

## ROUGH CONVERGENCE OF A SEQUENCE OF INTERVALS OF FUZZY NUMBERS

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**ABSTRACT.** In this paper, we introduce the concept of rough convergence of a sequence of intervals of numbers. Then, we obtain some rough convergence criteria for this type of sequences.

### 1. INTRODUCTION

Matloka [15] introduced the concepts of bounded and convergent sequences of fuzzy numbers and proved that every convergent sequence is bounded. In [19], Nanda studied the spaces of bounded and convergent sequences of fuzzy numbers and showed that they are complete metric spaces. In the literature, there are so many papers related to the theory of convergence of a sequence of fuzzy numbers (see [2, 12, 13, 20, 26, 27]). One of the recently studied papers is that of Akçay and Aytar's [1]. They defined the concept of rough convergence of a sequence of fuzzy numbers, obtained a relation between the rough limit set and the set of extreme limit points of a sequence of fuzzy numbers. The rough limit set of a sequence being closed, bounded and convex was also proved by them.

Phu [23] gave the definition of rough convergence of a sequence in finite dimensional normed spaces. He [23] also showed that a sequence which is not convergent in the usual sense might be convergent to a point with a certain degree of roughness. Subsequently, he has proved analogous results for infinite-dimensional spaces [24]. In 2008, Aytar [3] investigated the relations between the core and the  $r$ -limit set of a real sequence. Recently, the rough statistical convergence theory have been studied by many authors (see [4], [7], [9], [10], [14] and [22]).

The concept of interval arithmetic was first introduced by Dwyer [11] in 1951. This theory was progressed by Moore [17] and Moore and Yang [18]. In 2002, Chiao [6] introduced the notion of convergence of a sequence of interval numbers by using the metric on the set of all interval numbers. Recently, Ölmez and Aytar [21] have introduced the concept of rough convergence of a sequence of interval numbers, where these intervals consist of real numbers. They also proved the convexity of the rough limit set of a sequence.

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Aytar [5] introduced a function in order to measure the distance between two order intervals of fuzzy numbers, and showed that this function is a metric. In this work, we introduce the concept of rough convergence of a sequence of intervals of numbers (Throughout this paper, the phrase "sequence of intervals of fuzzy numbers" is abbreviated to SIFN) Then, we obtain some rough convergence criteria for this type sequences. Finally, we prove the closedness of set of rough limit set a SIFN.

## 2. PRELIMINARIES

We first recall some of the basic concepts and notations in the theory of fuzzy numbers. We refer to [8] and [16] for more details.

A *fuzzy number* is a function  $X : \mathbb{R} \rightarrow [0, 1]$  satisfying the following properties:

- (i)  $X$  is *normal*, i.e., there exists a  $t_0 \in \mathbb{R}$  such that  $X(t_0) = 1$ ;
- (ii)  $X$  is *fuzzy convex*, i.e., for any  $t, u \in \mathbb{R}$  and  $\lambda \in [0, 1]$  we have

$$X(\lambda t + (1 - \lambda)u) \geq \min\{X(t), X(u)\};$$

- (iii)  $X$  is *upper semi-continuous*;
- (iv) The *closure* of the set  $\{t \in \mathbb{R} : X(t) > 0\}$ , denoted by  $X^0$ , is *compact*.

These properties imply that for each  $\alpha \in (0, 1]$ , the  $\alpha$ -level set

$$X^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\} = [\underline{X}^\alpha, \bar{X}^\alpha]$$

is a non-empty compact convex subset of  $\mathbb{R}$ , as the support  $X^0$ .

A real number  $r$  can be considered as a fuzzy number  $r_1$  defined by

$$r_1(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \neq r \end{cases}.$$

The set of all fuzzy numbers is usually denoted by  $L(\mathbb{R})$ .

A partial order  $\preceq$  on  $L(\mathbb{R})$  can be defined via

$$X \preceq Y \text{ iff } \underline{X}^\alpha \leq \underline{Y}^\alpha \text{ and } \bar{X}^\alpha \leq \bar{Y}^\alpha$$

for each  $\alpha \in [0, 1]$ . We write  $X \prec Y$  if  $X \preceq Y$  and there exists an  $\alpha_0 \in [0, 1]$  such that  $\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0}$  or  $\bar{X}^{\alpha_0} < \bar{Y}^{\alpha_0}$ .

The equality of two fuzzy numbers is defined by

$$X = Y \Leftrightarrow \underline{X}^\alpha = \underline{Y}^\alpha \text{ and } \bar{X}^\alpha = \bar{Y}^\alpha$$

for each  $\alpha \in [0, 1]$ .

One basic problem encountered in fuzzy analysis is the distance between fuzzy numbers. There are many alternatives for this (for a brief review, see [8]). Here we will consider the one which is mostly used and easy to manipulate, namely, the function

$$D : L(\mathbb{R}) \times L(\mathbb{R}) \rightarrow \mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$$

defined by

$$D(X, Y) = \sup_{\alpha \in [0, 1]} \max \left\{ |\underline{X}^\alpha - \underline{Y}^\alpha|, |\bar{X}^\alpha - \bar{Y}^\alpha| \right\}.$$

The function  $D$  is called the *supremum metric*. Puri and Ralescu [25] proved that the pair  $(L(\mathbb{R}), D)$  is a complete metric space.

As in the case of the classical theory of partially ordered sets, an *order interval*  $[X, Y]$  of fuzzy numbers can be defined as

$$[X, Y] = \{Z \in L(\mathbb{R}) : X \preceq Z \preceq Y\},$$

where  $X, Y \in L(\mathbb{R})$ . This interval is an ordinary (non-fuzzy) set whose elements are fuzzy numbers (see also [12]).

We say that the order intervals of fuzzy numbers  $[X_1, Y_1]$  and  $[X_2, Y_2]$  are *equal* if, and only if,  $X_1 = X_2$  and  $Y_1 = Y_2$ .

Denote the set of all order intervals of fuzzy numbers by  $\mathbf{I}[L(\mathbb{R})]$ . We now recall the function

$$D_I : \mathbf{I}[L(\mathbb{R})] \times \mathbf{I}[L(\mathbb{R})] \longrightarrow \mathbb{R}_0^+$$

which is defined by

$$D_I([X_1, Y_1], [X_2, Y_2]) = \max\{D(X_1, X_2), D(Y_1, Y_2)\}. \quad (2.1)$$

The function  $D_I$  defined by (2.1) is a metric on  $\mathbf{I}[L(\mathbb{R})]$  (see [5] for more details).

Now let  $r$  be a nonnegative real number. Then we say that a sequence  $\{X_n\}$  of fuzzy numbers is *r-convergent* to a fuzzy number  $X$  and we write

$$X_n \xrightarrow{r} X \text{ as } n \rightarrow \infty,$$

provided that for every  $\varepsilon > 0$  there is an integer  $N(\varepsilon)$  so that

$$d(X_n, X_*) < r + \varepsilon$$

whenever  $n \geq N(\varepsilon)$ . The set

$$\text{LIM}^r X_n := \{X \in L(\mathbb{R}) : X_n \xrightarrow{r} X, \text{ as } n \rightarrow \infty\}$$

is called the *r-limit set* of the sequence  $\{X_n\}$ .

### 3. ROUGH CONVERGENCE OF A SIFN

In this section, we introduce the concept of rough convergence of a SIFN, and obtain some basic results.

**Definition 3.1.** Let  $r > 0$  be given. A SIFN  $\{[X_n, Y_n]\}$  is said to *rough converge* (*r-converge*) to an interval of fuzzy number  $[X, Y] \in \mathbf{I}[L(\mathbb{R})]$  if for every  $\varepsilon > 0$  there is a positive integer  $N$  (which in general depends on  $\varepsilon$ ) such that  $n \geq N$  implies  $D_I([X_n, Y_n], [X, Y]) < r + \varepsilon$ , and we denote this situation by  $[X_n, Y_n] \xrightarrow{r} [X, Y]$  or  $D_I([X_n, Y_n], [X, Y]) \xrightarrow{r} 0$ , as  $n \rightarrow \infty$ .

It is clear from the definition that  $[X_n, Y_n] \xrightarrow{r} [X, Y] \Leftrightarrow X_n \xrightarrow{r} X$  and  $Y_n \xrightarrow{r} Y$ , as  $n \rightarrow \infty$ .

The rough convergence theory for the SIFN is a generalization of the classical convergence theory. In order words, every convergent SIFN is also rough convergent for the roughness degree  $r = 0$ . But, the converse of this claim does not holds in general. The following example illustrate this fact.

Define

$$[X_n, Y_n] = \begin{cases} [X, Y] & , \text{ if } n \text{ is an odd number} \\ [Z, W] & , \text{ otherwise} \end{cases}$$

where

$$X = \begin{cases} x & , \text{ if } x \in [0, 1] \\ -x + 2 & , \text{ if } x \in (1, 2] \\ 0 & , \text{ otherwise} \end{cases} ,$$

$$Y = \begin{cases} x - 2 & , \text{ if } x \in [2, 3] \\ -x + 4 & , \text{ if } x \in (3, 4] \\ 0 & , \text{ otherwise} \end{cases} ,$$

$$Z = \begin{cases} x - 11 & , \text{ if } x \in [11, 12] \\ -x + 13 & , \text{ if } x \in (12, 13] \\ 0 & , \text{ otherwise} \end{cases}$$

and

$$W = \begin{cases} x - 13 & , \text{ if } x \in [13, 14] \\ -x + 15 & , \text{ if } x \in (14, 15] \\ 0 & , \text{ otherwise} \end{cases} .$$

It is clear that this sequence is not convergent to any interval of fuzzy numbers. But, for the roughness degree 6, this sequence is  $r$ -convergent to the interval  $[T, R]$ , where

$$T = \begin{cases} x - 5 & , \text{ if } x \in [5, 6] \\ -x + 7 & , \text{ if } x \in (6, 7] \\ 0 & , \text{ otherwise} \end{cases}$$

and

$$R = \begin{cases} x - 7 & , \text{ if } x \in [7, 8] \\ -x + 9 & , \text{ if } x \in (8, 9] \\ 0 & , \text{ otherwise} \end{cases} .$$

We calculate  $D_I([X, Y], [T, R]) = 5$  since  $D(X, T) = 5$  and  $D(Y, R) = 5$ . Similarly, we have  $D_I([Z, W], [T, R]) = 6$  since  $D(Z, T) = 6$  and  $D(W, R) = 6$ . Consequently, we get

$$D_I([X_n, Y_n], [T, R]) = \begin{cases} 5 & , \text{ if } n \text{ is an odd number} \\ 6 & , \text{ otherwise} \end{cases}$$

which shows that  $D_I([X_n, Y_n], [T, R]) < 6 + \varepsilon$  for each  $n$  and each  $\varepsilon > 0$ . Therefore the sequence  $\{[Z_n, W_n]\}$  is  $r$ -convergent to the interval  $[T, R]$  for  $r \geq 6$ .

**Theorem 3.1.** *If there exists a sequence  $\{[Z_n, W_n]\}$  of intervals of fuzzy numbers such that*

$$[Z_n, W_n] \rightarrow [X, Y], \text{ as } n \rightarrow \infty, \text{ and} \\ D_I([X_n, Y_n], [Z_n, W_n]) \leq r \text{ for each } n \in \mathbb{N}.$$

*then the sequence  $\{[X_n, Y_n]\}$  is  $r$ -convergent to the interval  $[X, Y]$  of fuzzy numbers.*

*Proof.* Since the SIFN  $\{[Z_n, W_n]\}$  converges to the interval  $[X, Y]$ , for every  $\varepsilon > 0$  there exists an  $N(\varepsilon)$  such that

$$D_I([Z_n, W_n], [X, Y]) < \varepsilon \text{ for all } n \geq N(\varepsilon).$$

We have

$$D_I([X_n, Y_n], [X, Y]) \leq D_I([X_n, Y_n], [Z_n, W_n]) + D_I([Z_n, W_n], [X, Y]) \\ < r + \varepsilon \text{ if } n \geq N(\varepsilon),$$

since  $D_I([X_n, Y_n], [Z_n, W_n]) \leq r$ . Therefore the sequence  $\{[X_n, Y_n]\}$  is  $r$ -convergent to the interval  $[X, Y]$ .  $\square$

Now we define the *rough limit set* of a SIFN  $\{[X_n, Y_n]\}$  (namely, the set of all  $r$ -limit points of  $\{[X_n, Y_n]\}$ ) by

$$LIM^r[X_n, Y_n] = \{[X, Y] \in \mathbf{I}[L(\mathbb{R})] : \text{the SIFN } [X_n, Y_n] \text{ is } r\text{-convergent to the interval } [X, Y]\}.$$

Using the metric boundedness for the metric space  $(\mathbf{I}[L(\mathbb{R})], D_I)$ , we can give the following theorem.

**Theorem 3.2.** *If there is a non-negative number  $r$  such that  $LIM^r[X_n, Y_n] \neq \emptyset$  then the sequence  $\{[X_n, Y_n]\}$  is metric bounded.*

*Proof.* Since  $LIM^r[X_n, Y_n] \neq \emptyset$  we have  $[X, Y] \in LIM^r[X_n, Y_n]$ . Take  $\varepsilon := 1$ . By definition, there exists a number  $N = N(1)$  such that

$$D_I([X_n, Y_n], [X, Y]) < r + 1 \text{ for each } n \geq N.$$

On the other hand, if we define  $k := \max\{D_I([X_n, Y_n], [0_1, 0_1]) : n < N\}$ , then we have  $D_I([X_n, Y_n], [0_1, 0_1]) < r + 1 + k$  for each  $n \in \mathbb{N}$ . Hence the proof is completed.  $\square$

The converse of this theorem is also valid. Since its proof is straightforward, we omit it.

Some readers might think that is there any roughness degree in order to rough converge for every SIFN? As can be seen in the following example, the answer of this question is negative.

**Example 3.1.** *Define the sequence  $\{[X_n, Y_n]\}$  as*

$$X_n = \begin{cases} x - (3n - 3) & , \text{ if } x \in [3n - 3, 3n - 2] \\ -x + (3n - 1) & , \text{ if } x \in (3n - 2, 3n - 1] \\ 0 & , \text{ otherwise} \end{cases}$$

and

$$Y_n = \begin{cases} x - (3n - 2) & , \text{ if } x \in [3n - 2, 3n - 1] \\ -x + 3n & , \text{ if } x \in (3n - 1, 3n] \\ 0 & , \text{ otherwise} \end{cases}.$$

*There is no roughness degree for this SIFN being  $r$ -convergent*

After we give a topological property of rough limit set, we put an end to our paper.

**Theorem 3.3.** *The set  $LIM^r[X_n, Y_n]$  is closed.*

*Proof.* Take an arbitrary convergent sequence  $\{[Z_n, W_n]\} \subset LIM^r[X_n, Y_n]$ . Let its limit be the interval  $[Z, W]$ . Fixed  $\varepsilon > 0$ . Then there exists a positive integer  $N = N(\varepsilon)$  such that

$$D_I([Z_n, W_n], [Z, W]) < \frac{\varepsilon}{2} \text{ for all } n \geq N.$$

Since  $[Z_N, W_N] \in LIM^r[X_n, Y_n]$  for this fixed  $N$ , we get

$$D_I([X_n, Y_n], [Z_N, W_N]) < r + \frac{\varepsilon}{2}$$

for each  $n \geq N$ . Therefore we have

$$\begin{aligned} D_I([X_n, Y_n], [X, Y]) &\leq D_I([X_n, Y_n], [Z_N, W_N]) + D_I([Z_N, W_N], [X, Y]) \\ &< r + \varepsilon \end{aligned}$$

for each  $n \geq N$ , which shows that  $[X, Y] \in LIM^r[X_n, Y_n]$ . Hence the set  $LIM^r[X_n, Y_n]$  is closed.  $\square$

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