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SHAPE AND TOPOLOGY DESIGN OF HEAT CONDUCTION USING TOPOLOGICAL SENSITIVITY ANALYSIS METHOD

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Abstract. In this paper, we propose to extend the notion of the topological sensitivity analysis for parabolic equations. It consists in deriving an asymptotic expansion of a shape function with respect to the presence of a small insulator with an adiabatic condition on its boundary in a homogeneous heat conductor. Based on the obtained theoretical results, we propose a fast and accurate one-step algorithm to demonstrate the efficiency of the suggested approach. Furthermore, in order to perform and deepen the theoretical results, one seeks to obtain the optimal design of a heat conductor.

1. Introduction

The classical methods of shape optimization have been studied in [22, 19]. In fact, they are very general method which can handle any type of shape functions and structural models, but they have two main drawbacks: they are computationally costly (because of remeshing) and they do not allow any topology changes. Recently, shape optimization techniques have progressed a lot. In particular, some topological optimization methods have been developed for designing domains whose topology is a priori unknown. Among them, the topological sensitivity analysis which gives a new perspective on shape optimization. It consists in studying the asymptotic behavior of a shape function with respect to the size of a small hole inserted inside the reference domain. Recently, the topological sensitivity analysis method has become a broad, rich and fascinating research area from both theoretical and numerical standpoint. It has proved to be extremely used in the treatment of a lot of applications such as inverse problems, imaging processing, mechanical modeling and damage evolution modeling. This approach was introduced rigorously by A. Schumacher in the context of compliance in linear elasticity with Neumann condition on the boundary of the inserted hole [27]. Generally, we can refer the reader to ([4, 6, 26, 2]) for a completely study of topological asymptotic expansion in order to include arbitrary shaped holes to various Partial Differential Equation; Laplace, Helmholtz, Stokes, Elasticity, Quasi-Stokes. Particularly, an asymptotic expansion with a Neumann condition on the boundary of the inserted hole has been already obtained for the Laplace equation in [9], for the Maxwell equations in [21], for the Helmholtz equations in [2], and for the Stokes equations in [12, 19]. Indeed, all these contributions were treated in a steady-state case and associated to elliptic equations. The aim of this chapter is to extend the notion of the topological sensitivity analysis for the parabolic equation. In this work, we will address two main questions. The first one concerns the theoretical part. We will derive a topological asymptotic expansion for the heat conduction problem with respect to the presence of a small insulator with an adiabatic condition on its boundary. More precisely, we consider a heated design domain $\mathcal{H}$ and a shape function $j(\mathcal{H}) = \int_0^T J(\theta, \mathcal{H})dt$ to be minimized, where θ is the solution to the evolutionary

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heat equation defined on $\mathcal{H}$. For $\varepsilon \geq 0$, let $\mathcal{H}_{z,\varepsilon} = \mathcal{H}\backslash\overline{z+\varepsilon\mathcal{I}}$, the modified domain obtained by inserting a small insulator $\overline{z+\varepsilon\mathcal{I}}$ with $z \in \mathcal{H}$ and where $\mathcal{I} \subset \mathbb{R}^d$ is a fixed bounded domain containing the origin. This insulator is obtained by removing or degenerating some conductive elements. Subsequently, an asymptotic expansion of the shape function is established in the following form:

$$j(\mathcal{H}\backslash\overline{z+\varepsilon\mathcal{I}}) - j(\mathcal{H}) = f(\varepsilon)\delta j(z) + o(f(\varepsilon)),$$

where ε and z denote the diameter and center of the insulator respectively, $f(\varepsilon)$ is an explicit positive function which is expected to vanish in the limit $\varepsilon \to 0$ and δj is the topological gradient. Thus, to minimize the shape function j, we have interest to insert an insulator inside the homogeneous heat conductor where δj is the smallest value. The basic idea is to say that the leading term of the topological asymptotic expansion requires the solution of the boundary integral problem of the stationary exterior Laplace problem and the fundamental solution of the Laplace operator. Concerning the insulator shape, the obtained theoretical results are available for any bounded domain $\mathcal{I} \subset \mathbb{R}^d$ containing the origin and having a connected boundary $\partial\mathcal{I}$ piecewise of class $\mathcal{C}^1$. However, to get an explicit expression of the boundary integral equation, we will take the case of a simple geometry: the unit ball. To the best of our knowledge, [13] was the first publication where this issue was addressed for a time-dependent problem. Yet, the proof presented there was merely formal and it is studied a restricted class of a shape function. In another context, one should mention the work of [8], which investigates the topological sensitivity analysis of shape function for time-dependent problems where an inhomogeneity in the coefficients was considered. The second question concerns the numerical aspect. Based on the obtained theoretical results, we propose a fast and accurate reconstruction algorithm to demonstrate the efficiency of the suggested approach. Furthermore, in order to demonstrate the performance of the obtained theoretical results and asymptotic behavior, one try to find the optimal shape of a heat conductor having one inlet and one outlet. The final shape is obtained using an iterative procedure building a sequence of geometries $(\mathcal{H}_k)_k$ starting with the initial domain $\mathcal{H}_0 = \mathcal{H}$. Knowing $\mathcal{H}_k$, the new domain $\mathcal{H}_{k+1}$ is obtained by inserting an insulator $\mathcal{I}_k$ in the domain. The placement of the insulator $\mathcal{I}_k$ and its shape are defined by a level curve of the topological gradient g_k:

$$\mathcal{I}_k = \Big\{x \in \mathcal{H}_k, \text{ such that } g_k(x) \leq c_k\Big\},$$

where the constant c_k is chosen in such a way that the shape function j decreases as much as possible.

The remainder of the article is arranged as follows: Section 2 is devoted to the model setting. We present the main results in Section 3. We examine the influence of the geometric perturbation on the direct and adjoint problems solutions. We derive a topological sensitivity analysis for the unsteady heat equation with respect to the presence of an arbitrary shaped insulator on which is applied a Neumann boundary condition. Some examples of shape functions are exhibited. In Section 4, some numerical simulations are presented to point out the efficiency and accuracy of the suggested one-step numerical precess. Based on the obtained asymptotic behavior, one try to find out the optimal design of a heat conductor. For the sake of readability, the proofs of all intermediate estimates are reported in Section 5.

2. Setting of the problem

Let $\mathcal{H}$ be a heated design domain fully occupied by conductive materials. We assume that $\mathcal{H}$ is an open and bounded domain of $\mathbb{R}^d, d = 2, 3$, with a smooth boundary Γ. The heat transfer across the domain is solution to the following boundary value problem

$$\begin{cases} \dfrac{\partial\theta}{\partial t} - \Delta\theta &= F \quad \text{in} \quad \mathcal{H} \times (0,T), \\ \theta &= 0 \quad \text{on} \quad \Gamma \times (0,T), \\ \theta(\cdot,0) &= 0 \quad \text{in} \quad \mathcal{H}, \end{cases}$$

where $F \in L^2(0,T,L^2(\mathcal{H}))$ is a generated heat source.

We denote by $\mathcal{H}_{z,\varepsilon} = \mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}$ the modified domain obtained by inserting a small insulator $\mathcal{I}_{z,\varepsilon}$ inside the conductive materials by removing or degenerating some conductive elements (see Fig.

1). We suppose that the insulator has the form $\mathcal{I}_{z,\varepsilon} = z+\varepsilon\mathcal{I} \subset \mathcal{H}$ and characterized by its location $z \in \mathcal{H}$, its size $\varepsilon > 0$ and its shape $\mathcal{I}$, where $\mathcal{I}$ is a fixed open and bounded subdomain of $\mathbb{R}^d$ containing the origin, whose boundary $\partial\mathcal{I}$ is of class $\mathcal{C}^1$.

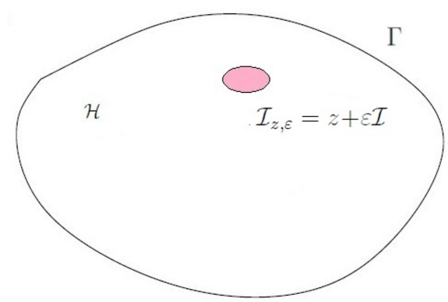


FIGURE 1. the design domain with the presence of a small insulator $\mathcal{I}_{z,\varepsilon}$.

In this work, we assume that the temperature field satisfies an adiabatic condition on the insulator boundary $\partial\mathcal{I}_{z,\varepsilon}$. More precisely, in the presence of the insulator, the temperature θ_ε is defined in the perturbed domain $\mathcal{H}_{z,\varepsilon} = \mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}$ and satisfies the following system:

$$\left\{\begin{array}{rcll} \dfrac{\partial\theta_\varepsilon}{\partial t} - \Delta\theta_\varepsilon & = F & \text{in} & \mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}} \times (0,T), \\ \theta_\varepsilon & = 0 & \text{on} & \Gamma \times (0,T), \\ \nabla\theta_\varepsilon . n & = 0 & \text{on} & \partial\mathcal{I}_{z,\varepsilon} \times (0,T), \\ \theta_\varepsilon(\cdot,0) & = 0 & \text{in} & \mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}. \end{array}\right. \tag{1}$$

Note that in the absence of the insulator (ie $\varepsilon = 0$), we have $\mathcal{H}_{z,\varepsilon} = \mathcal{H}$ and θ_0 is solution to

$$\left\{\begin{array}{rcll} \dfrac{\partial\theta_0}{\partial t} - \Delta\theta_0 & = F & \text{in} & \mathcal{H} \times (0,T), \\ \theta_0 & = 0 & \text{on} & \Gamma \times (0,T), \\ \theta_0(\cdot,0) & = 0 & \text{in} & \mathcal{H}. \end{array}\right. \tag{2}$$

Let us introduce the following functional spaces:

$$\mathcal{V}_\varepsilon = \mathcal{C}(0,T,L^2(\mathcal{H}_{z,\varepsilon})) \cap L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon})),$$

and

$$\mathcal{V}^0_\varepsilon = \{\theta \in \mathcal{V}_\varepsilon,\ \theta = 0 \text{ on } \Gamma \text{ and } \theta(\cdot,0) = 0 \text{ in } \mathcal{H}_{z,\varepsilon}\}.$$

From the weak formulation of the problem (1), we deduce that $\theta_\varepsilon \in \mathcal{V}_\varepsilon$ is a solution to

$$\mathcal{A}_\varepsilon(\theta_\varepsilon, w) \quad = \quad \mathcal{L}_\varepsilon(w), \quad \forall\, w \in \mathcal{V}^0_\varepsilon,$$

where the bilinear form $\mathcal{A}_\varepsilon$ is defined for every u, $w \in \mathcal{V}_\varepsilon$ by

$$\mathcal{A}_\varepsilon(u,w) = \int_0^T \int_{\mathcal{H}_{z,\varepsilon}} \frac{\partial u}{\partial t} w\, dx\, dt + \int_0^T \int_{\mathcal{H}_{z,\varepsilon}} \nabla u . \nabla w\, dx\, dt,$$

$\mathcal{L}_\varepsilon$ is the linear form defined for every $w \in \mathcal{V}_\varepsilon$ by

$$\mathcal{L}_\varepsilon(w) = \int_0^T \int_{\mathcal{H}_{z,\varepsilon}} F\, w\, dx\, dt.$$

It should be noted that, for any $F \in L^2(0,T,L^2(\mathcal{H})$, the problem (1) has a unique solution θ_ε,(For more details, one can see [15, 25, 29]).

Consider now a shape function j having the generic form

$$j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) = \int_0^T J_\varepsilon(\theta_\varepsilon(\cdot,t))dt, \tag{3}$$

where θ_ε is the solution to (1) and J_ε is a scalar function defined on $H^1(\mathcal{H}_{z,\varepsilon})$.

In this paper, we address two mains questions. The first one concerns the theoretical aspect. We will derive a topological sensitivity analysis for the heat transfer problem. This question has been already discussed for many problems such as elasticity [18], Laplace [9], Maxwell [21], Helmholtz [2], and Stokes [12, 19]. All these contribution concern the steady state case and associated to equations of elliptic type. In this paper, we extend this notion for the parabolic equation and we derive a topological asymptotic expansion for the heat transfer problem valid for all shape functions j having the generic form (3) and satisfying the following assumption:

Assumption ($\mathcal{A}$)

i) $\forall \varepsilon \geq 0$, $\forall \theta \in H^1(\mathcal{H}_{z,\varepsilon})$, $J_\varepsilon(\theta) \in L^1(0,T)$.

ii) The function J_ε is differentiable with respect to θ, its derivative being denoted by $DJ_\varepsilon(\theta)$ satisfies

$$\|DJ_\varepsilon(\theta_\varepsilon) - DJ_0(\theta_0)\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{d/2}). \tag{4}$$

iii) There exist a real number δJ, and a scalar function $f : \mathbb{R}_+ \to \mathbb{R}_+$ tending to zero with ε such that

$$\int_0^T (J_\varepsilon(\theta_\varepsilon(\cdot,t)) - J_0(\theta_0(\cdot,t)))dt = \int_0^T DJ_\varepsilon(\theta_\varepsilon)(\theta_\varepsilon - \theta_0)dt + f(\varepsilon)\delta J + o(f(\varepsilon)). \tag{5}$$

The second one concern the numerical aspect. Based on the theoretical results obtained in the first part, we propose a fast and accurate reconstruction algorithm.

To this end, we introduce the adjoint state associated to the minimization of J_ε, solution to

$$\mathcal{A}_\varepsilon(q,p_\varepsilon) = -\int_0^T DJ_\varepsilon(\theta_\varepsilon(\cdot,t))(q)dt, \quad \forall q \in \mathcal{V}_\varepsilon^T, \tag{6}$$

where the functional space $\mathcal{V}_\varepsilon^T$ is defined by:

$$\mathcal{V}_\varepsilon^T = \{q \in \mathcal{V}_\varepsilon, \text{ such that } q = 0 \text{ on } \Gamma \text{ and } q(\cdot,T) = 0\}.$$

Under the assumption ($\mathcal{A}$), one can easily check that the variation of j reads as follows:

$$\begin{aligned} j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) - j(\mathcal{H}) &= \int_0^T \Big(J_\varepsilon(\theta_\varepsilon(\cdot,t)) - J_0(\theta_0(\cdot,t))\Big)dt, \\ &= \int_0^T DJ_\varepsilon(\theta_\varepsilon)(\theta_\varepsilon - \theta_0)dt + f(\varepsilon)\delta J(z) + o(f(\varepsilon)). \end{aligned}$$

Using (6), the last shape function variation can be rewritten as:

$$j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) - j(\mathcal{H}) = \mathcal{A}_\varepsilon(\theta_0 - \theta_\varepsilon, p_\varepsilon) + f(\varepsilon)\delta J(z) + o(f(\varepsilon)), \tag{7}$$

In order to obtain the leading term of the variation $j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) - j(\mathcal{H})$, we start by studying the influence of the geometric perturbation on the direct and adjoint problems solutions.

3. Main results

3.1. **Influence of the geometry perturbation.** We examine here the influence of the geometric perturbation on the direct and adjoint problems solutions. We derive an asymptotic formula outlining the temperature and adjoint variation with respect to the perturbation size ε. These estimates play a fundamental role in the derivation of our topological asymptotic expansion. In order to derive the leading term of the direct and adjoint variation, we introduce the field vector $\psi = {}^t(\psi^1, \psi^2, .., \psi^d)$ where the components ψ^i are solutions to the following exterior problem

$$\left\{ \begin{array}{rlll} -\Delta\psi^i & = 0 & \text{in} & \mathbb{R}^d\backslash\overline{\mathcal{I}}, \\ \nabla\psi^i.n & = -e_i.n & \text{on} & \partial\mathcal{I}, \\ \psi^i & \to 0 & \text{at} & \infty, \end{array} \right. \tag{8}$$

where $\{e_i\}_{\{1\leq i\leq d\}}$ is the canonical basis in $\mathbb{R}^d$.

Based on the simple layer potential representation [16], the function ψ^i can be expressed as

$$\psi^i(y) = \int_{\partial\mathcal{I}} U(y-x)\eta_i(x)ds(x), \;\; \forall y \in \mathbb{R}^d\backslash\overline{\mathcal{I}}, \; 1 \leq i \leq d,$$

where $\eta_i \in H^{-1/2}(\partial\mathcal{I})$ is the unique solution to the boundary integral equation

$$\frac{-\eta_i(y)}{2} + \int_{\partial\mathcal{I}} \frac{\partial U}{\partial n}(y-x)\eta_i(x)ds(x) = -e_i.n, \;\; \forall y \in \partial\mathcal{I}. \tag{9}$$

Here U is the fundamental solution to the Laplace operator.

3.1.1. *Estimate of the temperature variation.* We study the asymptotic behavior of the temperature variation and their gradient which play a key role in the derivation of the topological asymptotic expansion. The following proposition 3.1 describes the behavior of the perturbed temperature θ_ε caused by the presence of a small insulator $\mathcal{I}_{z,\varepsilon}$.

Proposition 3.1. *The perturbed temperature θ_ε satisfies the following estimates:*

$$\|\theta_\varepsilon - \theta_0 - \Theta_\varepsilon\|_{L^\infty(0,T,L^2(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}))} + \|\theta_\varepsilon - \theta_0 - \Theta_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}))} = o(\varepsilon^{d/2}),$$

where Θ_ε is a the leading term of the temperature variation $\theta_\varepsilon - \theta_0$, defined by

$$\Theta_\varepsilon(x,t) = \varepsilon\psi(\tfrac{x-z}{\varepsilon}).\nabla\theta_0(z,t), \; (x,t) \in \mathbb{R}^d\backslash\overline{\mathcal{I}} \times (0,T).$$

The obtained estimates of the temperature variation $\theta_0 - \theta_\varepsilon$ are given by the following Lemma.

Lemma 3.2. *The temperature variation $\theta_\varepsilon - \theta_0$ satisfies the following estimates:*

$$\|\theta_\varepsilon - \theta_0\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} = O(\varepsilon^{d/2}),$$

$$\|\theta_\varepsilon - \theta_0\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{d/2}),$$

$$\|\theta_\varepsilon - \theta_0\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{d/2}),$$

$$\|\nabla(\theta_\varepsilon - \theta_0)\|_{L^2(0,T,L^2(\mathcal{H}\backslash\overline{B(z,R)}))} = o(\varepsilon^{d/2}),$$

where R is a positive real number such that $\overline{B(z,R)} \subset \mathcal{H}$ and $\overline{\mathcal{I}_{z,\varepsilon}} \subset B(z,R)$.

3.1.2. *Estimate of the perturbed adjoint state.* In the following proposition, we present an estimate of the perturbed adjoint state p_ε. This estimate is an indispensable tool for determining our topological asymptotic expansion.

Proposition 3.3. *The perturbed adjoint state p_ε satisfies the following estimate:*

$$\|p_\varepsilon - p_0 - P_\varepsilon\|_{L^\infty(0,T,L^2(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}))} + \|p_\varepsilon - p_0 - P_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}))} = o(\varepsilon^{d/2}),$$

where P_ε is a the leading term of the adjoint variation $p_\varepsilon - p_0$, defined by

$$P_\varepsilon(x,t) = \varepsilon\psi(\tfrac{x-z}{\varepsilon}).\nabla p_0(z,t), \; (x,t) \in \mathbb{R}^d\backslash\overline{\mathcal{I}} \times (0,T).$$

We are now ready to compute the sensitivity variation of the shape function j.

3.2. **Sensitivity variation.** In this section, we will derive a topological asymptotic expansion valid for all shape function verifying the assumption $(\mathcal{A})$. In (7), the term δJ depends on the expression of the function J_ε. This term will be discussed in Subsection 3.3 for some particular shape function example. In this section, we will examine the sensitivity analysis of the term $\mathcal{A}_\varepsilon(\theta_0 - \theta_\varepsilon, p_\varepsilon)$ with respect to ε.

Using the weak formulation of (29) and splitting p_ε into $p_\varepsilon = p_0 + (p_\varepsilon - p_0)$, the term $\mathcal{A}_\varepsilon(\theta_0 - \theta_\varepsilon, p_\varepsilon)$ in (7) can be decomposed as

$$\mathcal{A}_\varepsilon(\theta_0 - \theta_\varepsilon, p_\varepsilon) = \int_0^T \int_{\partial\mathcal{I}_{z,\varepsilon}} \nabla\theta_0.n\, p_0\, ds(x)\, dt + \int_0^T \int_{\partial\mathcal{I}_{z,\varepsilon}} \nabla\theta_0.n(p_\varepsilon - p_0)\, ds(x)\, dt. \tag{10}$$

Next, we will examine each term in (10) separately. The following lemma gives an estimate for the first integral in (10).

Lemma 3.4. *We have the estimate*

$$\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla\theta_0.n\, p_0\, ds\, dt = \varepsilon^d |\mathcal{I}| \Big[\int_0^T F(z,t)p_0(z,t)dt - \int_0^T \frac{\partial \theta_0}{\partial t}(z,t)p_0(z,t)dt - \int_0^T \nabla\theta_0(z,t).\nabla p_0(z,t)dt \Big] + o(\varepsilon^d).$$

The following lemma present the asymptotic behavior for the second integral in (10).

Lemma 3.5. *We have the estimate*

$$\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla\theta_0.n\,(p_\varepsilon - p_0)\, ds\, dt = -\varepsilon^d \int_0^T \Big[\nabla\theta_0(z,t).\Big(\int_{\partial\mathcal{I}} \eta(y) y ds(y)\Big)\nabla p_0(z,t)\Big]dt + \varepsilon^d |\mathcal{I}| \int_0^T \nabla\theta_0(z,t).\nabla p_0(z,t)dt + o(\varepsilon^d).$$

We are now ready to present the main theoretical result of this work. An asymptotic expansion is derived for the unsteady heat equation with respect to the presence of a small geometric perturbation $\mathcal{I}_{z,\varepsilon}$ inside the conductive materials $\mathcal{H}$. To this end, we introduce the polarization matrix $\mathcal{M}$, defined by

$$\mathcal{M}_{i,j} = \int_{\partial\mathcal{I}} \eta_i(y) y_j ds(y),\ 1 \le i,j \le d, \tag{11}$$

where y_j is the j^{th} coordinate of the point $y \in \mathbb{R}^d$ and the density η_i is solution to (9).

Theorem 3.6. *Let $z \in \mathcal{H}$ and j be a shape function on the form $j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) = \int_0^T J_\varepsilon(\theta_\varepsilon(.,t))dt$. If the scalar function J_ε satisfies the assumption $(\mathcal{A})$, then j admits the following asymptotic expansion*

$$j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) - j(\mathcal{H}) = \varepsilon^d \Big[|\mathcal{I}| \int_0^T F(z,t)p_0(z,t)dt - \int_0^T \nabla\theta_0(z,t).\mathcal{M}\nabla p_0(z,t)dt - |\mathcal{I}| \int_0^T \frac{\partial\theta_0}{\partial t}(z,t)p_0(z,t)dt + \delta J(z)\Big] + o(\varepsilon^d). \tag{12}$$

In the particular case where $\mathcal{I}$ is the unit disc $B(0,1)$, we can explicitly determine the density η_i. It is given by

$$\eta_i(y) = -2e_i \cdot y \ \forall y \in \partial\mathcal{I}.$$

From (11), one can deduce that the polarization matrix is given by

$$\mathcal{M} = 2\pi I_2$$

where I_2 denotes the 2×2 identity matrix.

The following corollary shows the asymptotic behavior of the shape function j in the circle shaped case.

Corollary 3.7. *(Circle shaped case) If $\mathcal{I}$ is the unit disc, under the same assumptions of Theorem* 3.6*, the shape function j has the following asymptotic expansion*

$$j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) - j(\mathcal{H}) = \varepsilon^2 \delta j(z) + o(\varepsilon^2),$$

where the topological gradient $\delta j(z)$ is given by:

$$\delta j(z) = \pi \int_0^T F(z,t)p_0(z,t)dt - 2\pi \int_0^T \nabla\theta_0(z,t).\nabla p_0(z,t)dt - \pi \int_0^T \frac{\partial\theta_0}{\partial t}(z,t)p_0(z,t)dt + \delta J(z).$$

3.3. **Shape function examples.** We present here some examples of shape functions having the following form

$$j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) = \int_0^T J_\varepsilon(\theta_\varepsilon(.,t))dt,$$

and satisfying assumption $(\mathcal{A})$ and we compute their variation δJ.

Proposition 3.8. *Consider the function*

$$J_\varepsilon(\theta) = \int_{\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}} |\nabla\theta|^2 dx.$$

Then, J_ε satisfies assumption $(\mathcal{A})$ with

$$DJ_\varepsilon(\theta_\varepsilon(\cdot,t))(w) = 2\int_{\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}} \nabla\theta_\varepsilon(\cdot,t).\nabla w(\cdot,t)dx, \ \ \forall w \in H^1(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}),$$

$$\delta J(z) = -\int_0^T \nabla\theta_0(z,t).\mathcal{M}\nabla\theta_0(z,t)dt, \ \ \forall z \in \mathcal{H}.$$

Proposition 3.9. *Consider the function*

$$J_\varepsilon(\theta) = \int_\Gamma |\theta - \theta_d|^2 dx,$$

where $\theta_d \in L^2(0,T,H^{1/2}(\Gamma))$ is a given state. Then, the function J_ε satisfies assumption $(\mathcal{A})$ with

$$DJ_\varepsilon(\theta_\varepsilon(\cdot,t))(w) = 2\int_\Gamma (\theta_\varepsilon(\cdot,t) - \theta_d(\cdot,t))w(\cdot,t)dx, \ \ \forall w \in H^1(\Gamma),$$

$$\delta J(z) = 0, \ \forall z \in \mathcal{H}.$$

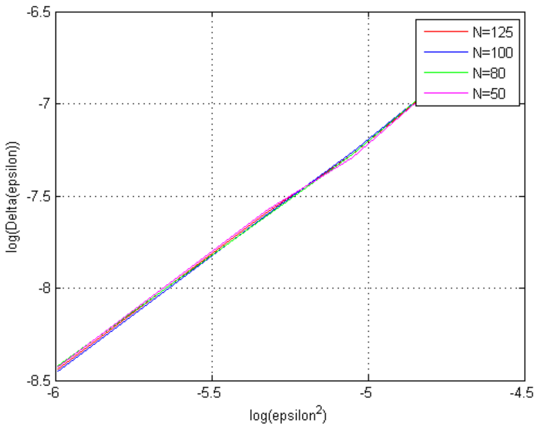
4. Numerical studies

The goal of this section is to point out, by several numerical results, the effectiveness of the main obtained theoretical result obtained in Theorem 3.6. For the sake of simplicity, we restrict ourselves to two-dimensional case. The numerical simulations are run under the software environment Freefem++ [30]. It is a free software based on the finite element method.

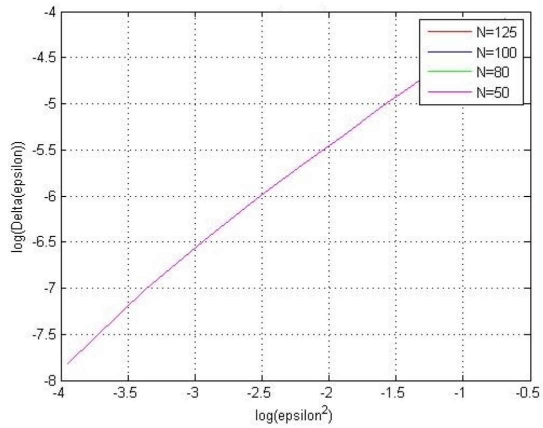
4.1. **Numerical validation.** We aims in this part to study the asymptotic behavior of the function $\Delta_z(\varepsilon)$ defined by $\Delta_z(\varepsilon) = j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) - j(\mathcal{H}) - \varepsilon^2\delta j(z)$ with respect to ε. We expect to prove numerically that the function $\Delta_z(\varepsilon)$ satisfies the theoretical estimate $\Delta_z(\varepsilon) = o(\varepsilon^2)$.

Next, we present some numerical results for arbitrary insulator $\mathcal{I}^i_{z_i,\varepsilon} = z_i + \varepsilon\, B(0,1)$ $i = 1,..,4$ inside the conductive materials $\mathcal{H}$ (We denote here that the initial domain $\mathcal{H} = B(0,1)$). Their location $z_i = (x_i, y_i)$ are described in Table 1.

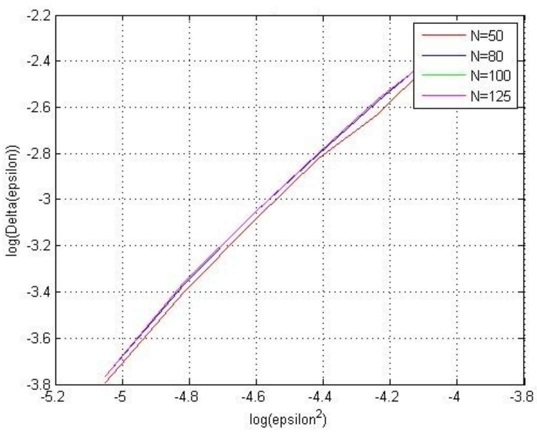
Denoting by β_i the parameter describing the behavior of $\Delta_{z_i}(\varepsilon)$ with respect to ε, i.e. $|\Delta_{z_i}(\varepsilon)| = O(|\varepsilon^2|^{\beta_i})$. Then, one can remark that β_i can be characterized as the slope of the line approximating the variation $\varepsilon \to \log(|\Delta_{z_i}(\varepsilon)|)$ with regard to $\log(|\varepsilon^2|^{\beta_i})$. Starting from this remark, we plot the behavior of the function $\log(|\Delta_{z_i}(\varepsilon)|)$, $i = 1,..,4$ in relation to $\log(\varepsilon^2)$ in Figure 2.



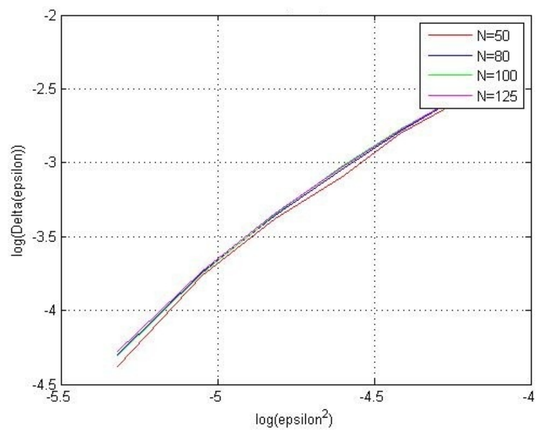
$a)$ Variation of $\log(|\Delta_{z_1}(\varepsilon)|)$ with respect to $\log(\varepsilon^2)$.



$b)$ Variation of $\log(|\Delta_{z_2}(\varepsilon)|)$ with respect to $\log(\varepsilon^2)$.



$c)$ Variation of $\log(|\Delta_{z_3}(\varepsilon)|)$ with respect to $\log(\varepsilon^2)$.



$d)$ Variation of $\log(|\Delta_{z_4}(\varepsilon)|)$ with respect to $\log(\varepsilon^2)$.

FIGURE 2. Variation of $\log(\Delta_{z_i}(\varepsilon))$ with respect to $\log(\varepsilon^2)$ for different value of the mesh $N = 50, 80, 100, 125$.

The obtained slopes β_i, $i = 1, .., 4$ of the curve $\log(|\Delta_{z_i}(\varepsilon)|)$ with respect to $\log(\varepsilon^2)$ are summarized in Table 1. From Table 1, one can observe that the obtained slopes β_i validates the obtained theoretical results: $\Delta_{z_i}(\varepsilon) = o(\varepsilon^2)$, $i = 1, .., 4$.

Insulator $\mathcal{I}^i_{z_i,\varepsilon}$	Emplacement $z_i = (x_i, y_i)$	Obtained slopes β_i
$\mathcal{I}^1_{z_1,\varepsilon}$	$z_1 = (0.3, 0.8)$	$\beta_1 = 1.31$
$\mathcal{I}^2_{z_2,\varepsilon}$	$z_2 = (1, 0.5)$	$\beta_2 = 1.11$
$\mathcal{I}^3_{z_3,\varepsilon}$	$z_3 = (0.5, 0.2)$	$\beta_3 = 1.33$
$\mathcal{I}^4_{z_4,\varepsilon}$	$z_4 = (1.8, 0.7)$	$\beta_4 = 1.52$

TABLE 1. Location of insulator $\mathcal{I}_{z_i,\varepsilon}$ and obtained slopes $\beta_i,\ i = 1, .., 4$.

4.2. **Algorithm and identification results.** We begin in this subsection by describing a simple and accurate numerical identification algorithm. Our numerical procedure is based on the asymptotic expansion established in Theorem 3.6. The main steps of our numerical procedure "One-iteration algorithm" are the following:

- Solve the direct problem (2) in $\mathcal{H}$.

- Solve the associated adjoint problem (6) in $\mathcal{H}$.
- Compute the topological gradient $\delta j(x)$, $\forall x \in \mathcal{H}$.
- Determine the location and shape of the insulator $\mathcal{I}_{z,\varepsilon}$.

The topological gradient gives information on the opportunity to create a small hole (insulator). In fact, the idea is that to insert a small insulator where the topological sensitivity is most negative.

Some illustrative numerical simulations are presented to demonstrate the efficiency of the proposed algorithm. We start by presenting some numerical results concerning the detection of regular insulator in Figures 3 and 4 with different locations and sizes. We consider the case of a small circular shape in Figure 3. In Figure 4, we test our one-step numerical process for the case of an elliptical shape. In one iteration, the location of the regular insulator in the homogeneous conductor is clearly pointed by the negative peak of the topological sensitivity, however, the observation of the isovalues gives a rough idea of its shape. To further emphasize the efficiency of our one-iteration detection procedure, we consider the case of a small insulator having a complex geometry. Figure 5 depicts the isovalues of the topological gradient. The result is quite efficient.

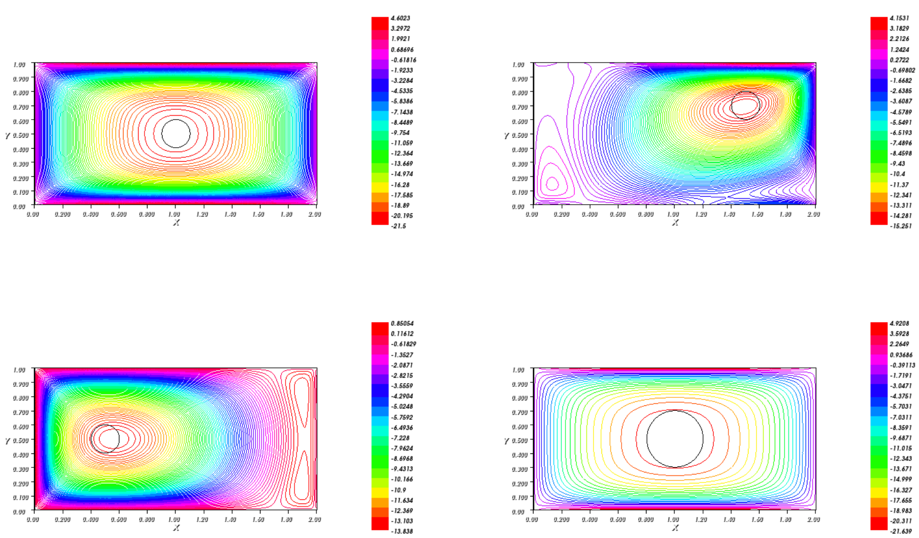


Figure 3. Isovalues of the topological gradient with various locations and sizes of a circular insulator $\mathcal{I}_{z,\varepsilon}$.

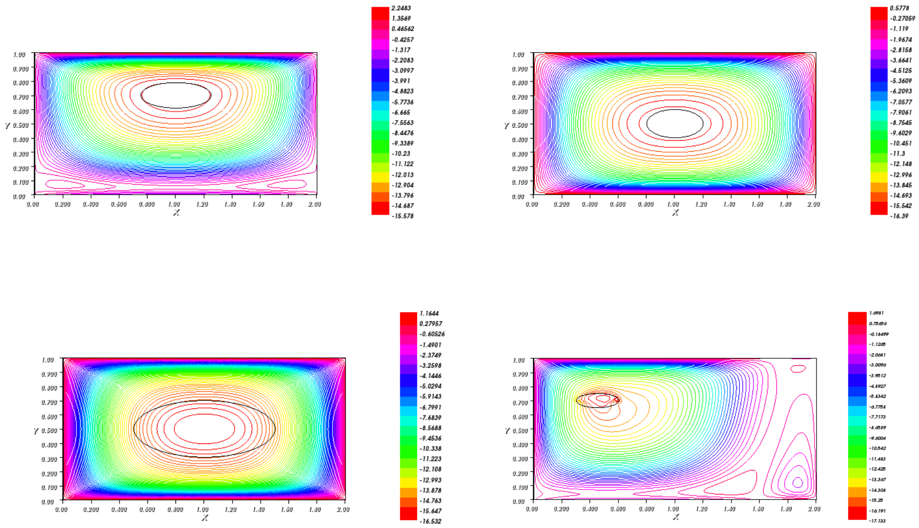


FIGURE 4. Isovalues of the topological gradient with various locations and sizes of an elliptical insulator $\mathcal{I}_{z,\varepsilon}$.

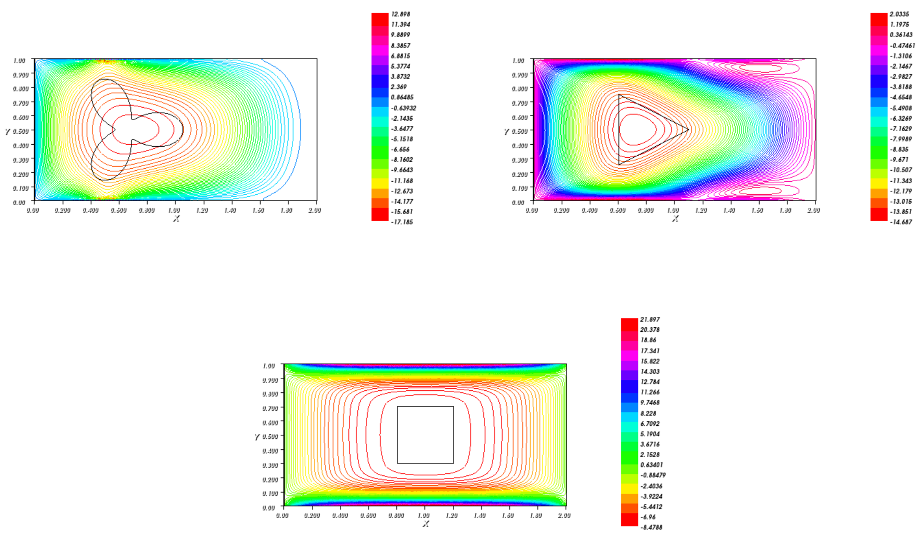


FIGURE 5. Isovalues of the topological gradient showing the location of irregular insulator $\mathcal{I}_{z,\varepsilon}$.

4.3. **Design of a thermal conductor.** In order to confirm the efficiency of the obtained theoretical results, one try to find the optimal design of a thermal conductor $\mathcal{H} = (0,1) \times (0,1)$, having a hole B_R in it center whose radius is $R = 0.2$ and one intlet Γ_1 and one outlet Γ_2. Figure 6 shows the disposition of Γ_1 and Γ_2, and the hole B_R. This test was treated by Novotny et al. in a steady-state case [24].

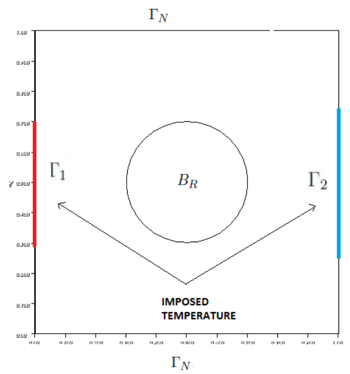


FIGURE 6. A theoretical model of a thermal conductor.

The inlet Γ_1 and the outlet Γ_2 are defined by

$$\Gamma_1 = \{(x, y) \in (0, 1) \times (0, 1),\ x = 0, y \in (0.3, 0.7)\},$$
$$\Gamma_2 = \{(x, y) \in (0, 1) \times (0, 1),\ x = 1, y \in (0.3, 0.7)\}.$$

The aim is to determine the optimal shape $\mathcal{C}^* \subset \mathcal{H}$ of a thermal conductor domain minimizing the design function

$$j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) = \int_0^T \int_{\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}} |\nabla\theta_\varepsilon|^2 dxdt + meas(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}), \tag{13}$$

where θ_ε is solution to (1).

The optimization problem consists in determining the optimal domain solution to

$$\min_{\mathcal{C}\in\mathcal{E}_{ad}} j(\mathcal{C}), \quad \text{such that } |\mathcal{C}| \leq V_{\text{desired}},$$

where $\mathcal{E}_{ad}$ is a set of admissible domains defined by:

$$\mathcal{E}_{ad} = \{\mathcal{C} \subset \mathcal{H} \text{ such that } \Gamma_1 \subset \partial\mathcal{H} \cap \partial\mathcal{C} \text{ and } \Gamma_2 \subset \partial\mathcal{H} \cap \partial\mathcal{C}\}.$$

For the boundary conditions, one has that $\nabla\theta.n = 0$ on Γ_N, $\theta = 100$ on Γ_1 and $\theta = 0$ on Γ_2 respectively. In the hole created via topological gradient, an adiabatic boundary condition is imposed, that is $\nabla\theta.n = 0$ on $\partial\mathcal{I}_{z,\varepsilon}$.

The variation of 13 is given by:

$$\begin{aligned} j(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) - j(\mathcal{H}) &= \int_0^T \int_{\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}} |\nabla\theta_\varepsilon|^2 dxdt + meas(\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}) - \int_0^T \int_{\mathcal{H}} |\nabla\theta_0|^2 dxdt - meas(\mathcal{H}) \\ &= \int_0^T \int_{\mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}} |\nabla\theta_\varepsilon|^2 dxdt - \int_0^T \int_{\mathcal{H}} |\nabla\theta_\varepsilon|^2 dxdt - meas(\mathcal{I}_{z,\varepsilon}), \end{aligned}$$

under proposition 3.8, one can deduce that the topological gradient δj of 13 reads as follows:

$$\begin{aligned} \delta j(z) = \pi \int_0^T F(z, t)p_0(z, t)dt - 2\pi \int_0^T \nabla\theta_0(z, t).\nabla p_0(z, t)dt \\ - \pi \int_0^T \frac{\partial\theta_0}{\partial t}(z, t)p_0(z, t)dt - 2\pi \int_0^T |\nabla\theta_0(z, t)|^2 dt - \pi. \end{aligned}$$

As stated in the works [1, 24], the function δj can be used similarly to descend direction in a topology optimization process. The optimal design is obtained iteratively. We apply an iterative process to build sequence of geometries $(\mathcal{C}_k)_{k\geq 0}$ with $\mathcal{C}_0 = \mathcal{H}$. At the k^{th} iteration the topological gradient δj is computed in $\mathcal{C}_k$ and the new geometry $\mathcal{C}_{k+1} = \mathcal{C}_k\backslash\overline{\mathcal{I}}_k$ is obtained by inserting a small insulator $\mathcal{I}_k$ in the design domain $\mathcal{C}_k$. The insulator $\mathcal{I}_k$ is defined by a level set curve of δj:

$$\mathcal{I}_k = \{x \in \mathcal{C}_k \text{ such that } \delta j(x) \leq c_k < 0\},$$

where c_k is chosen in such a way that the shape function decreases as much as possible. Numerically, the constant c_k depends on the most negative value of the topological gradient δj. We denote also that the adopted stop criterion is over the final volume to be obtained.

Our implementation is based on the following algorithm presented in the context of topological asymptotic in [1, 14].

The algorithm:

- Initialization: choose $\mathcal{C}_0 = \mathcal{H}$ and set $k = 0$.
- Repeat until $|\mathcal{C}_k| \leq V_{\text{desired}}$:
  - Solve the unsteady heat equation in $\mathcal{C}_k$,
  - Solve the associated adjoint problem in $\mathcal{C}_k$,
  - Compute the topological gradient $\delta j_k(x)$, $\forall x \in \mathcal{C}_k$,
  - Determine the insulator $\mathcal{I}_k$,
  - Set $\mathcal{C}_{k+1} = \mathcal{C}_k \backslash \overline{\mathcal{I}}_k$,
  - $k \leftarrow k + 1$.

This numerical process consists in inserting at each iteration an insulator, which their thermal conductivity is very small, where the topological gradient is the smallest value. We illustrate the temperature distribution and the geometries obtained during the optimization process in Figure 8. The final design corresponding to $V_{\text{desired}} = 0.7|\mathcal{H}|$, is obtained after 17 iterations. This academic example shows that topological gradient can be used to determine where the insulator $\mathcal{I}_{z,\varepsilon}$ must be placed, in order to direct the heat flux from Γ_1 (hotter region) to Γ_2 (colder region). Figure 7 describes the variation of the shape function during the optimization process.

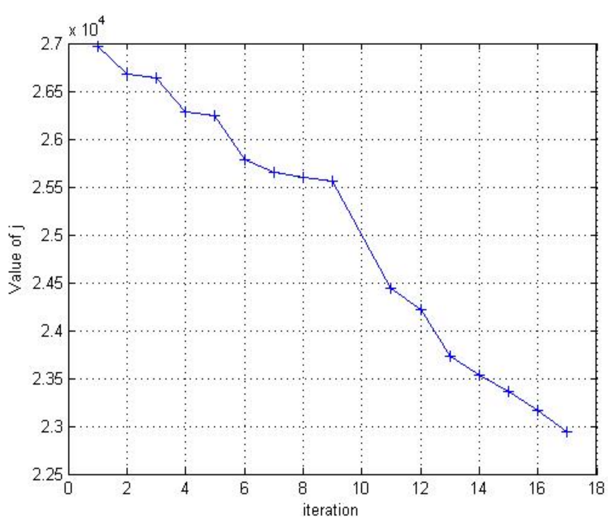


FIGURE 7. Variation of the shape function during optimization process.

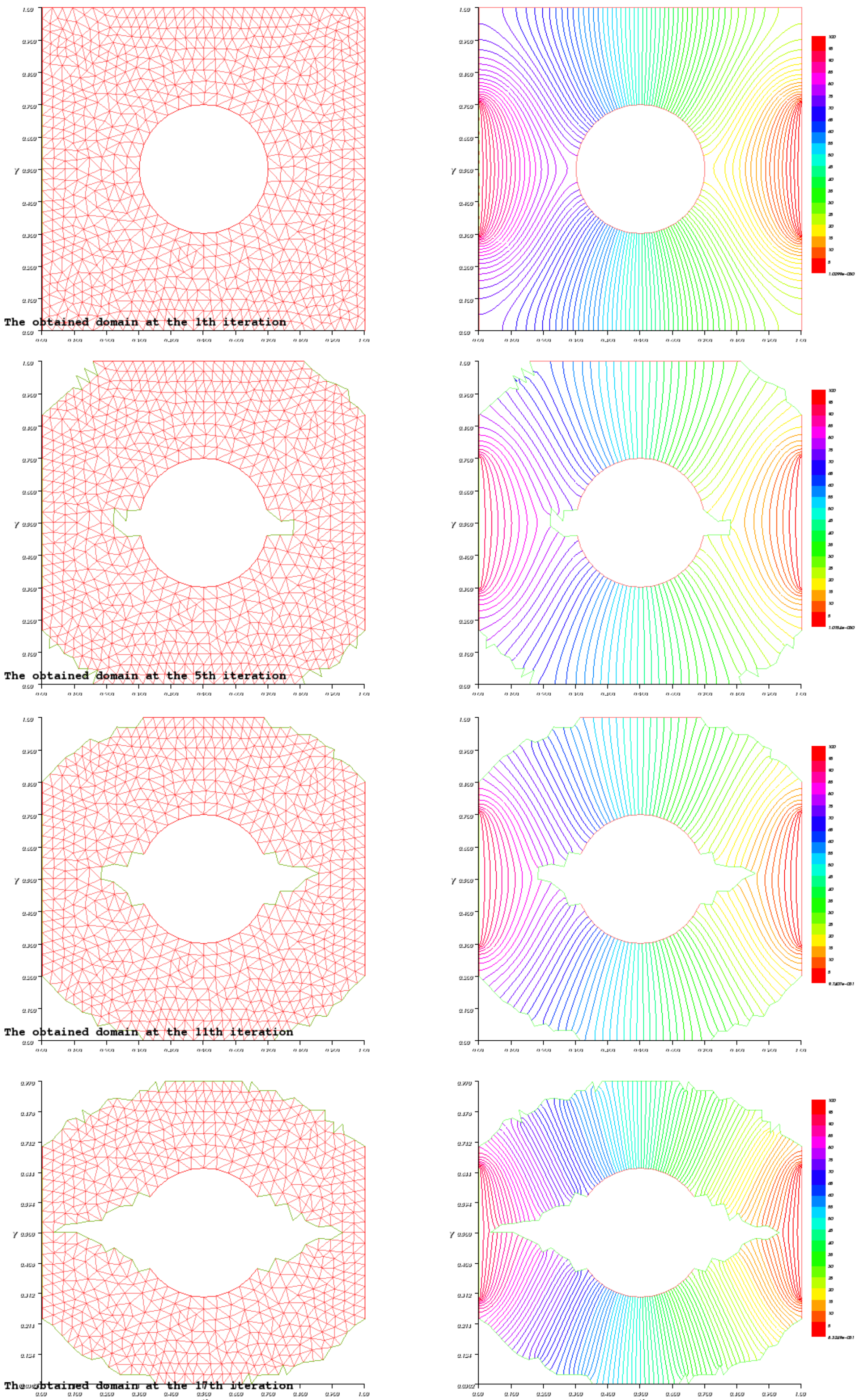


FIGURE 8. Obtained shape for different iterations $k = 0$, $k = 5$, $k = 11$ and $k = 17$.

5. Mathematical analysis

The main objective of this section is to present the proofs of Theorem 3.6, Lemmas 3.2, 3.4 and 3.5 and Propositions 3.1, 3.3, 3.8 and 3.9.

5.1. **Regularity assumptions and preliminary estimates.** In order to enable this study, we make some additional regularity assumptions on the direct and adjoint solutions.

There exists two neighborhood $\mathcal{I}_1$ and $\mathcal{I}_2$ of z such that

$$DJ_0(\theta_0) \in L^2(0,T;H^2(\mathcal{I}_1)) \cap H^1(0,T;L^2(\mathcal{I}_1)). \tag{14}$$

$$F \in L^2(0,T;H^2(\mathcal{I}_2)) \cap H^1(0,T;L^2(\mathcal{I}_2)) \tag{15}$$

If (14) and (15) hold, then we have

$$\begin{aligned} p_0 &\in L^2(0,T;H^3(\tilde{\mathcal{I}})) \cap H^2(0,T;L^2(\tilde{\mathcal{I}})),\\ \theta_0 &\in L^2(0,T;H^3(\tilde{\mathcal{I}})) \cap H^2(0,T;L^2(\tilde{\mathcal{I}})). \end{aligned}$$

for all subdomain $\tilde{\mathcal{I}}$ containing z and $\tilde{\mathcal{I}} \subset \mathcal{I}_1$, $\tilde{\mathcal{I}} \subset \mathcal{I}_2$.

Next, we give some preliminary results which are essential for our analysis. We first recall some estimates describing the behavior of the state ψ_ε, solution to (16).

Lemma 5.1. [7] *The state ψ_ε defined by*

$$\psi_\varepsilon(x) = \varepsilon\psi(\frac{x-z}{\varepsilon}) \ \forall x \in \mathbb{R}^d, \tag{16}$$

admits the following estimates

$$\begin{aligned} \|\psi_\varepsilon\|_{L^2(\mathcal{H}\setminus\overline{\mathcal{I}_{z,\varepsilon}})} &= o(\varepsilon^{\frac{d}{2}}),\\ \|\nabla\psi_\varepsilon\|_{L^2(\mathcal{H}\setminus\overline{\mathcal{I}_{z,\varepsilon}})} &= O(\varepsilon^{\frac{d}{2}}),\\ \|\nabla\psi_\varepsilon\|_{L^2(\mathcal{H}\setminus\overline{B(z,R)})} &= O(\varepsilon^{d}), \end{aligned}$$

where R is a positive real number such that $\overline{B(z,R)} \subset \mathcal{H}$ and $\overline{\mathcal{I}_{z,\varepsilon}} \subset B(z,R)$. In the sequel, C represents any constant, independent of ε, that may change from place to place.

Let us now study the asymptotic behavior of the perturbed temperature caused by the presence of a small geometric perturbation $\mathcal{I}_{z,\varepsilon}$ inside the conductive material $\mathcal{H}$.

5.2. **Proof of Proposition** 3.1. From (1), (2) and using the fact that $\Theta_\varepsilon(\cdot,0)=0$, we deduce that the temperature variation $\vartheta_\varepsilon = \theta_\varepsilon - \theta_0 - \Theta_\varepsilon$ satisfies the following system:

$$\left\{\begin{array}{lll} \dfrac{\partial\vartheta_\varepsilon}{\partial t} - \Delta\vartheta_\varepsilon = \dfrac{\partial\Theta_\varepsilon}{\partial t} & \text{in} & \mathcal{H}_{z,\varepsilon}\times(0,T),\\ \vartheta_\varepsilon = -\Theta_\varepsilon & \text{on} & \Gamma\times(0,T),\\ \nabla\vartheta_\varepsilon.n = -\nabla\theta_0.n + \nabla\theta_0(z,t).n & \text{on} & \partial\mathcal{I}_{z,\varepsilon}\times(0,T),\\ \vartheta_\varepsilon(\cdot,0) = 0 & \text{in} & \mathcal{H}_{z,\varepsilon}. \end{array}\right. \tag{17}$$

In order to demonstrate the estimate of the perturbed temperature, we begin by splitting ϑ_ε into

$$\vartheta_\varepsilon = \vartheta_{1,\varepsilon} + \vartheta_{2,\varepsilon},$$

where $\vartheta_{1,\varepsilon}$ and $\vartheta_{2,\varepsilon}$ are respectively solutions to the following systems:

$$\left\{\begin{array}{lll} \dfrac{\partial\vartheta_{1,\varepsilon}}{\partial t} - \Delta\vartheta_{1,\varepsilon} = \dfrac{\partial\Theta_\varepsilon}{\partial t} & \text{in} & \mathcal{H}_{z,\varepsilon}\times(0,T),\\ \vartheta_{1,\varepsilon} = 0 & \text{on} & \Gamma\times(0,T),\\ \nabla\vartheta_{1,\varepsilon}.n = -\nabla\theta_0.n + \nabla\theta_0(z,t).n & \text{on} & \partial\mathcal{I}_{z,\varepsilon}\times(0,T),\\ \vartheta_{1,\varepsilon}(\cdot,0) = 0 & \text{in} & \mathcal{H}_{z,\varepsilon}. \end{array}\right. \tag{18}$$

and

$$\left\{\begin{array}{rlll} \dfrac{\partial \vartheta_{2,\varepsilon}}{\partial t} - \Delta \vartheta_{2,\varepsilon} & = 0 & \text{in} & \mathcal{H}_{z,\varepsilon} \times (0,T), \\ \vartheta_{2,\varepsilon} & = -\Theta_\varepsilon & \text{on} & \Gamma \times (0,T), \\ \nabla \vartheta_{2,\varepsilon}.n & = 0 & \text{on} & \partial \mathcal{I}_{z,\varepsilon} \times (0,T), \\ \vartheta_{2,\varepsilon}(\cdot,0) & = 0 & \text{in} & \mathcal{H}_{z,\varepsilon}. \end{array}\right. \tag{19}$$

From the weak formulation of (18), we get for all $t_0 \in (0,T)$

$$\frac{1}{2}\int_{\mathcal{H}_{z,\varepsilon}} |\vartheta_{1,\varepsilon}(\cdot,t_0)|^2 dx + \int_0^{t_0}\int_{\mathcal{H}_{z,\varepsilon}} |\nabla \vartheta_{1,\varepsilon}|^2 dxdt \le \Big[\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} (\nabla\theta_0(z,t).n - \nabla\theta_0.n)\,\vartheta_{1,\varepsilon} dsdt$$
$$+ \int_0^T \int_{\mathcal{H}_{z,\varepsilon}} \frac{\partial \Theta_\varepsilon}{\partial t}\,\vartheta_{1,\varepsilon} dxdt\Big].$$

Using Cauchy-Schwarz and Poincaré inequalities, we obtain

$$\frac{1}{2}\int_{\mathcal{H}_{z,\varepsilon}} |\vartheta_{1,\varepsilon}(\cdot,t_0)|^2 dx + \int_0^{t_0}\int_{\mathcal{H}_{z,\varepsilon}} |\nabla \vartheta_{1,\varepsilon}|^2 dxdt \le \Big[\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} (\nabla\theta_0(z,t).n - \nabla\theta_0.n)\,\vartheta_{1,\varepsilon} dsdt$$
$$+ \left\|\frac{\partial \Theta_\varepsilon}{\partial t}\right\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}\Big],$$

Furthermore, using Poincaré inequality and taking the supremum for all $t_0 \in (0,T)$, we get

$$\|\vartheta_{1,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} \le \Big[\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} (\nabla\theta_0(z,t).n - \nabla\theta_0.n)\,\vartheta_{1,\varepsilon} dsdt$$
$$+ \left\|\frac{\partial \Theta_\varepsilon}{\partial t}\right\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}\Big].$$

Next, we will derive an estimate of each term of the right side of the above inequality separately:

- Estimate of the term $\left\|\dfrac{\partial \Theta_\varepsilon}{\partial t}\right\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))}$:

  We recall that
  $$\Theta_\varepsilon(x,t) = \psi_\varepsilon(x).\nabla\theta_0(z,t), \;\; \forall (x,t) \in \mathbb{R}^d \times (0,T),$$
  then, we have
  $$\begin{aligned} \left\|\frac{\partial \Theta_\varepsilon}{\partial t}\right\|^2_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} &= \int_0^T \left|\frac{\partial \nabla\theta_0(z,t)}{\partial t}\right|^2 \|\psi_\varepsilon\|^2_{L^2(\mathcal{H}_{z,\varepsilon})} dt \\ &\le C\|\psi_\varepsilon\|^2_{L^2(\mathcal{H}_{z,\varepsilon})} \|\nabla\theta_0(z,\cdot)\|^2_{H^1(0,T)}. \end{aligned}$$

  Using the fact that $\nabla\theta_0(z,\cdot) \in H^1(0,T)$ and Lemma 5.1, we obtain
  $$\left\|\frac{\partial \Theta_\varepsilon}{\partial t}\right\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \le C\varepsilon^{\frac{d}{2}+1}. \tag{20}$$

- Estimate of the term $\displaystyle\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} (\nabla\theta_0(z,t).n - \nabla\theta_0.n)\,\vartheta_{1,\varepsilon} ds\, dt$:

Using Cauchy-Schwarz inequality and Trace theorem, we obtain

$$\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} (\nabla\theta_0(z,t).n - \nabla\theta_0.n)\vartheta_{1,\varepsilon} dsdt \le C\int_0^T \|\theta_0(z,t) - \theta_0\|_{H^1(\mathcal{I}_{z,\varepsilon})} \|\vartheta_{1,\varepsilon}\|_{H^1(\mathcal{H}_{z,\varepsilon})} dt,$$

then, based on the Cauchy-Schwarz inequality and the Taylor's Theorem in a neighborhood of the point z, we have

$$\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} (\nabla\theta_0(z,t).n - \nabla\theta_0.n)\vartheta_{1,\varepsilon} dsdt \le C\varepsilon \|\nabla\theta_0(\xi_y,t)y\|_{L^2(0,T,H^1(\mathcal{I}_{z,\varepsilon}))} \|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))},$$

Moreover, from the change of variable $x = z + \varepsilon y$ and the fact that $\nabla\theta_0$ is regular near z, we get

$$\int_0^T \int_{\partial\mathcal{I}_{z,\varepsilon}} (\nabla\theta_0(z,t).n - \nabla\theta_0.n)\vartheta_{1,\varepsilon} ds dt \leq C\varepsilon^{\frac{d}{2}+1}\|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}. \tag{21}$$

Gathering the previous results (20) and (21), we have

$$\|\vartheta_{1,\varepsilon}\|^2_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\vartheta_{1,\varepsilon}\|^2_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} \leq C\,\varepsilon^{\frac{d}{2}+1}\|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}. \tag{22}$$

From Young's inequality, we deduce

$$\begin{aligned}\|\vartheta_{1,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))}\|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} &\leq \frac{1}{2}\|\vartheta_{1,\varepsilon}\|^2_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \frac{1}{2}\|\vartheta_{1,\varepsilon}\|^2_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}\\ &\leq C\,\varepsilon^{\frac{d}{2}+1}\|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))},\end{aligned}$$

hence

$$\|\vartheta_{1,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \leq C\varepsilon^{\frac{d}{2}+1}.$$

From (22), we have

$$\|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} \leq C\varepsilon^{\frac{d}{2}+1}.$$

Thus, we get

$$\|\vartheta_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} + \|\vartheta_{1,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}}). \tag{23}$$

In order to estimate $\vartheta_{2,\varepsilon}$, we consider a smooth function $e : \mathcal{H}_{z,\varepsilon} \to \mathbb{R}$ such that $e = 0$ in $\mathcal{H}_{z,\varepsilon} \cup \partial\mathcal{I}_{z,\varepsilon}$, and $e = 1$ on Γ. Then we set

$$\tilde{\Theta}_\varepsilon(x,t) = \Theta_\varepsilon(x,t)e(x), \tag{24}$$

$$\tilde{\vartheta}_{2,\varepsilon}(x,t) = \vartheta_{2,\varepsilon}(x,t) + \tilde{\Theta}_\varepsilon(x,t). \tag{25}$$

In fact, from equations (24) and (25), we deduce that the state $\tilde{\vartheta}_{2,\varepsilon}$ satisfies the following system:

$$\begin{cases} \dfrac{\partial\tilde{\vartheta}_{2,\varepsilon}}{\partial t} - \Delta\tilde{\vartheta}_{2,\varepsilon} &= -\dfrac{\partial\tilde{\Theta}_\varepsilon}{\partial t} - \Delta\tilde{\Theta}_\varepsilon & \text{in} & \mathcal{H}_{z,\varepsilon}\times(0,T),\\ \tilde{\vartheta}_{2,\varepsilon} &= 0 & \text{on} & \Gamma\times(0,T),\\ \nabla\tilde{\vartheta}_{2,\varepsilon}.n &= 0 & \text{on} & \partial\mathcal{I}_{z,\varepsilon}\times(0,T),\\ \tilde{\vartheta}_{2,\varepsilon}(\cdot,0) &= 0 & \text{in} & \mathcal{H}_{z,\varepsilon}. \end{cases} \tag{26}$$

Using the weak formulation, we obtain that

$$\|\tilde{\vartheta}_{2,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\tilde{\vartheta}_{2,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} \leq C\Big[\Big\|\frac{\partial\tilde{\Theta}_\varepsilon}{\partial t}\Big\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \Big\|\tilde{\Theta}_\varepsilon\Big\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}\Big].$$

Taking into account (25), we get

$$\begin{aligned}\|\vartheta_{2,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\vartheta_{2,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} &\leq \Big(\|\tilde{\vartheta}_{2,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\tilde{\vartheta}_{2,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}\\ &\quad + \|\tilde{\Theta}_\varepsilon\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\tilde{\Theta}_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}\Big)\\ &\leq C\Big(\Big\|\tilde{\Theta}_\varepsilon\Big\|_{H^1(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \Big\|\tilde{\Theta}_\varepsilon\Big\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}\Big)\\ &\leq C\|\nabla\theta_0(z,\cdot)\|_{H^1(0,T)}\|\psi_\varepsilon\|_{H^1(\mathcal{H}\setminus\overline{B(z,R)})}.\end{aligned}$$

It results from Lemma 5.1 that

$$\|\vartheta_{2,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\vartheta_{2,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}}). \tag{27}$$

Therefore, combining (23) and (27), we obtain the desired estimate

$$\|\vartheta_\varepsilon\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\vartheta_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}}). \tag{28}$$

□

5.3. **Proof of Lemma** 3.2. It follows from (1) and (2) that $\mathcal{T}_\varepsilon = \theta_\varepsilon - \theta_0$ is solution to the following system:

$$\begin{cases} \dfrac{\partial \mathcal{T}_\varepsilon}{\partial t} - \Delta \mathcal{T}_\varepsilon &= 0 \quad \text{in} \quad \mathcal{H}_{z,\varepsilon} \times (0,T), \\ \mathcal{T}_\varepsilon &= 0 \quad \text{on} \quad \Gamma \times (0,T), \\ \nabla \mathcal{T}_\varepsilon . n &= -\nabla \theta_0 . n \quad \text{on} \quad \partial \mathcal{I}_{z,\varepsilon} \times (0,T), \\ \mathcal{T}_\varepsilon(\cdot, 0) &= 0 \quad \text{in} \quad \mathcal{H}_{z,\varepsilon}. \end{cases} \tag{29}$$

From the weak formulation of the previous system, we obtain for all $t_0 \in (0,T)$

$$\frac{1}{2}\int_{\mathcal{H}_{z,\varepsilon}} |\mathcal{T}_\varepsilon(\cdot,t_0)|^2 dx + \int_0^{t_0}\int_{\mathcal{H}_{z,\varepsilon}} |\nabla \mathcal{T}_\varepsilon(\cdot,t)|^2 dxdt \leq \int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla\theta_0 . n \, \mathcal{T}_\varepsilon \, ds \, dt.$$

It then follows from Cauchy-Schwarz inequality and Trace theorem that

$$\frac{1}{2}\int_{\mathcal{H}_{z,\varepsilon}} |\mathcal{T}_\varepsilon(\cdot,t_0)|^2 dx + \int_0^{t_0}\int_{\mathcal{H}_{z,\varepsilon}} |\nabla \mathcal{T}_\varepsilon(\cdot,t)|^2 dxdt \leq \int_0^T \|\theta_0(\cdot,t)\|_{H^1(\mathcal{I}_{z,\varepsilon})} \|\mathcal{T}_\varepsilon(\cdot,t)\|_{H^1(\mathcal{H}_{z,\varepsilon})} dt. \tag{30}$$

One can easily see from (30) that

$$\int_0^{t_0}\int_{\mathcal{H}_{z,\varepsilon}} |\nabla \mathcal{T}_\varepsilon(\cdot,t)|^2 dxdt \leq \int_0^T \|\theta_0(\cdot,t)\|_{H^1(\mathcal{I}_{z,\varepsilon})} \|\mathcal{T}_\varepsilon(\cdot,t)\|_{H^1(\mathcal{H}_{z,\varepsilon})} dt.$$

Using Cauchy-Schwarz inequality, the change of variable $x = z + \varepsilon y$, and the regularity of θ_0 near z, we get

$$\int_0^{T}\int_{\mathcal{H}_{z,\varepsilon}} |\nabla \mathcal{T}_\varepsilon(\cdot,t)|^2 dxdt \leq C\varepsilon^{\frac{d}{2}} \|\mathcal{T}_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}. \tag{31}$$

Based on Poincaré's inequality, we obtain

$$\|\mathcal{T}_\varepsilon\|^2_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} \leq C\varepsilon^{\frac{d}{2}} \|\mathcal{T}_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))},$$

hence

$$\|\mathcal{T}_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} = O(\varepsilon^{\frac{d}{2}}). \tag{32}$$

It results from (30) that

$$\frac{1}{2}\int_{\mathcal{H}_{z,\varepsilon}} |\mathcal{T}_\varepsilon(\cdot,t_0)|^2 dx \leq C\varepsilon^{\frac{d}{2}} \|\mathcal{T}_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}, \tag{33}$$

Taking the supremum for all $t_0 \in (0,T)$ and using (32), we obtain

$$\|\mathcal{T}_\varepsilon\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}}).$$

Integrating (33) between 0 and T, we get

$$\int_0^{T}\int_{\mathcal{H}_{z,\varepsilon}} |\mathcal{T}_\varepsilon(\cdot,t)|^2 dxdt \leq C\varepsilon^{\frac{d}{2}} \|\mathcal{T}_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}, \tag{34}$$

From (32), we obtain immediately that

$$\|\mathcal{T}_\varepsilon\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}}). \tag{35}$$

For the last estimate, we recall that

$$\mathcal{T}_\varepsilon = \Theta_\varepsilon + \vartheta_\varepsilon,$$

then we have

$$\begin{aligned} \|\nabla \mathcal{T}_\varepsilon\|_{L^2(0,T,L^2(\mathcal{H}\setminus\overline{B(z,R)}))} &= \|\nabla\vartheta_\varepsilon + \nabla\Theta_\varepsilon\|_{L^2(0,T,L^2(\mathcal{H}\setminus\overline{B(z,R)}))} \\ &\leq \|\nabla\vartheta_\varepsilon\|_{L^2(0,T,L^2(\mathcal{H}\setminus\overline{B(z,R)}))} + \|\nabla\Theta_\varepsilon\|_{L^2(0,T,L^2(\mathcal{H}\setminus\overline{B(z,R)}))} \\ &\leq \|\vartheta_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} + \|\nabla\theta_0(z,\cdot)\|_{L^2(0,T)} \|\nabla\psi_\varepsilon\|_{L^2(\mathcal{H}\setminus\overline{B(z,R)})}. \end{aligned}$$

Moreover, based on Proposition 3.1 and Lemma 5.1, we deduce the following estimate

$$\|\nabla \mathcal{T}_\varepsilon\|_{L^2(0,T,L^2(\mathcal{H}\setminus\overline{B(z,R)}))} = o(\varepsilon^{\frac{d}{2}}).$$

□

In this section, we will discuss the asymptotic behavior of the perturbed adjoint state with respect to the presence of a small insulator $\mathcal{I}_{z,\varepsilon}$ inside the heated domain $\mathcal{H}$.

5.4. **Proof of Proposition** 3.3. One can easily see from (6) that the adjoint variation $z_\varepsilon = p_\varepsilon - p_0$ satisfies the following system:

$$\begin{cases} -\dfrac{\partial z_\varepsilon}{\partial t} - \Delta z_\varepsilon & = DJ_0(\theta_0) - DJ_\varepsilon(\theta_\varepsilon) \quad \text{in} \quad \mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}} \times (0,T), \\ z_\varepsilon & = 0 \quad \text{on} \quad \Gamma \times (0,T), \\ \nabla z_\varepsilon . n & = -\nabla p_0 . n \quad \text{on} \quad \partial\mathcal{I}_{z,\varepsilon} \times (0,T), \\ z_\varepsilon(\cdot, T) & = 0 \quad \text{in} \quad \mathcal{H}\backslash\overline{\mathcal{I}_{z,\varepsilon}}. \end{cases} \tag{36}$$

Then, denoting by $\mathcal{P}_\varepsilon = p_\varepsilon - p_0 - P_\varepsilon$ and using the fact that $P_\varepsilon(\cdot, T) = 0$, then we deduce that $\mathcal{P}_\varepsilon$ is solution to the following system:

$$\begin{cases} -\dfrac{\partial \mathcal{P}_\varepsilon}{\partial t} - \Delta \mathcal{P}_\varepsilon & = DJ_0(\theta_0) - DJ_\varepsilon(\theta_\varepsilon) + \dfrac{\partial P_\varepsilon}{\partial t} \quad \text{in} \quad \mathcal{H}_{z,\varepsilon} \times (0,T), \\ \mathcal{P}_\varepsilon & = -P_\varepsilon \quad \text{on} \quad \Gamma \times (0,T), \\ \nabla \mathcal{P}_\varepsilon . n & = -\nabla p_0 . n + \nabla p_0(z,t) . n \quad \text{on} \quad \partial\mathcal{I}_{z,\varepsilon} \times (0,T), \\ \mathcal{P}_\varepsilon(\cdot, T) & = 0 \quad \text{in} \quad \mathcal{H}_{z,\varepsilon}. \end{cases} \tag{37}$$

In order to separate difficulties, we split $\mathcal{P}_\varepsilon$ into

$$\mathcal{P}_\varepsilon = \mathcal{P}_{1,\varepsilon} + \mathcal{P}_{2,\varepsilon},$$

where $\mathcal{P}_{1,\varepsilon}$ and $\mathcal{P}_{2,\varepsilon}$ satisfy respectively the following systems:

$$\begin{cases} -\dfrac{\partial \mathcal{P}_{1,\varepsilon}}{\partial t} - \Delta \mathcal{P}_{1,\varepsilon} & = DJ_0(\theta_0) - DJ_\varepsilon(\theta_\varepsilon) + \dfrac{\partial P_\varepsilon}{\partial t} \quad \text{in} \quad \mathcal{H}_{z,\varepsilon} \times (0,T), \\ \mathcal{P}_{1,\varepsilon} & = 0 \quad \text{on} \quad \Gamma \times (0,T), \\ \nabla \mathcal{P}_{1,\varepsilon} . n & = -\nabla p_0 . n + \nabla p_0(z,t) . n \quad \text{on} \quad \partial\mathcal{I}_{z,\varepsilon} \times (0,T), \\ \mathcal{P}_{1,\varepsilon}(\cdot, T) & = 0 \quad \text{in} \quad \mathcal{H}_{z,\varepsilon}. \end{cases} \tag{38}$$

and

$$\begin{cases} -\dfrac{\partial \mathcal{P}_{2,\varepsilon}}{\partial t} - \Delta \mathcal{P}_{2,\varepsilon} & = 0 \quad \text{in} \quad \mathcal{H}_{z,\varepsilon} \times (0,T), \\ \mathcal{P}_{2,\varepsilon} & = -P_\varepsilon \quad \text{on} \quad \Gamma \times (0,T), \\ \nabla \mathcal{P}_{2,\varepsilon} . n & = 0 \quad \text{on} \quad \partial\mathcal{I}_{z,\varepsilon} \times (0,T), \\ \mathcal{P}_{2,\varepsilon}(\cdot, T) & = 0 \quad \text{in} \quad \mathcal{H}_{z,\varepsilon}. \end{cases} \tag{39}$$

From the weak formulation of (38) and applying Cauchy-Schwarz inequality, we get for all $t_0 \in (0,T)$

$$\frac{1}{2}\int_{\mathcal{H}_{z,\varepsilon}} |\mathcal{P}_{1,\varepsilon}(\cdot, t_0)|^2 dx + \int_{t_0}^T \int_{\mathcal{H}_{z,\varepsilon}} |\nabla \mathcal{P}_{1,\varepsilon}|^2 dx dt \leq \Big[\int_0^T \int_{\partial\mathcal{I}_{z,\varepsilon}} (\nabla p_0(z,t).n - \nabla p_0 . n)\, \mathcal{P}_{1,\varepsilon} ds dt$$
$$+ \Big(\|DJ_0(\theta_0) - DJ_\varepsilon(\theta_\varepsilon)\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \left\| \frac{\partial P_\varepsilon}{\partial t} \right\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \Big) \|\nabla \mathcal{P}_{1,\varepsilon}\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \Big].$$

Taking the supremum for all $t_0 \in (0,T)$ and applying Poincaré inequality, we obtain

$$\|\mathcal{P}_{1,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\mathcal{P}_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} \leq \Big[\int_0^T \int_{\partial\mathcal{I}_{z,\varepsilon}} (\nabla p_0(z,t).n - \nabla p_0 . n) \mathcal{P}_{1,\varepsilon} ds dt$$
$$+ \Big(\|DJ_0(\theta_0) - DJ_\varepsilon(\theta_\varepsilon)\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \left\| \frac{\partial P_\varepsilon}{\partial t} \right\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \Big) \|\mathcal{P}_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} \Big].$$

Next, we shall estimate each term of the right side of the above inequality: Firstly, from assumption $(\mathcal{A})$, we have

$$\|DJ_0(\theta_0) - DJ_\varepsilon(\theta_\varepsilon)\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}}). \tag{40}$$

Secondly, we will provide an estimate of the term $\left\| \dfrac{\partial P_\varepsilon}{\partial t} \right\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))}$:

We denote by

$$P_\varepsilon(x,t) = \psi_\varepsilon(x) \cdot \nabla p_0(z,t), \quad \forall (x,t) \in \mathbb{R}^d \times (0,T),$$

therefore, we get

$$\left\| \frac{\partial P_\varepsilon}{\partial t} \right\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \leq \|\psi_\varepsilon\|_{L^2(\mathcal{H}_{z,\varepsilon})} \|\nabla p_0(z,\cdot)\|_{H^1(0,T)}.$$

Due to Lemma 5.1 and the fact that $\nabla p_0(z,\cdot) \in H^1(0,T)$, we obtain

$$\left\| \frac{\partial P_\varepsilon}{\partial t} \right\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \leq C\varepsilon^{\frac{d}{2}+1}. \tag{41}$$

Now, we will estimate the term $\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} (\nabla p_0(z,t).n - \nabla p_0.n)\mathcal{P}_{1,\varepsilon} dsdt$:
It results from Cauchy-schwarz inequality, Trace theorem, the change of variable $x = z + \varepsilon y$, and the fact that ∇p_0 is regular near z that

$$\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} (\nabla p_0(z,t).n - \nabla p_0.n)\mathcal{P}_{1,\varepsilon} dsdt \leq C\varepsilon^{\frac{d}{2}+1} \|\mathcal{P}_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} \tag{42}$$

Collecting results (40), (41) and (42) produces

$$\|\mathcal{P}_{1,\varepsilon}\|^2_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\mathcal{P}_{1,\varepsilon}\|^2_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} \leq C\, \varepsilon^{\frac{d}{2}+1} \|\mathcal{P}_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}. \tag{43}$$

Thanks to Young's inequality, we obtain

$$\|\mathcal{P}_{1,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} + \|\mathcal{P}_{1,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}}). \tag{44}$$

Following the analysis already used to estimate $\vartheta_{2,\varepsilon}$, we get

$$\|\mathcal{P}_{2,\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} + \|\mathcal{P}_{2,\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}}). \tag{45}$$

Hence, from equations (44) and (45), one finds the desired estimate

$$\|\mathcal{P}_{\varepsilon}\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))} + \|\mathcal{P}_{\varepsilon}\|_{L^\infty(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}}). \tag{46}$$

□

We search now to compute the asymptotic behavior of the first integral in (10).

5.5. **Proof of Lemma** 3.4. From (2), we have

$$\frac{\partial \theta_0}{\partial t}(x,t) - \Delta\theta_0(x,t) = F(x,t), \;\; (x,t) \in \mathcal{I}_{z,\varepsilon} \times (0,T).$$

Besides, using Green's formula and taking into account the normal orientation, we get

$$\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla\theta_0.n\, p_0\, ds\, dt = \int_0^T \int_{\mathcal{I}_{z,\varepsilon}} F\, p_0\, dx\, dt - \int_0^T \int_{\mathcal{I}_{z,\varepsilon}} \frac{\partial \theta_0}{\partial t} p_0\, dx\, dt - \int_0^T \int_{\mathcal{I}_{z,\varepsilon}} \nabla\theta_0.\nabla p_0\, dx\, dt. \tag{47}$$

Using Taylor's theorem and the change of variable $x = z + \varepsilon y$, the first integral in (47) may be written as

$$\begin{aligned} \int_0^T \int_{\mathcal{I}_{z,\varepsilon}} F\, p_0\, dx\, dt &= \varepsilon^d |\mathcal{I}| \int_0^T F(z,t)p_0(z,t)dt \\ &\quad + \varepsilon^d \int_0^T \int_{\mathcal{I}} \Big[F(z+\varepsilon y,t)p_0(z+\varepsilon y,t) - F(z,t)p_0(z,t)\Big] dydt. \end{aligned}$$

The regularity of F and p_0 near z allows to write

$$\varepsilon^d \int_0^T \int_{\mathcal{I}} \Big[F(z+\varepsilon y,t)p_0(z+\varepsilon y,t) - F(z,t)p_0(z,t)\Big] dydt = o(\varepsilon^d),$$

hence

$$\int_0^T \int_{\mathcal{I}_{z,\varepsilon}} F\, p_0\, dx\, dt = \varepsilon^d |\mathcal{I}| \int_0^T F(z,t)p_0(z,t)dt + o(\varepsilon^d). \tag{48}$$

With the help of Taylor's theorem and the change of variable $x = z + \varepsilon y$, the second integral in (47) can be written as

$$\int_0^T \int_{\mathcal{I}_{z,\varepsilon}} \frac{\partial \theta_0}{\partial t} p_0 \, dx \, dt = \varepsilon^d |\mathcal{I}| \int_0^T \frac{\partial \theta_0}{\partial t}(z,t) p_0(z,t) dt + \varepsilon^d \int_0^T \int_{\mathcal{I}} \Big[\frac{\partial \theta_0}{\partial t}(z+\varepsilon y,t) p_0(z+\varepsilon y,t) - \frac{\partial \theta_0}{\partial t}(z,t) p_0(z,t) \Big] dydt.$$

Due to the smoothness of $\dfrac{\partial \theta_0}{\partial t}$ and p_0 near z, we have

$$\varepsilon^d \int_0^T \int_{\mathcal{I}} \Big[\frac{\partial \theta_0}{\partial t}(z+\varepsilon y,t) p_0(z+\varepsilon y,t) - \frac{\partial \theta_0}{\partial t}(z,t) p_0(z,t) \Big] dydt = o(\varepsilon^d).$$

Consequently,

$$\int_0^T \int_{\mathcal{I}_{z,\varepsilon}} \frac{\partial \theta_0}{\partial t} p_0 \, dx \, dt = \varepsilon^d |\mathcal{I}| \int_0^T \frac{\partial \theta_0}{\partial t}(z,t) p_0(z,t) dt + o(\varepsilon^d). \tag{49}$$

To estimate the third integral in (47), we again use Taylor's theorem and the change of variable $x = z + \varepsilon y$, then we have

$$\int_0^T \int_{\mathcal{I}_{z,\varepsilon}} \nabla \theta_0 . \nabla p_0 \, dx \, dt = \varepsilon^d |\mathcal{I}| \int_0^T \nabla \theta_0(z,t) . \nabla p_0(z,t) dt + \varepsilon^d \int_0^T \int_{\mathcal{I}} \Big[\nabla \theta_0(z+\varepsilon y,t) . \nabla p_0(z+\varepsilon y,t) - \nabla \theta_0(z,t) . \nabla p_0(z,t) \Big] dydt.$$

The regularity of $\nabla \theta_0$ and ∇p_0 near z results in

$$\int_0^T \int_{\mathcal{I}_{z,\varepsilon}} \nabla \theta_0 . \nabla p_0 \, dx \, dt = \varepsilon^d |\mathcal{I}| \int_0^T \nabla \theta_0(z,t) . \nabla p_0(z,t) dt + o(\varepsilon^d). \tag{50}$$

Gathering (48), (49) and (50) leads to the desired expansion

$$\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla \theta_0 . n p_0 ds dt = \varepsilon^d |\mathcal{I}| \Big[\int_0^T F(z,t) p_0(z,t) dt - \int_0^T \frac{\partial \theta_0}{\partial t}(z,t) p_0(z,t) dt - \int_0^T \nabla \theta_0(z,t) . \nabla p_0(z,t) dt \Big] + o(\varepsilon^d).$$

□

Let us now turn to compute the asymptotic behavior of the second integral in (10).

5.6. **Proof of Lemma** 3.5. We have

$$\begin{aligned} \int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla \theta_0 . n \, (p_\varepsilon - p_0) \, ds \, dt &= \int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla \theta_0(x,t) . n \, P_\varepsilon(x,t) \, ds(x) dt + R_1(\varepsilon), \\ &= \int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla \theta_0(z,t) . n \, P_\varepsilon(x,t) \, ds(x) dt + R_1(\varepsilon) + R_2(\varepsilon). \end{aligned}$$

with

$$R_1(\varepsilon) = \int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla \theta_0 . n (p_\varepsilon - p_0 - P_\varepsilon) \, ds \, dt.$$

$$R_2(\varepsilon) = \int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla (\theta_0(x,t) - \theta_0(z,t)) . n \, P_\varepsilon \, ds \, dt.$$

From the definition of P_ε and the change of variable $x = z + \varepsilon y$, we get

$$\begin{aligned}\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla\theta_0(z,t).n\, P_\varepsilon(x,t) ds(x) dt &= \varepsilon^d \int_0^T \int_{\partial \mathcal{I}} \nabla\theta_0(z,t).n\, \psi(y) \nabla p_0(z,t) ds(y) dt \\ &= \varepsilon^d \int_0^T \int_{\partial \mathcal{I}} \nabla\theta_0(z,t).n\, Q(y) \nabla p_0(z,t) ds(y) dt,\end{aligned}$$

where Q^i is an extension of ψ^i on $\mathcal{I}$, solution to

$$\begin{cases} -\Delta_y Q^j &= 0 \quad \text{in} \quad \mathcal{I}, \\ \quad Q^i &= \psi^i \quad \text{on} \quad \partial\mathcal{I}. \end{cases}$$

Moreover, Green's formula and the regularity of θ_0 provide

$$\begin{aligned}\int_0^T \int_{\partial \mathcal{I}} \nabla\theta_0(z,t).n\, Q(y) \nabla p_0(z,t) ds(y) dt &= \int_0^T \int_{\mathcal{I}} \nabla\theta_0(z,t).\nabla_y (Q(y) \nabla p_0(z,t)) ds(y) dt, \\ &= \int_0^T \int_{\partial \mathcal{I}} \nabla\theta_0(z,t) y \nabla_y (Q(y) \nabla p_0(z,t)).n\, ds(y)\, dt.\end{aligned}$$

Then, we have

$$\begin{aligned}\int_0^T \int_{\partial \mathcal{I}} \nabla\theta_0(z,t) y \nabla_y (Q(y) \nabla_x p_0(z,t)).n ds(y)\, dt &= \sum_{i,j=1}^d \int_0^T \int_{\partial \mathcal{I}} \frac{\partial \theta_0}{\partial x_j}(z,t) y_j \nabla_y Q^i(y).n\, \frac{\partial p_0}{\partial x_i}(z,t) ds dt \\ &= \sum_{i,j=1}^d \int_0^T \frac{\partial \theta_0}{\partial x_j}(z,t) \frac{\partial p_0}{\partial x_i}(z,t) \int_{\partial \mathcal{I}} \nabla_y Q^i(y).n\, y_j ds dt.\end{aligned}$$

It follows from the jump relation on $\partial\mathcal{I}$ that [7]

$$\eta_i(y) = -e_i.n - \nabla_y Q^i(y).n, \ \ \forall y \in \partial\mathcal{I}.$$

Therefore, we obtain

$$\begin{aligned}&\int_0^T \int_{\partial \mathcal{I}} \nabla\theta_0(z,t) y \nabla_y (Q(y) \nabla p_0(z,t)).n ds(y)\, dt \\ &= \sum_{i,j=1}^d \int_0^T \frac{\partial \theta_0}{\partial x_j}(z,t) \frac{\partial p_0}{\partial x_i}(z,t) \int_{\partial \mathcal{I}} (-\eta_i(y) - e_i.n) y_j ds dt \\ &= -\sum_{i,j=1}^d \int_0^T \frac{\partial \theta_0}{\partial x_j}(z,t) \frac{\partial p_0}{\partial x_i}(z,t) \int_{\partial \mathcal{I}} \eta_i(y) y_j ds dt - \sum_{i,j=1}^d \int_0^T \frac{\partial \theta_0}{\partial x_j}(z,t) \frac{\partial p_0}{\partial x_i}(z,t) \int_{\partial \mathcal{I}} e_i.n\, y_j ds dt\end{aligned} \tag{51}$$

An integration by parts and taking into account the normal orientation provides

$$\int_{\partial \mathcal{I}} e_i.n\, y_j ds(y) = -|\mathcal{I}| I_{i,j}, \tag{52}$$

where $I_{i,j}$ are the entires of identity matrix.

Gathering results (51) and (52), we obtain

$$\begin{aligned}\int_0^T \int_{\partial \mathcal{I}} \nabla\theta_0(z,t) y \nabla_y (Q(y) \nabla p_0(z,t)).n ds(y)\, dt &= -\int_0^T \nabla\theta_0(z,t) \Big(\int_{\partial \mathcal{I}} \eta(y) y ds \Big) \nabla p_0(z,t) dt \\ &\quad + |\mathcal{I}| \int_0^T \nabla\theta_0(z,t).\nabla p_0(z,t) dt.\end{aligned}$$

Then,

$$\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla\theta_0.n\,(p_\varepsilon - p_0)\,ds\,dt = -\varepsilon^d \int_0^T \nabla\theta_0(z,t)\Big(\int_{\partial\mathcal{I}} \eta(y)yds\Big)\nabla p_0(z,t)dt$$
$$+\,\varepsilon^d|\mathcal{I}|\int_0^T \nabla\theta_0(z,t).\nabla p_0(z,t)dt + R_1(\varepsilon) + R_2(\varepsilon).$$

Next, we will successively prove that $R_i(\varepsilon) = o(\varepsilon^d)$, $i = 1, 2$.

• **Estimate of** $R_1(\varepsilon)$:

Due to Trace theorem, we have

$$\begin{aligned}|R_1(\varepsilon)| &\le C \int_0^T \|\nabla\theta_0.n(\cdot,t)\|_{H^{-1/2}(\partial\mathcal{I}_{z,\varepsilon})} \|p_\varepsilon - p_0 - P_\varepsilon(\cdot,t)\|_{L^2(0,T,H^{1/2}(\partial\mathcal{I}_{z,\varepsilon}))}\,dt \\ &\le C\,\|\nabla\theta_0.n(\cdot,t)\|_{L^2(0,T,H^1(\mathcal{I}_{z,\varepsilon}))} \|p_\varepsilon - p_0 - P_\varepsilon(\cdot,t)\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}\,.\end{aligned}$$

Changing variable $x = z + \varepsilon y$ and using Proposition 3.3

$$R_1(\varepsilon) \le C\varepsilon^{\frac{d}{2}}\,o(\varepsilon^{\frac{d}{2}}),$$

then we deduce

$$R_1(\varepsilon) = o(\varepsilon^d). \tag{53}$$

• **Estimate of** $R_2(\varepsilon)$:

Due to Trace theorem, we obtain

$$|R_2(\varepsilon)| \le C\,\|(\theta_0(z+\varepsilon y,t) - \theta_0(z,t)\|_{L^2(0,T,H^1(\mathcal{I}_{z,\varepsilon}))} \|P_\varepsilon\|_{L^2(0,T,H^1(\mathcal{H}_{z,\varepsilon}))}\,.$$

Changing variable $x = z + \varepsilon y$, using the definition of P_ε

$$|R_2(\varepsilon)| \le C\varepsilon^{d/2}\,\|\theta_0(z+\varepsilon y,t) - \theta_0(z,t)\|_{L^2(0,T,H^1(\mathcal{I}))} \|\nabla p_0(z,\cdot)\|_{L^2(0,T)} \|\psi_\varepsilon\|_{H^1(\mathcal{H}\setminus\overline{B(z,R)})}\,.$$

Using Lemma 5.1 and the fact that $\nabla p_0(z,\cdot) \in L^2(0,T)$ leads to

$$R_2(\varepsilon) = o(\varepsilon^d). \tag{54}$$

Then, according to (53) and (54), we have

$$\int_0^T \int_{\partial \mathcal{I}_{z,\varepsilon}} \nabla\theta_0.n\,(p_\varepsilon - p_0)\,ds\,dt = -\varepsilon^d \int_0^T \nabla\theta_0(z,t)\Big(\int_{\partial\mathcal{I}} \eta(y)yds\Big)\nabla p_0(z,t)dt$$
$$+\,\varepsilon^d|\mathcal{I}|\int_0^T \nabla\theta_0(z,t).\nabla p_0(z,t)dt + o(\varepsilon^d).$$

□

Let us now turn to proof the main theoretical result given by Theorem 3.6.

5.7. **Proof of Theorem** 3.6**.** Now, it is possible to complete the evaluation of the asymptotic behavior of the shape function j. Combining the results of Lemmas 3.4 and 3.5 and a few simplifications, we obtain the following asymptotic expansion

$$j(\mathcal{H}\setminus\overline{\mathcal{I}_{z,\varepsilon}}) - j(\mathcal{H}) = \varepsilon^d|\mathcal{I}|\Big[\int_0^T F(z,t)p_0(z,t)dt - \int_0^T \frac{\partial\theta_0}{\partial t}(z,t)p_0(z,t)dt\Big]$$
$$-\,\varepsilon^d\int_0^T \Big[\nabla\theta_0(z,t).\Big(\int_{\partial\mathcal{I}}\eta(y)yds(y)\Big)\nabla p_0(z,t)\Big]dt + \varepsilon^d\delta J + o(\varepsilon^d).$$

This ends the proof of the theorem.

□

5.8. **Proof of Proposition** 3.8**.** The function J_ε is differentiable and we have

$$DJ_\varepsilon(\theta_\varepsilon(\cdot,t))(w) = 2\int_{\mathcal{H}_{z,\varepsilon}} \nabla\theta_\varepsilon(\cdot,t).\nabla w dx,\ w \in H^1(\mathcal{H}_{z,\varepsilon}).$$

From the definition of j, we have

$$\begin{aligned}
j(\mathcal{H}_{z,\varepsilon}) - j(\mathcal{H}) &= \int_0^T \int_{\mathcal{H}_{z,\varepsilon}} |\nabla\theta_\varepsilon|^2 dxdt - \int_0^T \int_{\mathcal{H}} |\nabla\theta_0|^2 dxdt \\
&= 2\int_0^T \int_{\mathcal{H}_{z,\varepsilon}} \nabla\theta_\varepsilon(\nabla\theta_\varepsilon - \nabla\theta_0)dxdt + \int_0^T \int_{\mathcal{H}_{z,\varepsilon}} |\nabla\theta_\varepsilon - \nabla\theta_0|^2 dxdt \\
&- \int_0^T \int_{\mathcal{I}_{z,\varepsilon}} |\nabla\theta_0|^2 dxdt \\
&= \int_0^T DJ_\varepsilon(\theta_\varepsilon)(\theta_\varepsilon - \theta_0)dt + \int_0^T \int_{\mathcal{H}_{z,\varepsilon}} |\nabla\theta_\varepsilon - \nabla\theta_0|^2 dxdt \\
&- \int_0^T \int_{\mathcal{I}_{z,\varepsilon}} |\nabla\theta_0|^2 dxdt.
\end{aligned}$$

Using Taylor's theorem and the change of variable $x = z + \varepsilon y$, and the regularity of $\nabla\theta_0$, we have that

$$\int_0^T \int_{\mathcal{I}_{z,\varepsilon}} |\nabla\theta_0|^2 dxdt = \varepsilon^d |\mathcal{I}| \int_0^T |\nabla\theta_0(z,t)|^2 dt + o(\varepsilon^d).$$

Based on the weak formulation of (29), we obtain for all $t_0 \in (0,T)$

$$\int_0^T \int_{\mathcal{H}_{z,\varepsilon}} \nabla(\theta_\varepsilon - \theta_0).\nabla(\theta_\varepsilon - \theta_0)\, dx\, dt = -\frac{1}{2}\int_{\mathcal{H}_{z,\varepsilon}} |(\theta_\varepsilon - \theta_0)(\cdot,t_0)|^2 dx + \int_0^T \int_{\partial\mathcal{I}_{z,\varepsilon}} \nabla\theta_0.n(\theta_0 - \theta_\varepsilon) dsdt.$$

It follows from Lemma 3.2 that

$$-\frac{1}{2}\int_{\mathcal{H}_{z,\varepsilon}} |(\theta_\varepsilon - \theta_0)(\cdot,t_0)|^2 dx = o(\varepsilon^d). \tag{55}$$

Moreover, by an adaptation of the same technique in Lemma 3.5 and using $\theta_\varepsilon - \theta_0$ instead $p_\varepsilon - p_0$, we get

$$\begin{aligned}
\int_0^T \int_{\partial\mathcal{I}_{z,\varepsilon}} \nabla\theta_0.n(\theta_\varepsilon - \theta_0)dsdt = -\varepsilon^d \int_0^T \nabla\theta_0(z,t)\Big(\int_{\partial\mathcal{I}} \eta(y) y ds(y)\Big)\nabla\theta_0(z,t)dt \\
+ \varepsilon^d |\mathcal{I}| \int_0^T |\nabla\theta_0(z,t)|^2 dt + o(\varepsilon^d).
\end{aligned} \tag{56}$$

Gathering the previous results, then the asymptotic expansion is given by

$$j(\mathcal{H}_{z,\varepsilon}) - j(\mathcal{H}) = \int_0^T DJ_\varepsilon(\theta_\varepsilon))(\theta_\varepsilon - \theta_0)dt + \varepsilon^d \delta J(z) + o(\varepsilon^d),$$

with

$$\delta J(z) = -\int_0^T \nabla\theta_0(z,t)\Big(\int_{\partial\mathcal{I}} \eta(y) y ds(y)\Big)\nabla\theta_0(z,t)dt.$$

Finally, for any $w \in L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))$

$$\begin{aligned}
\int_0^T (DJ_\varepsilon(\theta_\varepsilon) - DJ_0(\theta_0))w dt &= 2\int_0^T \int_{\mathcal{H}_{z,\varepsilon}} \nabla\theta_\varepsilon.\nabla w\, dx\, dt - 2\int_0^T \int_{\mathcal{H}} \nabla\theta_0.\nabla w\, dx\, dt \\
&= 2\int_0^T \int_{\mathcal{H}_{z,\varepsilon}} \nabla(\theta_\varepsilon - \theta_0).\nabla w\, dx\, dt - 2\int_0^T \int_{\mathcal{I}_{z,\varepsilon}} \nabla\theta_0.\nabla w\, dx\, dt.
\end{aligned}$$

Using Cauchy-Schwarz inequality, one can deduce that

$$\int_0^T \int_{\mathcal{H}_{z,\varepsilon}} \nabla(\theta_\varepsilon - \theta_0).\nabla w \, dx \, dt \leq \|\nabla(\theta_\varepsilon - \theta_0)\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \|\nabla w\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))}$$
$$\leq \Big(\|\nabla(\theta_\varepsilon - \theta_0 - \Theta_\varepsilon)\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} + \|\Theta_\varepsilon\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} \Big) \|\nabla w\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))}.$$

We recall that

$$\Theta_\varepsilon(x,t) = \psi_\varepsilon(x) \cdot \nabla\theta_(z,t) \;\; (x,t) \in \mathbb{R}^d \times (0,T).$$

Based on Proposition 3.1 and Lemma 5.1, and using the fact $\nabla\theta(z,\cdot) \in L^2(0,T)$, we obtain

$$\int_0^T \int_{\mathcal{H}_{z,\varepsilon}} \nabla(\theta_\varepsilon - \theta_0).\nabla w \, dx \, dt \leq \varepsilon^{\frac{d}{2}+1} \|\nabla w\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))}.$$

Using the change of variable $x = z + \varepsilon y$ and thanks to the regularity of $\nabla\theta_0$ in $\mathcal{I}_{z,\varepsilon}$, one obtains

$$\int_0^T \int_{\mathcal{I}_{z,\varepsilon}} \nabla\theta_0.\nabla w \, dx \, dt \leq C\varepsilon^d.$$

Then, under the previous results, we obtain

$$\|DJ_\varepsilon(\theta_\varepsilon) - DJ_0(\theta_0)\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))} = o(\varepsilon^{\frac{d}{2}})$$

which achieves the proof of proposition (3.8).

□

5.9. **Proof of Proposition** 3.9**.** The function J_ε is differentiable and we have

$$DJ_\varepsilon(\theta_\varepsilon(.,t))(w) = 2\int_\Gamma (\theta_\varepsilon(\cdot,t) - \theta_d(\cdot,t)) w dx, \;\; w \in H^1(\mathcal{H}_{z,\varepsilon}).$$

From the definition of j, we have

$$j(\mathcal{H}_{z,\varepsilon}) - j(\mathcal{H}) = \int_0^T \int_\Gamma |\theta_\varepsilon - \theta_d|^2 \, dx \, dt - \int_0^T \int_\Gamma |\theta_0 - \theta_d|^2 \, dx \, dt$$
$$= \int_0^T DJ_\varepsilon(\theta_\varepsilon)(\theta_\varepsilon - \theta_0) dt + \int_0^T \int_\Gamma |\theta_\varepsilon - \theta_0|^2 \, dx \, dt.$$

Due to Trace theorem and Lemma 3.2, we get

$$\int_0^T \int_\Gamma |\theta_\varepsilon - \theta_0|^2 \, dx \, dt \leq C\|(\theta_\varepsilon - \theta_0)\|^2_{L^2(0,T,L^2(\mathcal{H}\setminus\overline{B(z,R)}))} = o(\varepsilon^d).$$

Then, we obtain the desired expansion

$$j(\mathcal{H}_{z,\varepsilon}) - j(\mathcal{H}) = \int_0^T DJ_\varepsilon(\theta_\varepsilon)(\theta_\varepsilon - \theta_0) dt + o(\varepsilon^d),$$

where

$$\delta J(z) = 0.$$

Finally, for any $w \in L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))$

$$\int_0^T (DJ_\varepsilon(\theta_\varepsilon) - DJ_0(\theta_0))(\theta_\varepsilon - \theta_0)(\cdot,t) w(\cdot,t) dt = 2\int_0^T \int_\Gamma (\theta_\varepsilon - \theta_0) w \, dx \, dt$$

Using Trace theorem, we obtain

$$|\int_0^T \int_\Gamma (\theta_\varepsilon - \theta_0) w \, dx \, dt| \leq C\|(\theta_\varepsilon - \theta_0)\|_{L^2(0,T,L^2(\mathcal{H}\setminus\overline{B(z,R)}))} \|w\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))},$$

and based on Lemma 3.2, we get

$$|\int_0^T \int_\Gamma (\theta_\varepsilon - \theta_0) w \, dx \, dt| \leq C\varepsilon^{\frac{d}{2}+1} \| = w\|_{L^2(0,T,L^2(\mathcal{H}_{z,\varepsilon}))},$$

from which we deduce the desired result.

□

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