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HIGHER ORDER NON-DIFFERENTIABLE MULTI-OBJECTIVE SYMMETRIC DUALITY INVOLVING GENERALIZED $K-(\Phi,\rho)$-CONVEX FUNCTIONS

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Abstract. In this paper, a new class of generalized $K-(\Phi,\rho)$ convex function is introduced, in which the sub linearity property of F as in literature is relaxed by imposing the convexity assumption on Φ in its third argument with an example. This new class of generalized convex function is more generalized than the (F,α,ρ,d)-convex functions, (C,α,ρ,d)-convex functions and $K-(F,\alpha,\rho,d)$ convex functions studied in literature. Also, a new model of higher order Wolfe type non-differentiable multi-objective symmetric dual programs is presented and the weak, strong and converse duality theorem under higher order $K-(\Phi,\rho)$ convex functions are established. Some special cases which generalizes our results is discussed.

1. Introduction

The duality theory for convex multi-objective optimization problem is useful both theoretically and practically. Unlike the linear programming problems, there is no unique dual formulation for the nonlinear programming. The study of second and higher order duality is significantly developed due to the computational advantages over the first order duality as it provides tighter bounds for the value of the objective function when approximation are used. Higher order duality in nonlinear programming has been studied by many researchers like Suneja et al. [[19, 20]], Gulati and Gupta [[5, 6]], Kim and Lee [[13]] and Gulati and Sani [[7]], Gupta and Jaysal [[8]], Mishra [[15, 16]], Kassem and Hady[[2]], Gupta et al. [[9, 10]], Padhan and Nahak [[17]], Tripathy and Devi [[21]], Agarwal et al. [[1]] and many more. On the other hand, to relax convexity assumptions imposed on the functions in theorems on optimality conditions and duality, various generalized convexity notations have been introduced. A significant generalization of convex function is that of invex function introduced by Hanson [[11]] and Craven [[4]]. After the work of Hanson and Craven, other types of differentiable function have been introduced with the intent of generalizing invex function from different point of view. Hanson and Mond [[20]] introduced the concept of F-convex which is a generalization of

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invex function. Preda [[18]] generalized F-convexity to (F, ρ)-convexity. The (F, ρ)-convexity was recently generalized to (Φ, ρ)-invexity by Caristi et al.[[3]] in which Φ is convex in its third argument and replaces the sublinear property of F in the third argument. Liang et al.[[14]] introduced a unified formulation of generalization convexity called (F, α, ρ, d)-convex and Yuan et al. [[22]] introduced the concept of (C, α, ρ, d)-convexity by relaxing the sub linearity of F to convexity of C. Tripathy and Devi[[21]] introduced the concepts of higher order (Φ, ρ)-invex function by relaxing the convexity as well as sub linearity assumption on Φ. Gupta and Jayswal [[8]] introduced K-preinvex/K-pseudo invex functions,where as Agarwal et al. [[1]] defined higher order K-F convex functions. Gupta et al. [[9]] introduced the concept of $K-(F, \alpha, \rho, d)$ convexity.

In this paper, motivated by **Yuan et al.** [[14]] and **Gupta et al.** [[9]], we introduced a new class of **generalized** $K-(\Phi, \rho)$ **convex function** in which the sub linearity property imposed on F as in **Gupta et al.** [[9]] is relaxed by imposing **the convexity assumption on Φ in its third argument**. This new class of generalized convex function is more generalized than the (F, α, ρ, d)-convex functions as in [[14]] , (C, α, ρ, d)-convex functions as in [[22]] and $K-(F, \alpha, \rho, d)$ convex functions as in [[9]]. Also, we have presented a new model of **higher order Wolfe type non-differentiable multi-objective symmetric dual programs** and established the weak, strong and converse duality theorem under higher order $K-(\Phi, \rho)$ convex functions.

2. Preliminaries and Definitions

Throughout this paper, we denote by R^n the n-dimensional Euclidean space and R^n_+ be its non-negative orthant. Let C_1 and C_2 be closed convex cones in R^n and R^m respectively, with nonempty interiors. Let K be a pointed convex cone with nonempty interior in R^k. Then for $y, z \in R^k$, we denote following convention:
$x \leqq y \Leftrightarrow y - x \in K$; $x \leq y \Leftrightarrow y - x \in K \setminus \{0\}$; $x < y \Leftrightarrow y - x \in intK$.

Definition 2.1 A non empty set $C \subset R^n$ is called a cone, if for each $x \in C$ and $\lambda \in R, \lambda \geq 0$, we have $\lambda x \in C$. More over if C in convex, then it is called convex cone.

Definition 2.2 The positive polar cone C^* of C is defined as

$$C^* = \{z \in R^n \mid x^T z \geq 0, \forall x \in C\}.$$

Consider the multi-objective programming problem:

First we consider the following multi-objective programming problem:

(MP) K-Minimize $f(x)$

Subject to $-g(x) \in Q, x \in S$,

where $f : R^n \to R^k, g : R^n \to R^m$ and $S \subset R^n, K$ and Q are closed convex cone with nonempty interior in R^k and R^m, respectively.

Let $X = \{x \in S : -g(x) \in Q\}$ be the set of feasible solutions of (MP). Further let K_0 denote the set $K \setminus \{0\}$.

Since the objectives in multi-objective problems generally conflict with one another, an optimal solution is chosen from the set of efficient/weak efficient solutions.

Definition 2.3 A point $\overline{x} \in X$ is an efficient (Pareto optimal) solution of (MP), if there does not exist $x \in X$ such that $f(\overline{x}) - f(x) \in K_0$.

Let C_1 and C_2 be closed convex cones in R^n and R^m, respectively. Also, $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ be open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

Definition 2.4 Let $x, y \in R^n$ and $A \in R^n \times R^n$ be a positive semi-definite matrix, then $x^T Ay \leq (x^T Ay)^{\frac{1}{2}}$. Equality holds if for some $\lambda \geq 0, Ax \geq \lambda Ay$.

Definition 2.5 A function $\Phi : S \times S \times R^{n+1} \to R$ is said to be convex in the third argument iff for any fixed $(x, u) \in S \times S$ the inequality

$$\Phi(x, u; \lambda a_1 + (1-\lambda)a_2) \leq \lambda \Phi(x, u; a_1) + (1-\lambda)\Phi(x, u; a_2), \forall \lambda \in (0,1)$$

holds for all $a_1, a_2 \in R^{n+1}$.

Throughout this paper, we assume that $\Phi(x, u; 0) = 0$.

Lemma 2.1(Jensen's Inequality)
Let $f : (a, b) \to R$ be convex function and let $x_1, x_2, ..., x_n \in (a, b)$. Then

$$f(\sum_{i=1}^{n} \lambda_i x_i) \leq \sum_{i=1}^{n} \lambda_i f(x_i),$$

for any $\lambda_1, \lambda_2, ..., \lambda_n \in [0, 1]$ satisfying $\sum_{i=1}^{n} \lambda_i = 1$.

Definition 2.6 Let $S \subseteq R^n$ and $g_i : S \times R^n \to R,\ i = 1, 2, ..., k$, is a differentiable function and let $\Phi : S \times S \times R^{n+1} \to R$ is convex function in its third argument. Then a differentiable function $f = (f_1, f_2, ..., f_k) : S \to R^k$ is said to be higher order $K - (\Phi, \rho)$-convex at $u \in S$ with respect to $g = (g_1, g_2, ..., g_k)$ such that for $x \in S, q_i \in R^n, i = 1, 2, ..., k$,

$$\begin{pmatrix} f_1(x) - f_1(u) - g_1(u, q_1) + q_1^T \nabla_{q_1} g_1(u, q_1) \\ \quad -\Phi(x, u; (\nabla_x f_1(u) + \nabla_{q_1} g_1(u, q_1), \rho_1), \quad . \quad . \quad ., \\ \\ f_k(x) - f_k(u) - g_k(u, q_k) + q_k^T \nabla_{q_k} g_k(u, q_k) \\ \quad -\Phi(x, u; (\nabla_x f_k(u) + \nabla_{q_k} g_k(u, q_k), \rho_k) \end{pmatrix} \in K.$$

Definition 2.7 Let $S \subseteq R^n$ and $g_i : S \times R^n \to R,\ i = 1, 2, ..., k$, is a differentiable function and let $\Phi : S \times S \times R^{n+1} \to R$ is convex function in its third argument. Then a differentiable function $f = (f_1, f_2, ..., f_k) : S \to R^k$ is said to be higher order $K - (\Phi, \rho)$-pseudo convex at $u \in X$ with respect to $g = (g_1, g_2, ..., g_k)$ such that for $x \in S, q_i \in R^n, i = 1, 2, ..., k$,

$$\begin{pmatrix} \Phi(x, u; (\nabla_x f_1(u) + \nabla_{q_1} g_1(u, q_1), \rho_1), \quad . \quad . \quad ., \\ \\ \Phi(x, u; (\nabla_x f_k(u) + \nabla_{q_k} g_k(u, q_k), \rho_k) \end{pmatrix} \in K$$

$$\Rightarrow \begin{pmatrix} f_1(x) - f_1(u) - g_1(u, q_1) + q_1^T \nabla_{q_1} g_1(u, q_1), \quad . \quad . \quad ., \\ \\ f_k(x) - f_k(u) - g_k(u, q_k) + q_k^T \nabla_{q_k} g_k(u, q_k), \end{pmatrix} \in K.$$

Remark 2.1
If $K = R_+$, then the Definition reduces to that of higher order (Φ, ρ)-invex and higher order (Φ, ρ)-pseudo invex function in **Jayswal and Kumari** [[12]].

Example 2.1
Let $K = \{(x, y) : x \leq 0, y \leq 0\}$ and $f = (f_1, f_2) : R \to R^2$ defined as $f_1(x) = x \sin x$ and $f_2 = x^2 + 8 \cos x$. Let $g = (g_1, g_2) : R \times R \to R^2$ defined as $g_1(x, q) = -q(x^2 + 2)$ and $g_2(x, q) = q(x + 1)$.
Let $\Phi : R \times R \times R^2 \to$ defined by $\Phi(x, u; (a, \rho)) = (\rho^2 + 1)|a|(x^2 + u^2)$ and
$F : R \times R \times R \to R$ defined by $F = \frac{|a|}{4}(x^2 + u^2)$.
It is clear that Φ is convex but not sub-linear in third argument, but F is sub-linear as well as convex in third argument.
Now, at $u = 0$,$\forall x \in R, q \in R$ and $\rho \in R$,
$f_1(x) - f_1(u) - g_1(u, q) + q^T \nabla_q g_1(u, q) - \Phi(x, u; (\nabla f_1(u) + \nabla_q g_1(u, q), \rho_1))$
$= x \sin x - 2(\rho^2 + 1)x^2 \leq 0$
and
$f_2(x) - f_2(u) - g_2(u, q) + q^T \nabla_q g_2(u, q) - \Phi(x, u; (\nabla f_2(u) + \nabla_q g_2(u, q), \rho_2))$
$= 8 \cos x - 8 - \rho^2 x^2 \leq 0.$
So

$$\begin{pmatrix} f_1(x) - f_1(u) - g_1(u, q) + q^T \nabla_q g_1(u, q) - \Phi(x, u; (\nabla_x f_1(u) + \nabla_q g_1(u, q), \rho_1), \\ f_2(x) - f_2(u) - g_2(u, q) + q^T \nabla_q g_2(u, q) - \Phi(x, u; (\nabla_x f_2(u) + \nabla_q g_2(u, q), \rho_2) \end{pmatrix} \in K.$$

Hence, f is higher order $K - (\Phi, \rho)$-convex function with respect to $g(x, q)$ at $u = 0$.
But for $x \in \left(0, \frac{\pi}{2}\right]$ and $u = 0$,
$f_1(x) - f_1(u) - g_1(u, q) + q^T \nabla_q g_1(u, q) - F(x, u; (\nabla f_1(u) + \nabla_q g_1(u, q)))$
$= x \sin x - \frac{1}{2}x^2 \geq 0$

and for all $x \geq 4.1$ and $u = 0$,
$f_2(x) - f_2(u) - g_2(u, q) + q^T \nabla_q g_2(u, q) - F(x, u; (\nabla f_2(u) + \nabla_q g_2(u, q)))$
$= \frac{3}{4}x^2 + 8 \cos x - 8 \geq 0.$

Hence f is not higher order $K - F$-convex function with respect to $g(x, q)$ at $u = 0$ as in **Agarwal et al.**[[1]].

Again, if we take $\rho \geq \frac{1}{2}$ and $d^2(x, u) = x^2 + u^2$, then $x \in \left(0, \frac{\pi}{2}\right]$ and $u = 0$,

$f_1(x) - f_1(u) - g_1(u, q) + q^T \nabla_q g_1(u, q) - F(x, u; \alpha(x, u)(\nabla f_1(u) + \nabla_q g_1(u, q)) + \rho d^2(x, u)$

$= x \sin x + (\rho - \frac{1}{2})x^2 \geq 0.$

Also, for all $x \geq 4.1$,$\rho \geq 0$,$d^2(x, u) = x^2 + u^2$ and $u = 0$,

$f_2(x) - f_2(u) - g_2(u, q) + q^T \nabla_q g_2(u, q) - F(x, u; \alpha(x, u)(\nabla f_2(u) + \nabla_q g_2(u, q)) + \rho d^2(x, u)$

$= (\rho + \frac{3}{4})x^2 + 8 \cos x - 8 \geq 0.$

Hence f is not higher order $K - (F, \alpha, \rho, d)$-convex function with respect to $g(x, q)$ at $u = 0$ for all x as in **Gupta et al.**[[9]].

Definition 2.8 Let$S_1 \subseteq R^n$ and $S_2 \subseteq R^m$. Let $g_i : S_1 \times S_2 \times R^n \to R, i = 1, 2, ..., k$, is a differentiable function and let $\Phi_1 : S_1 \times S_1 \times R^{n+1} \to R$ is convex function in its third argument. Then a differentiable function $f = (f_1, f_2, ..., f_k) : S_1 \times S_2 \to R^k$ is said to be higher order $K - (\Phi_1, \rho)$-convex in the first variable $u \in S_1$ at fixed $y \in S_2$ with respect to $g = (g_1, g_2, ..., g_k)$ such that for $x \in S_1, q_i \in R^n, i = 1, 2, ..., k$,

$$\begin{pmatrix} f_1(x,y) - f_1(u,y) - g_1(u,y,q_1) + q_1^T \nabla_{q_1} g_1(u,y,q_1) \\ -\Phi_1(x,u;(\nabla_x f_1(u,y) + \nabla_{q_1} g_1(u,y,q_1),\rho_1), \quad . \quad . \quad ., \\ \\ f_k(x,y) - f_k(u,y) - g_k(u,y,q_k) + q_k^T \nabla_{q_k} g_k(u,y,q_k) \\ -\Phi_1(x,u;(\nabla_x f_k(u,y) + \nabla_{q_k} g_k(u,y,q_k),\rho_k) \end{pmatrix} \in K.$$

Definition 2.9 Let$S_1 \subseteq R^n$ and $S_2 \subseteq R^m$. Let $g_i : S_1 \times S_2 \times R^n \to R, i = 1, 2, ..., k$, is a differentiable function and let $\Phi_1 : S_1 \times S_1 \times R^{n+1} \to R$ is convex function in its third argument. Then a differentiable function $f = (f_1, f_2, ..., f_k) : S_1 \times S_2 \to R^k$ is said to be higher order $K - (\Phi_1, \rho)$-pseudo convex in first variable $u \in S_1$ at fixed $y \in S_2$ with respect to $g = (g_1, g_2, ..., g_k)$ such that for $x \in S_1, q_i \in R^n, i = 1, 2, ..., k$,

$$\begin{pmatrix} \Phi_1(x,u;(\nabla_x f_1(u,y) + \nabla_{q_1} g_1(u,y,q_1),\rho_1), \quad . \quad . \quad ., \\ \\ \Phi_1(x,u;(\nabla_x f_k(u,y) + \nabla_{q_k} g_k(u,y,q_k),\rho_k) \end{pmatrix} \in K$$

$$\Rightarrow \begin{pmatrix} f_1(x,y) - f_1(u,y) - g_1(u,y,q_1) + q_1^T \nabla_{q_1} g_1(u,y,q_1), \quad . \quad . \quad ., \\ \\ f_k(x,y) - f_k(u,y) - g_k(u,y,q_k) + q_k^T \nabla_{q_k} g_k(u,y,q_k \end{pmatrix} \in K.$$

Remark 2.2
Similarly higher order $K - (\Phi_2, \sigma)$-convex and higher order $K - (\Phi_2, \sigma)$-pseudo convex in second variable $v \in S_2$ at fixed $x \in S_1$ can be defined.

Remark 2.3
If Φ is replaced by $F : S_1 \times S_1 \times R^n \to R$ a sub-linear function in its third argument, then Definition 2.8 and Definition 2.9 reduces to that of higher order $K - F$-convex and higher order $K - F$-pseudo convex function in **Agarwal et al.**[[1]].

3. **Wolfe type higher order multi-objective symmetric duality**

Now, we consider the following pair of non-differentiable multi-objective higher order symmetric dual programs:

• **Primal(WHMP)**:

$$L(x,y,\lambda,w,p) = \text{K-Minimize} \begin{pmatrix} f_1(x,y) + (x^T B_1 x)^{\frac{1}{2}} - y^T D_1 w_1 \\ +\Sigma_{i=1}^k \lambda_i h_i(x,y,p_i) - \Sigma_{i=1}^k \lambda_i [p_i^T \nabla_{p_i} h_i(x,y,p_i)] \\ -y^T \Sigma_{i=1}^k \lambda_i [\nabla_y f_i(x,y) - D_i w_i + \nabla_{p_i} h_i(x,y,p_i)], \\ ..., \\ f_k(x,y) + (x^T B_k x)^{\frac{1}{2}} - y^T D_k w_k \\ +\Sigma_{i=1}^k \lambda_i h_i(x,y,p_i) - \Sigma_{i=1}^k \lambda_i [p_i^T \nabla_{p_i} h_i(x,y,p_i)] \\ -y^T \Sigma_{i=1}^k \lambda_i [\nabla_y f_i(x,y) - D_i w_i + \nabla_{p_i} h_i(x,y,p_i)] \end{pmatrix}$$

Subject to

$$-\sum_{i=1}^{k}\lambda_i[\nabla_y f_i(x,y)-D_iw_i+\nabla_{p_i}h_i(x,y,p_i)]\in C_2^*, \tag{3.1}$$

$$w_i^TD_iw_i\le 1, i=1,2,...,k, \tag{3.2}$$

$$x\in C_1, w_i\in R^m, i=1,2,...,k, \tag{3.3}$$

$$\lambda=(\lambda_1,\lambda_1,...,\lambda_k)\in intK^*, \sum_{i=1}^{k}\lambda_i=1. \tag{3.4}$$

• **Dual(WHMD)**:

$$M(u,v,\lambda,z,q)=\text{ K-Maxmize }\begin{pmatrix} f_1(u,v)-(v^TD_1v)^{\frac{1}{2}}+u^TB_1z_1\\ +\Sigma_{i=1}^k\lambda_ig_i(u,v,q_i)-\Sigma_{i=1}^k\lambda_i[q_i^T\nabla_{q_i}g_i(u,v,q_i)]\\ -u^T\Sigma_{i=1}^k\lambda_i[\nabla_uf_i(u,v)+B_iz_i+\nabla_{q_i}g_i(u,v,q_i)],\\ ...,\\ f_k(u,v)-(v^TD_kv)^{\frac{1}{2}}+u^TB_kz_k\\ +\Sigma_{i=1}^k\lambda_ig_i(u,v,q_i)-\Sigma_{i=1}^k\lambda_i[q_i^T\nabla_{q_i}g_i(u,v,q_i)]\\ -u^T\Sigma_{i=1}^k\lambda_i[\nabla_uf_i(u,v)+B_iz_i+\nabla_{q_i}g_i(u,v,q_i)] \end{pmatrix}$$

Subject to

$$\sum_{i=1}^{k}\lambda_i[\nabla_u f_i(u,v)+B_iz_i+\nabla_{q_i}g_i(u,v,q_i)]\in C_1^*, \tag{3.5}$$

$$z_i^TB_iz_i\le 1, i=1,2,...,k, \tag{3.6}$$

$$v\in C_2, z_i\in R^n, i=1,2,...,k, \tag{3.7}$$

$$\lambda=(\lambda_1,\lambda_1,...,\lambda_k)\in intK^*, \sum_{i=1}^{k}\lambda_i=1. \tag{3.8}$$

where (i) $f_i:S_1\times S_2\to R,$, $h_i:S_1\times S_2\times R^m\to R^k$ and $g_i:S_1\times S_2\times R^n\to R^k, i=1,2,...,k$; are continuously differentiable functions,
(ii) C_1 and C_2 are closed convex cones in R^n and R^m with nonempty interior respectively,
(iii) C_1^* andC_2^* are polar cones of C_1 and C_2 respectively,
(iv) B_i and $D_i, i=1,2,...,k$, are positive semi-definite symmetric matrix of order $n\times n$ and $m\times m$respectively.

Theorem 3.1 (Weak Duality) Let (x,y,λ,w,p) and (u,v,λ,z,q) be the feasible solution for Primal (WHMP) and Dual (WHMD) respectively. If
(i) $f_i(.,v)+(.)^TB_iz_i$ is higher order $K-(\Phi,\rho)$-convex at u with respect to $g_i(u,v,q_i), i=1,2,...,k$,
(ii) $-[f_i(x,.)-(.)^TD_iw_i]$ is higher order $K-(\Phi,\rho)$-convex at v with respect to $-h_i(x,y,p_i), i=1,2,...,k$,
(iii) $\Phi_1(x,u;(a,\rho))+u^Ta\ge 0, \forall a\in C_1^*$ and
(iv) $\Phi_2(v,y;(b,\rho))+y^Tb\ge 0, \forall b\in C_2^*$,
where $\Phi_1:S_1\times S_1\times R^{n+1}\to R$ and $\Phi_2:S_1\times S_1\times R^{m+1}\to R$ are convex function in their third argument and $g_i:S_1\times S_2\times R^n\to R$, and $h_i:S_1\times S_2\times R^m\to R$, $i=1,2,...,k$, are a differentiable function.
Then $M(u,v,\lambda,z,q)-L(x,y,\lambda,w,p)\notin K\setminus\{0\}$.

Proof: Suppose that contradiction holds. That is
$M(u,v,\lambda,z,q) - L(x,y,\lambda,w,p) \in K \setminus \{0\}$

$$\Rightarrow \begin{pmatrix} f_1(u,v) - f_1(x,y) - (v^T D_1 v)^{\frac{1}{2}} - (x^T B_1 x)^{\frac{1}{2}} + u^T B_1 z_1 + y^T D_1 w_1 \\ +\Sigma_{i=1}^k \lambda_i g_i(u,v,q_i) - \Sigma_{i=1}^k \lambda_i q_i^T \nabla_{q_i} g_i(u,v,q_i) - u^T \Sigma_{i=1}^k \lambda_i [\nabla_u f_i(u,v) \\ +B_i z_i + \nabla_{q_i} g_i(u,v,q_i)] - \Sigma_{i=1}^k \lambda_i [h_i(x,y,p_i) - p_i^T \nabla_{p_i} h_i(x,y,p_i)] \\ +y^T \Sigma_{i=1}^k \lambda_i [\nabla_y f_i(x,y) - D_i w_i + \nabla_{p_i} h_i(x,y,p_i)], \\ ..., \\ f_k(u,v) - f_k(x,y) - (v^T D_k v)^{\frac{1}{2}} - (x^T B_k x)^{\frac{1}{2}} + u^T B_k z_k + y^T D_k w_k \\ +\Sigma_{i=1}^k \lambda_i g_i(u,v,q_i) - \Sigma_{i=1}^k \lambda_i q_i^T \nabla_{q_i} g_i(u,v,q_i) - u^T \Sigma_{i=1}^k \lambda_i [\nabla_u f_i(u,v) \\ +B_i z_i + \nabla_{q_i} g_i(u,v,q_i)] - \Sigma_{i=1}^k \lambda_i [h_i(x,y,p_i) - p_i^T \nabla_{p_i} h_i(x,y,p_i)] \\ +y^T \Sigma_{i=1}^k \lambda_i [\nabla_y f_i(x,y) - D_i w_i + \nabla_{p_i} h_i(x,y,p_i)] \end{pmatrix} \in K \setminus \{0\}.$$

Since $\lambda \in intK^*$,we get

$$\sum_{i=1}^k \lambda_i \begin{bmatrix} f_i(u,v) - f_i(x,y) - (v^T D_i v)^{\frac{1}{2}} - (x^T B_i x)^{\frac{1}{2}} + u^T B_i z_i + y^T D_i w_i \\ +\Sigma_{i=1}^k \lambda_i g_i(u,v,q_i) - \Sigma_{i=1}^k \lambda_i q_i^T \nabla_{q_i} g_i(u,v,q_i) - u^T \Sigma_{i=1}^k \lambda_i [\nabla_u f_i(u,v) \\ +B_i z_i + \nabla_{q_i} g_i(u,v,q_i)] - \Sigma_{i=1}^k \lambda_i h_i(x,y,p_i) + \Sigma_{i=1}^k \lambda_i p_i^T \nabla_{p_i} h_i(x,y,p_i) \\ +y^T \Sigma_{i=1}^k \lambda_i [\nabla_y f_i(x,y) - D_i w_i + \nabla_{p_i} h_i(x,y,p_i)] \end{bmatrix} > 0. \tag{3.9}$$

Now using (3.2) and (3.6) in Schwartz inequality,we obtain

$$x^T B_i z_i \leq (x^T B_i x)^{\frac{1}{2}}, v^T D_i w_i \leq (v^T D_i v)^{\frac{1}{2}}, i = 1, 2, ..., k. \tag{3.10}$$

So, (3.9) in lieu of (3.10) becomes

$$\sum_{i=1}^k \lambda_i \begin{bmatrix} f_i(u,v) - f_i(x,y) - v^T D_i w_i - x^T B_i z_i + g_i(u,v,q_i) \\ -q_i^T \nabla_{q_i} g_i(u,v,q_i) - u^T [\nabla_u f_i(u,v) + \nabla_{q_i} g_i(u,v,q_i)] \\ -h_i(x,y,p_i) + p_i^T \nabla_{p_i} h_i(x,y,p_i) + y^T [\nabla_y f_i(x,y) + \nabla_{p_i} h_i(x,y,p_i)] \end{bmatrix} > 0. \tag{3.11}$$

From hypothesis (i), we have $f_i(.,v) + (.)^T B_i z_i$ is higher order $K - (\Phi, \rho)$-convex at u with respect to $g_i(u,v,q_i), i = 1, 2, ..., k$.
So, we get

$$\begin{pmatrix} f_1(x,v) + x^T B_1 z_1 - f_1(u,v) - u^T B_1 z_1 - g_1(u,v,q_1) + q_1^T \nabla_{q_1} g_1(u,v,q_1) \\ -\Phi_1(x,u;(\nabla_u f_1(u,v) + B_1 z_1 + \nabla_{q_1} g_1(u,v,q_1), \rho_1)), ..., \\ f_k(x,v) + x^T B_k z_k - f_k(u,v) - u^T B_k z_k - g_k(u,v,q_k) + q_k^T \nabla_{q_k} g_k(u,v,q_k) \\ -\Phi_1(x,u;(\nabla_u f_k(u,v) + B_k z_k + \nabla_{q_k} g_k(u,v,q_k), \rho_k)) \end{pmatrix} \in K. \tag{3.12}$$

As $\lambda \in intK^*$, from (3.12) we get

$$\sum_{i=1}^k \lambda_i [f_i(x,v) + x^T B_i z_i - f_i(u,v) - u^T B_i z_i - g_i(u,v,q_i) + q_i^T \nabla_{q_i} g_i(u,v,q_i)]$$
$$- \sum_{i=1}^k \lambda_i \Phi_1(x,u;(\nabla_u f_i(u,v) + B_i z_i + \nabla_{q_i} g_i(u,v,q_i), \rho_i)) \geq 0.$$

$$\Rightarrow \sum_{i=1}^{k} \lambda_i[f_i(x,v) + x^T B_i z_i - f_i(u,v) - u^T B_i z_i - g_i(u,v,q_i) + q_i^T \nabla_{q_i} g_i(u,v,q_i)]$$
$$\geq \sum_{i=1}^{k} \lambda_i \Phi_1(x,u;(\nabla_u f_i(u,v) + B_i z_i + \nabla_{q_i} g_i(u,v,q_i),\rho_i)). \tag{3.13}$$

Using Jension's inequality (lemma 2.1) in (3.13), we get

$$\sum_{i=1}^{k} \lambda_i[f_i(x,v) + x^T B_i z_i - f_i(u,v) - u^T B_i z_i - g_i(u,v,q_i) + q_i^T \nabla_{q_i} g_i(u,v,q_i)]$$
$$\geq \Phi_1(x,u;\sum_{i=1}^{k} \lambda_i(\nabla_u f_i(u,v) + B_i z_i + \nabla_{q_i} g_i(u,v,q_i),\rho_i)). \tag{3.14}$$

From hypothesis (iii) of theorem 3.1, constraint (3.5) and inequality (3.14), we obtain

$$\sum_{i=1}^{k} \lambda_i[f_i(x,v) + x^T B_i z_i - f_i(u,v) - u^T B_i z_i - g_i(u,v,q_i) + q_i^T \nabla_{q_i} g_i(u,v,q_i)]$$
$$\geq -u^T \sum_{i=1}^{k} \lambda_i[\nabla_u f_i(u,v) + B_i z_i + \nabla_{q_i} g_i(u,v,q_i]$$
$$\Rightarrow \sum_{i=1}^{k} \lambda_i \left[\begin{array}{c} f_i(x,v) + x^T B_i z_i - f_i(u,v) - g_i(u,v,q_i) \\ +q_i^T \nabla_{q_i} g_i(u,v,q_i) + u^T(\nabla_u f_i(u,v) + \nabla_{q_i} g_i(u,v,q_i)) \end{array} \right] \geq 0. \tag{3.15}$$

Again from hypothesis (ii), we have $-[f_i(x,.) - (.)^T D_i w_i]$ is higher order $K-(\Phi_2,\rho)$-convex at y with respect to $-h_i(x,y,p_i), i = 1,2,...,k$.
So, we get

$$\left(\begin{array}{c} -f_1(x,v) + v^T D_1 w_1 + f_1(x,y) - y^T D_1 w_1 + h_1(x,y,p_1) - p_1^T \nabla_{p_1} h_1(x,y,p_1) \\ -\Phi_2(v,y;(-[\nabla_y f_1(x,y) - D_1 w_1 + \nabla_{p_1} h_1(x,y,p_1)],\rho_1)),..., \\ -f_k(x,v) + v^T D_k w_k + f_k(x,y) - y^T D_k w_k + h_k(x,y,p_k) - p_k^T \nabla_{p_k} h_k(x,y,p_k) \\ -\Phi_2(v,y;(-[\nabla_y f_k(x,y) - D_k w_k + \nabla_{p_k} h_k(x,y,p_k)],\rho_k)) \end{array} \right) \in K. \tag{3.16}$$

As $\lambda \in intK^*$, from (3.16) we get

$$-\sum_{i=1}^{k} \lambda_i[f_i(x,v) - v^T D_i w_i - f_i(x,y) + y^T D_i w_i - h_i(x,y,p_i) + p_i^T \nabla_{p_i} h_i(x,y,p_i)]$$
$$-\sum_{i=1}^{k} \lambda_i \Phi_2(v,y;(-[\nabla_y f_i(x,y) - D_i w_i + \nabla_{p_i} h_i(x,y,p_i)],\rho_i)) \geq 0$$

$$\Rightarrow -\sum_{i=1}^{k}\lambda_i[f_i(x,v)-v^TD_iw_i-f_i(x,y)+y^TD_iw_i-h_i(x,y,p_i)+p_i^T\nabla_{p_i}h_i(x,y,p_i)]$$

$$\geq \sum_{i=1}^{k}\lambda_i\Phi_2(v,y;(-[\nabla_y f_i(x,y)-D_iw_i+\nabla_{p_i}h_i(x,y,p_i)],\rho_i)). \quad (3.17)$$

Using Jension's inequality (lemma 2.1) in (3.17), we get

$$-\sum_{i=1}^{k}\lambda_i[f_i(x,v)-v^TD_iw_i-f_i(x,y)+y^TD_iw_i-h_i(x,y,p_i)+p_i^T\nabla_{p_i}h_i(x,y,p_i)]$$

$$\geq \Phi_2(v,y;(-\sum_{i=1}^{k}\lambda_i([\nabla_y f_i(x,y)-D_iw_i+\nabla_{p_i}h_i(x,y,p_i)],\rho_i))). \quad (3.18)$$

From hypothesis (iii) of theorem 3.1, constraint (3.1) and inequality (3.18), we obtain

$$-\sum_{i=1}^{k}\lambda_i[f_i(x,v)-v^TD_iw_i-f_i(x,y)+y^TD_iw_i-h_i(x,y,p_i)+p_i^T\nabla_{p_i}h_i(x,y,p_i)]$$

$$\geq -y^T[-\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)-D_iw_i+\nabla_{p_i}h_i(x,y,p_i)$$

$$\Rightarrow \sum_{i=1}^{k}\lambda_i\begin{bmatrix} -f_i(x,v)+v^TD_iw_i+f_i(x,y)+h_i(x,y,p_i) \\ -p_i^T\nabla_{p_i}h_i(x,y,p_i)-y^T(\nabla_y f_i(x,y)+\nabla_{p_i}h_i(x,y,p_i)) \end{bmatrix}\geq 0. \quad (3.19)$$

Adding (3.15) and (3.19), we obtained

$$\sum_{i=1}^{k}\lambda_i\begin{bmatrix} -f_i(u,v)+f_i(x,y)+v^TD_iw_i+x^TB_iz_i-g_i(u,v,q_i) \\ +q_i^T\nabla_{q_i}g_i(u,v,q_i)+u^T[\nabla_u f_i(u,v)+\nabla_{q_i}g_i(u,v,q_i)] \\ +h_i(x,y,p_i)-p_i^T\nabla_{p_i}h_i(x,y,p_i)-y^T[\nabla_y f_i(x,y)+\nabla_{p_i}h_i(x,y,p_i)] \end{bmatrix}\geq 0$$

$$\Rightarrow \sum_{i=1}^{k}\lambda_i\begin{bmatrix} f_i(u,v)-f_i(x,y)-v^TD_iw_i-x^TB_iz_i+g_i(u,v,q_i) \\ -q_i^T\nabla_{q_i}g_i(u,v,q_i)-u^T[\nabla_u f_i(u,v)+\nabla_{q_i}g_i(u,v,q_i)] \\ -h_i(x,y,p_i)+p_i^T\nabla_{p_i}h_i(x,y,p_i)+y^T[\nabla_y f_i(x,y)+\nabla_{p_i}h_i(x,y,p_i)] \end{bmatrix}\leq 0. \quad (3.20)$$

This is a contradiction to (3.11). Hence we proved.

Remark 3.1 If we replace (i) and (ii) of Theorem 3.1 by
(a) $f_i(.,v)+(.)^TB_iz_i$ is higher order $K-(\Phi,\rho)$-pseudo convex at u with respect to $g_i(u,v,q_i), i=1,2,...,k$,
(b) $-[f_i(x,.)-(.)^TD_iw_i]$ is higher order $K-(\Phi,\rho)$-pseudo convex at u with respect to $-h_i(x,y,p_i), i=1,2,...,k$, then the same conclusion of Theorem 3.1 also holds.

In order to prove the strong duality theorem, we shall make use of the following lemma established by Suneja et al. [[20]]. It gives Fritz John type necessary optimality conditions for a weakly efficient solution of (WHMP).

Lemma 3.1
If x^* is a weakly efficient solution of (MP), then there exist $\alpha^*\in K^*, \beta^*\in Q^*$, not both zero, such that $(x-\overline{x})^T[\alpha^{*T}\nabla f(\overline{x})+\beta^{*T}\nabla g(\overline{x})]\geq 0, \forall x\in C$ and $\beta^{*T}g(\overline{x})=0$.

Theorem 3.2 (Strong Duality) Let $f : S_1 \times S_2 \to R^k$ be a twice differentiable function and let $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ be a weak efficient solution of primal (WHMP). Suppose that
(i) the matrix $\nabla_{p_i p_i} h_i(\overline{x}, \overline{y}, \overline{p_i}), i = 1, 2, ..., k$, is positive definite or negative definite,
(ii) the set of vectors $\{\nabla_y f_1(\overline{x}, \overline{y}) - D_1 w_1, ..., \nabla_y f_k(\overline{x}, \overline{y}) - D_k w_k\}$ are linearly independent,
(iii)the vectors $\sum_{i=1}^{k} \overline{\lambda}_i [\nabla_y (h_i(\overline{x}, \overline{y}, \overline{p_i})) - \nabla_{p_i} h_i(\overline{x}, \overline{y}, \overline{p_i})) + \nabla_{yy} f_i(\overline{x}, \overline{y}) \overline{p_i}] \notin$ $span \left\{ \nabla_y f_1(\overline{x}, \overline{y}) - D_1 w_1, ..., \nabla_y f_k(\overline{x}, \overline{y}) - D_k w_k \right\}$,

(iv)for some $\overline{\lambda} \in intK^*$ and $\overline{p_i} \in R^n, \overline{p_i} \neq 0 (i = 1, 2, ..., k)$ implies that $\nabla_y (\overline{\lambda}^T h(\overline{x}, \overline{y}, \overline{p}) - \nabla_p (\overline{\lambda}^T h(\overline{x}, \overline{y}, \overline{p}) + \nabla_{yy} (\overline{\lambda}^T f(\overline{x}, \overline{y}) \overline{p} \neq 0$,

(v) $h_i(\overline{x}, \overline{y}, 0) = 0, g_i(\overline{x}, \overline{y}, 0) = 0, \nabla_{p_i} h_i(\overline{x}, \overline{y}, 0) = 0, \nabla_y h_i(\overline{x}, \overline{y}, 0) = 0$, and $\nabla_x h_i(\overline{x}, \overline{y}, 0) = \nabla_{q_i} g_i(\overline{x}, \overline{y}, 0), i = 1, 2, ..., k$;
(vi) K is a closed convex pointed cone with $R^k_+ \subseteq K$.
Then **(a)** $\overline{p}_i = 0, \forall i$ and **(b)**there exist $\overline{z}_i \in R^n$ such that $(\overline{x}, \overline{y}, \overline{z}, \overline{\lambda}, \overline{q} = 0)$ is feasible solution for dual (WHMD) and two objective values are equal. Also, if the hypothesis of theorem 3.1 are satisfied for all feasible solution of primal (WHMP)and dual (WHMD), then $(\overline{x}, \overline{y}, \overline{z}, \overline{\lambda}, \overline{q} = 0)$ is an efficient solution of dual (WHMD).
Proof: Since $(\overline{x}, \overline{y}, \overline{w}, \overline{\lambda}, \overline{p})$ is weakly efficient solution of (WHMP), by lemma 3.1, there exist $\overline{\alpha} \in K^*, \sum_{i=1}^{k} \overline{\alpha}_i = \mu, \overline{\beta} \in C_2, \overline{\gamma} \in R_+, \overline{\delta} \in R_+, \overline{z} \in R^n$ such that

$$(x - \overline{x})^T \left\{ \begin{array}{c} \sum_{i=1}^{k} \overline{\alpha}_i [\nabla_x f_i(\overline{x}, \overline{y}) + B_i \overline{z}_i] + [\sum_{i=1}^{k} \overline{\lambda}_i (\nabla_{xy} f_i(\overline{x}, \overline{y}))] [\overline{\beta} - \mu \overline{y}] \\ + \sum_{i=1}^{k} \overline{\lambda}_i (\nabla_x h_i(\overline{x}, \overline{y}, \overline{p}_i)) \mu + \sum_{i=1}^{k} \overline{\lambda}_i [\nabla_{p_i x} h_i(\overline{x}, \overline{y}, \overline{p}_i)(\overline{\beta} - \mu(\overline{y} + \overline{p}_i))] \end{array} \right\} \geq 0, \tag{3.21}$$

$$(y - \overline{y})^T \left\{ \begin{array}{c} \sum_{i=1}^{k} ([\overline{\alpha}_i - \mu \lambda_i][\nabla_y f_i(\overline{x}, \overline{y}) - D_i \overline{w}_i]) \\ + [\sum_{i=1}^{k} \overline{\lambda}_i (\nabla_{yy} f_i(\overline{x}, \overline{y}))] [\overline{\beta} - \mu \overline{y}] \\ + \sum_{i=1}^{k} \overline{\lambda}_i [\nabla_y h_i(\overline{x}, \overline{y}, \overline{p}_i)) - \nabla_{p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i)] \mu \\ + \sum_{i=1}^{k} \overline{\lambda}_i [\nabla_{p_i y} h_i(\overline{x}, \overline{y}, \overline{p}_i)(\overline{\beta} - \mu(\overline{y} + \overline{p}_i))] \end{array} \right\} \geq 0, \forall y \in R^m, \tag{3.22}$$

$$(\lambda - \overline{\lambda})^T \left\{ \begin{array}{c} [\nabla_y f_1(\overline{x}, \overline{y}) - D_1 \overline{w}_1][\overline{\beta} - \mu \overline{y}] + (\nabla_{p_1} h_1(\overline{x}, \overline{y}, \overline{p}_1))[\overline{\beta} - \mu(\overline{y} + \overline{p}_1)] \\ + h_1(\overline{x}, \overline{y}, \overline{p}_1)\mu + \delta, ..., [\nabla_y f_k(\overline{x}, \overline{y}) - D_k \overline{w}_k][\overline{\beta} - \mu \overline{y}] \\ + (\nabla_{p_k} h_k(\overline{x}, \overline{y}, \overline{p}_k))[\overline{\beta} - \mu(\overline{y} + \delta + \overline{p}_k)] + h_k(\overline{x}, \overline{y}, \overline{p}_k)\mu + \delta \end{array} \right\}^T \geq 0, \quad \forall \lambda \in intK^*. \tag{3.23}$$

$$th \nabla_{p_i p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i)(\overline{\beta} - \mu(\overline{y} + \overline{p}_i))] = 0, i = 1, 2, ..., k, \tag{3.24}$$

$$\overline{\beta}^T \sum_{i=1}^{k} \overline{\lambda}_i [\nabla_y f_i(\overline{x}, \overline{y}) - D_i \overline{w}_i + \nabla_{p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i)] = 0, \tag{3.25}$$

$$(D_i \overline{\beta} + \overline{\gamma} D_i \overline{w}_i) = 0, i = 1, 2, ..., k, \tag{3.26}$$

$$\overline{\gamma}(\overline{w}_i^T D_i \overline{w}_i - 1), i = 1, 2, ..., k, \tag{3.27}$$

$$\overline{\delta}(\sum_{i=1}^{k} \overline{\lambda}_i - 1) = 0, \tag{3.28}$$

$$\overline{x} B_i \overline{z}_i = (\overline{x} B_i \overline{x})^{\frac{1}{2}}, i = 1, 2, ..., k, \tag{3.29}$$

$$\overline{z}_i B_i \overline{z}_i \leq 1, \tag{3.30}$$

$$(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}) \neq 0. \tag{3.31}$$

From hypothesis i and equation (4), we observed that

$$\overline{\beta} = \mu(\overline{y} + \overline{p}_i), i = 1, 2, ..., k. \tag{3.32}$$

Also inequality (3.22) and (3.23) imply, respectively

$$\left\{ \begin{array}{c} \sum_{i=1}^{k}([\overline{\alpha}_i - \mu\lambda_i][\nabla_y f_i(\overline{x}, \overline{y}) - D_i \overline{w}_i]) \\ +[\sum_{i=1}^{k} \overline{\lambda}_i(\nabla_{yy} f_i(\overline{x}, \overline{y}))][\overline{\beta} - \mu\overline{y}] \\ + \sum_{i=1}^{k} \overline{\lambda}_i[\nabla_y h_i(\overline{x}, \overline{y}, \overline{p}_i)) - \nabla_{p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i)]\mu \\ + \sum_{i=1}^{k} \overline{\lambda}_i[\nabla_{p_i y} h_i(\overline{x}, \overline{y}, \overline{p}_i)(\overline{\beta} - \mu(\overline{y} + \overline{p}_i))] \end{array} \right\} = 0, \tag{3.33}$$

and for each i

$$\nabla_y f_i(\overline{x}, \overline{y}) - D_i \overline{w}_i][\overline{\beta} - \mu\overline{y}] + (\nabla_{p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i))[\overline{\beta} - \mu(\overline{y} + \overline{p}_i)] + h_i(\overline{x}, \overline{y}, \overline{p}_i)\mu + \delta = 0. \tag{3.34}$$

Now we claim that $\overline{\alpha} \neq 0$. To do so, suppose $\overline{\alpha} = 0 \Rightarrow \overline{\alpha}_i = 0, \forall i$.
So $\mu = \sum_{i=1}^{k} \overline{\alpha}_i = 0$. Then (3.32) gives $\overline{\beta} = 0$, which along with equation (3.34), yields $\overline{\delta} = 0$.
From (3.26) and (3.27), we have
$\overline{\gamma} = \overline{\gamma}(\overline{w}_i^T D_i \overline{w}_i) = \overline{w}_i^T(\overline{\gamma} D_i \overline{w}_i) = \overline{w}_i^T D_i \overline{\beta} = 0$.
Thus $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}) = 0$, which contradicts inequality (3.31).
Hence $\overline{\alpha} \neq 0$.
Since $\overline{\alpha} \in K^*$ and $R_+^k \subseteq K$ implies $K^* \subseteq R_+^k$, we therefor get $\overline{\alpha} \geq 0$ or

$$\mu = \sum_{i=1}^{k} \overline{\alpha}_i > 0. \tag{3.35}$$

Now, using (3.32) and (3.35) in equation (3.33), we get

$$\sum_{i=1}^{k} \overline{\lambda}_i[\nabla_y h_i(\overline{x}, \overline{y}, \overline{p}_i)) - \nabla_{p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i) + (\nabla_{yy} f_i(\overline{x}, \overline{y}))\overline{p}_i] = \frac{1}{\mu} \sum_{i=1}^{k}([\overline{\alpha}_i - \mu\lambda_i][\nabla_y f_i(\overline{x}, \overline{y}) - D_i \overline{w}_i]). \tag{3.36}$$

Suppose that for each i, $\overline{p}_i \neq 0$, then hypothesis (iv) imply that
$\sum_{i=1}^{k} \overline{\lambda}_i[\nabla_y h_i(\overline{x}, \overline{y}, \overline{p}_i)) - \nabla_{p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i) + (\nabla_{yy} f_i(\overline{x}, \overline{y}))\overline{p}_i] \neq 0$, which in view of equation (3.36) contradicts hypothesis (iii).
Therefore

$$\overline{p}_i = 0, \forall i. \tag{3.37}$$

So equation (3.36) and (3.37), yields

$$\sum_{i=1}^{k}([\overline{\alpha}_i-\mu\lambda_i][\nabla_y f_i(\overline{x},\overline{y})-D_i\overline{w}_i])=0. \tag{3.38}$$

Since the set of vectors $\{\nabla_y f_1(\overline{x},\overline{y})-D_1\overline{w}_1,...,\nabla_y f_k(\overline{x},\overline{y})-D_k\overline{w}_k\}$ are linearly independent (by hypothesis (ii)), equation (3.38) yields

$$\overline{\alpha}_i=\mu\overline{\lambda}_i, i=1,2,...,k. \tag{3.39}$$

Again, using (3.37) in (3.32), we get

$$\overline{\beta}=\mu\overline{y}. \tag{3.40}$$

Using (3.35), (3.37),(3.39) and (3.40) in (3.21), we get

$$(x-\overline{x})^T[\sum_{i=1}^{k}\overline{\lambda}_i[\nabla_x f_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_x h_i(\overline{x},\overline{y},\overline{p}_i)]]\geq 0,\forall x\in C_1. \tag{3.41}$$

Let $x\in C_1$. Then $x+\overline{x}\in C_1$. So (3.41) implies

$$x^T[\sum_{i=1}^{k}\overline{\lambda}_i[\nabla_x f_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_x h_i(\overline{x},\overline{y},\overline{p}_i)]]\geq 0,\forall x\in C_1.$$

Therefore

$$x^T[\sum_{i=1}^{k}\overline{\lambda}_i[\nabla_x f_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_x h_i(\overline{x},\overline{y},\overline{p}_i)]]\geq 0,\forall x\in C_1. \tag{3.42}$$

Also, from (3.35), (3.40) and $\overline{\beta}\in C_2$, we obtained

$$\overline{y}\in C_2. \tag{3.43}$$

Also, from (3.35) and (3.39), we get

$$\sum_{i=1}^{k}\overline{\lambda}_i=1,\ and\ \lambda>0. \tag{3.44}$$

Hence from (3.30), (3.42), (3.43) and (3.44), we obtained that $(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{q}=0)$ satisfies the dual constraint (3.5)-(3.8) i.e. it is a feasible solution of the dual (WHMD).

Now, putting $x=0$ and $x=2\overline{x}$, simultaneously in (3.41), we get

$$\overline{x}^T[\sum_{i=1}^{k}\overline{\lambda}_i[\nabla_x f_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_x h_i(\overline{x},\overline{y},\overline{p}_i)]]\leq 0$$

and

$$\overline{x}^T[\sum_{i=1}^{k}\overline{\lambda}_i[\nabla_x f_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_x h_i(\overline{x},\overline{y},\overline{p}_i)]]\geq 0,$$

which implies

$$\overline{x}^T[\sum_{i=1}^{k}\overline{\lambda}_i[\nabla_x f_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_x h_i(\overline{x},\overline{y},\overline{p}_i)]]=0. \tag{3.45}$$

Again using (3.35) and (3.40) in (3.25), we get

$$\overline{y}^T[\sum_{i=1}^{k}\overline{\lambda}_i[\nabla_y f_i(\overline{x},\overline{y}) - D_i\overline{w}_i + \nabla_{p_i} h_i(\overline{x},\overline{y},\overline{p}_i)]] = 0. \tag{3.46}$$

From (3.26),(3.35) and (3.40), we get

$$D_i\overline{y} = \frac{\overline{\gamma}}{\mu}D_i\overline{w_i}. \tag{3.47}$$

Or

$$D_i\overline{y} = aD_i\overline{w_i},\ for\ a = \frac{\overline{\gamma}}{\mu} > 0. \tag{3.48}$$

Under this condition, the Schwartz inequality holds as equality. Therefore

$$\overline{y}D_i\overline{w}_i = (\overline{y}^T D_i\overline{y})^{\frac{1}{2}}(\overline{w}_i^T D_i\overline{w}_i)^{\frac{1}{2}}. \tag{3.49}$$

In case, $\overline{\gamma} > 0$, from (3.27), we get $\overline{w}_i^T D_i\overline{w}_i = 1$ for each i. So, (3.49) implies

$$\overline{y}D_i\overline{w}_i = (\overline{y}^T D_i\overline{y})^{\frac{1}{2}}.$$

In case, $\overline{\gamma} > 0$, from (3.47), we get $D_i\overline{y} = 0$ and so $\overline{y}^T D_i\overline{w}_i = 0 = (\overline{y}^T D_i\overline{y})^{\frac{1}{2}}$. Thus in either case,

$$\overline{y}D_i\overline{w}_i = (\overline{y}^T D_i\overline{y})^{\frac{1}{2}}. \tag{3.50}$$

So, from (3.29), (3.37), (3.45), (3.50) and hypothesis (v), we conclude that for every $i \in \{1, 2, ..., k\}$,
$f_i(\overline{x},\overline{y}) + (\overline{x}^T B_i\overline{x})^{\frac{1}{2}} - \overline{y}^T D_i\overline{w}_i + \sum_{i=1}^{k}\overline{\lambda}_i h_i(\overline{x},\overline{y},0) - \sum_{i=1}^{k}\overline{\lambda}_i[\nabla_{p_i} h_i(\overline{x},\overline{y},0)]$

$-\overline{y}^T\sum_{i=1}^{k}\overline{\lambda}_i[\nabla_y f_i(\overline{x},\overline{y}) - D_i\overline{w}_i + \nabla_{p_i} h_i(\overline{x},\overline{y},0)]$

$= f_i(\overline{x},\overline{y}) + \overline{x}^T B_i\overline{z}_i - (\overline{y}^T D_i\overline{y})^{\frac{1}{2}}$

$= f_i(\overline{x},\overline{y}) + \overline{x}^T B_i\overline{z}_i - (\overline{y}^T D_i\overline{y})^{\frac{1}{2}} + \sum_{i=1}^{k}\overline{\lambda}_i g_i(\overline{x},\overline{y},0) - \sum_{i=1}^{k}\overline{\lambda}_i[\nabla_{q_i} g_i(\overline{x},\overline{y},0)]$

$-\overline{y}^T\sum_{i=1}^{k}\overline{\lambda}_i[\nabla_y f_i(\overline{x},\overline{y}) - D_i\overline{w}_i + \nabla_{q_i} g_i(\overline{x},\overline{y},0)]$.

That is

$$L(\overline{x},\overline{y},\overline{\lambda},\overline{w},\overline{p}=0) = M(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{q}=0) \tag{3.51}$$

So, the two objective values are equal.
Now, we claim that $(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{q}=0)$ is an efficient solution of dual(WHMD). If this would not be the case, then there would exist a feasible solution $(\overline{u},\overline{v},\overline{\lambda},\overline{z},\overline{q}=0)$.

such that

$$\begin{pmatrix} (f_1(\overline{u},\overline{v})-(\overline{v}^TD_1\overline{v})^{\frac{1}{2}}+\overline{u}^TB_1\overline{z}_1+\sum_{i=1}^k\overline{\lambda_i}[g_i(\overline{u},\overline{v},\overline{q}_i=0)] \\ -\sum_{i=1}^k\overline{\lambda_i}[\overline{q}_i^T\nabla_{\overline{q}_i}g_i(\overline{u},\overline{v},\overline{q}_i=0)] \\ -\overline{u}^T\sum_{i=1}^k\overline{\lambda}_i[\nabla_uf_i(\overline{u},\overline{v})+B_i\overline{z}_i+\nabla_{q_i}g_i(\overline{u},\overline{v},\overline{q}_i=0)]) \\ -(f_1(\overline{x},\overline{y})-(\overline{y}^TD_1\overline{y})^{\frac{1}{2}}+\overline{x}^TB_1\overline{z}_1 \\ -\sum_{i=1}^k\overline{\lambda}_i[g_i(\overline{x},\overline{y},\overline{q}_i=0)+\overline{q}_i^T\nabla_{q_i}g_i(\overline{x},\overline{y},\overline{q}_i=0)] \\ +\overline{x}^T\sum_{i=1}^k\overline{\lambda_i}[\nabla_xf_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_{q_i}g_i(\overline{x},\overline{y},\overline{q}_i=0)]), \\ ..., \\ (f_k(\overline{u},\overline{v})-(\overline{v}^TD_k\overline{v})^{\frac{1}{2}}+\overline{u}^TB_k\overline{z}_k+\sum_{i=1}^k\overline{\lambda_i}[g_i(\overline{u},\overline{v},\overline{q}_i=0)] \\ -\sum_{i=1}^k\overline{\lambda_i}[\overline{q}_i^T\nabla_{\overline{q}_i}g_i(\overline{u},\overline{v},\overline{q}_i=0)] \\ -\overline{u}^T\sum_{i=1}^k\overline{\lambda}_i[\nabla_uf_i(\overline{u},\overline{v})+B_i\overline{z}_i+\nabla_{q_i}g_i(\overline{u},\overline{v},\overline{q}_i=0)]) \\ -(f_k(\overline{x},\overline{y})-(\overline{y}^TD_k\overline{y})^{\frac{1}{2}}+\overline{x}^TB_k\overline{z}_k \\ -\sum_{i=1}^k\overline{\lambda}_i[g_i(\overline{x},\overline{y},\overline{q}_i=0)+\overline{q}_i^T\nabla_{q_i}g_i(\overline{x},\overline{y},\overline{q}_i=0)] \\ +\overline{x}^T\sum_{i=1}^k\overline{\lambda_i}[\nabla_xf_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_{q_i}g_i(\overline{x},\overline{y},\overline{q}_i=0)]) \end{pmatrix} \in K\setminus\{0\}. \quad \textbf{(3.52)}$$

From hypothesis (v) and (3.45), we obtained

$$\overline{x}^T[\sum_{i=1}^k\overline{\lambda}_i[\nabla_xf_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_{q_i}g_i(\overline{x},\overline{y},\overline{q}_i)]]=0. \tag{3.53}$$

So, using (3.29),(3.46),(3.50) and (3.53), we can write for each $i\in\{1,2,...,k\}$

$$\begin{aligned} f_i(\overline{x},\overline{y})-(\overline{y}^TD_i\overline{y})^{\frac{1}{2}}+\overline{x}^TB_i\overline{z}_k-\sum_{i=1}^k\overline{\lambda}_i[g_i(\overline{x},\overline{y},\overline{q}_i=0)+\overline{q}_i^T\nabla_{q_i}g_i(\overline{x},\overline{y},\overline{q}_i=0)] \\ +\overline{x}^T\sum_{i=1}^k\overline{\lambda_i}[\nabla_xf_i(\overline{x},\overline{y})+B_i\overline{z}_i+\nabla_{q_i}g_i(\overline{x},\overline{y},\overline{q}_i=0)] \\ =f_i(\overline{x},\overline{y})+(\overline{x}^TB_i\overline{x})^{\frac{1}{2}}-\overline{y}^TD_i\overline{w}_i-\sum_{i=1}^k\overline{\lambda}_i[h_i(\overline{x},\overline{y},\overline{p}_i=0)+\overline{p}_i^T\nabla_{p_i}h_i(\overline{x},\overline{y},\overline{p}_i=0)] \\ +\overline{y}^T\sum_{i=1}^k\overline{\lambda_i}[\nabla_yf_i(\overline{x},\overline{y})-D_i\overline{w}_i+\nabla_{p_i}h_i(\overline{x},\overline{y},\overline{p}_i=0)]. \end{aligned} \tag{3.54}$$

Hence using (3.54) in (3.52), we get

$$\begin{pmatrix} (f_1(\overline{u},\overline{v})-(\overline{v}^TD_1\overline{v})^{\frac{1}{2}}+\overline{u}^TB_1\overline{z}_1+\sum_{i=1}^k\overline{\lambda_i}[g_i(\overline{u},\overline{v},\overline{q}_i=0)] \\ -\sum_{i=1}^k\overline{\lambda_i}[\overline{q}_i^T\nabla_{\overline{q}_i}g_i(\overline{u},\overline{v},\overline{q}_i=0)] \\ -\overline{u}^T\sum_{i=1}^k\overline{\lambda}_i[\nabla_uf_i(\overline{u},\overline{v})+B_i\overline{z}_i+\nabla_{q_i}g_i(\overline{u},\overline{v},\overline{q}_i=0)]) \\ -(f_1(\overline{x},\overline{y})+(\overline{x}^TB_1\overline{x})^{\frac{1}{2}}-\overline{y}^TD_1\overline{w}_1 \\ -\sum_{i=1}^k\overline{\lambda}_i[h_i(\overline{x},\overline{y},\overline{p}_i=0)+\overline{p}_i^T\nabla_{p_i}h_i(\overline{x},\overline{y},\overline{p}_i=0)] \\ +\overline{y}^T\sum_{i=1}^k\overline{\lambda_i}[\nabla_yf_i(\overline{x},\overline{y})-D_i\overline{w}_i+\nabla_{p_i}h_i(\overline{x},\overline{y},\overline{p}_i=0)]), \\ ..., \\ (f_k(\overline{u},\overline{v})-(\overline{v}^TD_k\overline{v})^{\frac{1}{2}}+\overline{u}^TB_k\overline{z}_k+\sum_{i=1}^k\overline{\lambda_i}[g_i(\overline{u},\overline{v},\overline{q}_i=0)] \\ -\sum_{i=1}^k\overline{\lambda_i}[\overline{q}_i^T\nabla_{\overline{q}_i}g_i(\overline{u},\overline{v},\overline{q}_i=0)] \\ -\overline{u}^T\sum_{i=1}^k\overline{\lambda}_i[\nabla_uf_i(\overline{u},\overline{v})+B_i\overline{z}_i+\nabla_{q_i}g_i(\overline{u},\overline{v},\overline{q}_i=0)]) \\ -(f_k(\overline{x},\overline{y})+(\overline{x}^TB_k\overline{x})^{\frac{1}{2}}-\overline{y}^TD_k\overline{w}_k \\ -\sum_{i=1}^k\overline{\lambda}_i[h_i(\overline{x},\overline{y},\overline{p}_i=0)+\overline{p}_i^T\nabla_{p_i}h_i(\overline{x},\overline{y},\overline{p}_i=0)] \\ +\overline{y}^T\sum_{i=1}^k\overline{\lambda_i}[\nabla_yf_i(\overline{x},\overline{y})-D_i\overline{w}_i+\nabla_{p_i}h_i(\overline{x},\overline{y},\overline{p}_i=0)]) \end{pmatrix} \in K\setminus\{0\},$$

which contradicts the weak duality theorem. Hence $\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{q}=0)$ is an efficient solution of dual (WHMD)

Theorem 3.3 (**Converse Duality**) Let $f: S_1\times S_2\to R^k$ be a twice differentiable function and let $(\overline{u},\overline{v},\overline{\lambda},\overline{z},\overline{q})$ be a weak efficient solution of primal (WHMP). Suppose that

(i) the matrix $\nabla_{q_iq_i}g_i(\overline{u},\overline{v},\overline{q_i}), i=1,2,...,k$, is positive definite or negative definite,
(ii) the set of vectors $\{\nabla_xf_1(\overline{u},\overline{v})-B_1z_1,...,\nabla_xf_k(\overline{u},\overline{v})-B_kz_k\}$ are linearly independent,
(iii)the vectors $\sum_{i=1}^k\overline{\lambda}_i[\nabla_x(g_i(\overline{u},\overline{v},\overline{q_i}))-\nabla_{q_i}g_i(\overline{u},\overline{v},\overline{q_i}))+\nabla_{xx}f_i(\overline{u},\overline{v})\overline{q_i}]\notin$ $span\left\{\ \nabla_xf_1(\overline{u},\overline{v})-B_1z_1,...,\nabla_xf_k(\overline{u},\overline{v})-B_kz_k\ \right\}$,

(iv)for some $\overline{\lambda}\in intK^*$ and $\overline{q_i}\in R^n,\overline{q_i}\neq 0(i=1,2,...,k)$ implies that
$\nabla_x(\overline{\lambda}^Tg(\overline{u},\overline{v},\overline{q})-\nabla_q(\overline{\lambda}^Tg(\overline{u},\overline{v},\overline{q})+\nabla_{xx}(\overline{\lambda}^Tf(\overline{u},\overline{v})\overline{q}\neq 0$,

(v) $g_i(\overline{u},\overline{v},0)=0, h_i(\overline{u},\overline{v},0)=0,\nabla_{q_i}g_i(\overline{u},\overline{v},0)=0,\nabla_xg_i(\overline{u},\overline{v},0)=0$, and $\nabla_{p_i}h_i(\overline{u},\overline{v},0)=\nabla_yg_i(\overline{u},\overline{v},0), i=1,2,...,k$;
(vi) K is a closed convex pointed cone with $R_+^k\subseteq K$.
Then (a) $\overline{q}_i=0,\forall i$ and (b)there exist $\overline{w}_i\in R^m$ such that $(\overline{u},\overline{v},\overline{w},\overline{\lambda},\overline{p}=0)$ is feasible solution for dual (WHMD) and two objective values are equal. Also, if the hypothesis of theorem 3.1 are satisfied for all feasible solution of primal (WHMP) and dual(WHMD), then $(\overline{u},\overline{v},\overline{w},\overline{\lambda},\overline{p}=0)$ is an efficient solution of dual (WHMP).

4. Special Cases

In this section, we consider some special cases of our problems as follows:
(i) If B_i and $D_i, i=1,2,...,k$ are null matrices, then (WHMP) and dual(WHMD) are reduced to the problem (WP) and (WD) considered by **Gupta et al.** [[9]]
(ii) If we take $(x^TB_ix)^{\frac{1}{2}}=s(x|D_i)$ and$(v^TD_iv)^{\frac{1}{2}}=s(v|E_i)$ with $D_i=\{B_ix: x^TB_ix\leq 1\}$ and $E_i=\{D_iv: v^TD_iv\leq 1\}$ and $h_i(x,y,p_i)=g_i(x,y,q_i)=0, i=1,2,...,k$, then our problem (WHMP) and dual(WHMD)are reduced a pair of Wolfe type nond-ifferentiable symmetric dual problem studied by **Kim and Lee** [[13]].

(iii) If $k = 1$ with B and D are null matrices, then our problem (WHMP) and dual(WHMD) are reduced to the problem studied by **Gulati and Gupta** [[5]].

5. Conclusion

In this paper, a new class of generalized $K-(\Phi,\rho)$ convex function is introduced, in which the sub linearity property of F as in literature is relaxed by imposing the convexity assumption on Φ in its third argument with example. This new class of generalized convex function is more generalized than the (F,α,ρ,d)-convex functions, (C,α,ρ,d)-convex functions and $K-(F,\alpha,\rho,d)$ convex functions. Also, a new model of higher order Wolfe type non-differentiable multi-objective symmetric dual programs is formulated and the weak, strong and converse duality theorem under higher order $K-(\Phi,\rho)$ convex functions are established. Based on this concept, higher order minmax mixed integer programming and higher order fractional programming over cone can be established.

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References

[1] R. P. Agarwal, I. Ahmad, A. Jayswal,: Higher order symmetric duality in non-differentiable multi-objective programming problems involving generalized cone convex functions. Mathematical and Computer Modeling, 52,1644-1650, 2010.

[2] M. Kassem, Abd EI-Hady : Higher-order symmetric duality in vector optimization problem involving generalized cone-invex functions. Applied Mathematics and Computations, 209,405-409,2009.

[3] G. Carsiti, M. Ferrara, A. Stefanescu : Mathematical programming with (Φ,ρ)-invexity, In: I.V. Konnov, D. I. Luc, A. Rubinov (eds) Generalized Convexity and Related Topics, Lecture Notes in Economics and Mathematical System, 583,167-176, 2006.

[4] B. D. Craven: Invex function and constrained local minima. Bulletin of Australia Mathematical Society, 24, 357-366, 1981.

[5] T. R. Gulati, S. K. Gupta : Higher-order non-differentiable symmetric duality with generalized F-convexity, Journal of Mathematical Analysis and Application, 329, 229-237,2007.

[6] T. R. Gulati, S. K. Gupta: Higher-order symmetric duality with cone constraints, Applied Mathematics Letters, 22, 776-781,2009.

[7] T. R. Gulati, H. Sani.: Higher order (F,α,β,ρ,d)-convexity and its applications in fractional programming. European J. Pure Appl. Math., 4, 3, 266-275, 2011.

[8] S. K. Gupta, A. Jayswal : Multi-objective higher-order symmetric duality involving generalized cone-invex functions, Computers and Mathematics with Applications, 60, 3187-3192,2010.

[9] S. K. Gupta, N. Kailey, M. K. Sharma : Higher order (F,α,ρ,d)-convexity and symmetric duality in multi-objective programming, Computers and Mathematics with Applications, 60,2373-2381,2010.

[10] S. K. Gupta, N. Kailey, S. Kumar: Duality for non-differentiable multi-objective higher order symmetric programs over cones involving generalized (F,α,ρ,d)-convexity, Journal of Inequality and Applications, 2012,298-312, 2012.

[11] M. A. Hanson : On sufficiency of Kuhn-Tucker conditions, J. Math. Anal. Appl., 80,545-550,1981.

[12] A. Jayswal,K. Kumari : Higher order duality for multiobjective programming problem involving (Φ,ρ)-invex functions, J. Egyptian Math. Soc., 23,12-19,2015.

[13] D. S. Kim, Y. J. Lee: Non-differentiable higher order duality in multi-objective programming involving cones, Nonlinear Anal. 71, e2474-e2480,2009.

[14] Z. A. Liang, H. X. Huang, P. M. Pardalos: Optimality conditions and duality for a class of nonlinear fractional programming problems, Journal of Optimization Theory and Applications, 110, 611-619,2001.

[15] S. K. Mishra,: Higher order generalized invexity and duality in mathematical programming, J. Math. Anal. Appl., 247, 173-182,2000.
[16] S. K. Mishra : Non-differentiable higher-order symmetric duality in mathematical programming with generalized invexity, European Journal of Operational Research, 167, 28-34,2005.
[17] S. K. Padhan, C. Nahak: Higher-order symmetric duality in multi-objective programming problems under higher-order invexity, Applied Mathematics and Computation, 218,1705-1712,2011.
[18] V. Preda: On efficiency and duality for multi-objective programs, Journal of Mathematical Analysis and Applications, 166,2,365-377,1992.
[19] S. K. Suneja, M. K. Srivastava, M. Bhatia: Higher order duality in multi-objective fractional programming with support functions. J. Math. Anal. Appl., 347, 8-17,2008.
[20] S. K. Suneja, S. Aggarwal, S. Davar: Multi-objective symmetric duality involving cones, European Journal of Operational Research, 141,471-479,2002.
[21] A. K. Tripathy, G. Devi: Wolfe type higher order multiple objective non-differentiable symmetric dual programming with generalized invex functions, Journal of Mathematical Modelling and Algorithms in Operations Research, 13,4,557-577,2014.
[22] D. Yuan,P. M. Pardalos, X. Liu, A. Chinchuluun: Non-differentiable mini-max fractional programming problem with (C, α, ρ, d)-convexity. Journal of Optimization Theory and Applications, 129,1,185-199,2006.

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