A STUDY OF INTUITIONISTIC FUZZY *I*-CONVERGENT DOUBLE SEQUENCE SPACES DEFINED BY COMPACT OPERATOR

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ABSTRACT. After the pioneering work of Zadeh[35], a huge number of research papers have been appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. It has a wide range of applications in various fields: population dynamics[1], chaos control[4], computer programming[5], nonlinear dynamical system[6], etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations where the use of classical theories breaks down.In this paper, we introduce ideal convergence of double sequences in intuitionistic fuzzy sequences defined by compact operator and study the fuzzy topology on the said spaces.

1. INTRODUCTION AND PRELIMINARIES:

The concept of intuitionistic fuzzy normed space[32] and of intuitionistic fuzzy 2-normed space[27] are the latest developments in fuzzy topology. Recently V. A. Khan and Yasmeen[[7],[9], [10],[11],[12]] studied the intuitionistic fuzzy Zweier *I*-convergent sequence spaces defined by paranorm, modulus function and Orlicz function. And recently Totur, Çanak and Önder [[2],[31], [34]] studied and defined some summable double sequence of fuzzy numbers.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical problems/matrices(double sequences) through the concept of density. The notion of I-convergence, which is a generalization of statistical convergence[[3],[19],[20],[21],[22],[23],[24],[25],[26],[28]] was introduced by Kostyrko, Salat and Wilczynski[15] by using the idea of I of subsets of the set of natural numbers $\mathbb{N}$ and further studied in[[30],[8],[13]]. Recently, the notion of statistical convergence of double sequences $x = (x_{ij})$ has been defined and studied by Mursaleen and Edely[18], and for fuzzy numbers by Savaş and Mursaleen[33], M. Mursaleen, H. M. Srivastava and S. K. Sharma[29]. Quite recently, Das et al.[16]

studied the notion of I and I^*- convergence of double sequences in $\mathbb{R}$.

We recall some notations and basic definitions used in this paper.

Definition 1.1 Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_k)$ is said to be I-convergent to a number L if, for every $\epsilon > 0$, the set

$$\{k \in \mathbb{N} :\mid x_k - L \mid \geq \epsilon\} \in I.$$

Definition 1.2 Let X be a non empty set. Then $\mathcal{F} \subset 2^X$ is said to be a *filter* on X if and only if $\phi \notin \mathcal{F}$, for $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

Definition 1.3 Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_k)$ is said to be I-Cauchy if, for each $\epsilon > 0$,there exists a number $N = N(\epsilon)$ such that the set $\{k \in \mathbb{N} :\mid x_k - x_N \mid \geq \epsilon\} \in I$.

Definition 1.4(See [8]) The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space(for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

   (1) $\mu(x,t) + \nu(x,t) \leq 1$,
   (2) $\mu(x,t) > 0$,
   (3) $\mu(x,t) = 1$ if and only if $x = 0$,
   (4) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
   (5) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,
   (6) $\mu(x,.) : (0, \infty) \to [0,1]$ is continuous,
   (7) $\lim_{t \to \infty} \mu(x,t) = 1$ and $\lim_{t \to 0} \mu(x,t) = 0$,
   (8) $\nu(x,t) < 1$,
   (9) $\nu(x,t) = 0$ if and only if $x = 0$,
   (10) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
   (11) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$,
   (12) $\nu(x,.) : (0, \infty) \to [0,1]$ is continuous,
   (13) $\lim_{t \to \infty} \nu(x,t) = 0$ and $\lim_{t \to 0} \nu(x,t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm.

Definition 1.5 Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \epsilon$ and $\nu(x_k - L, t) < \epsilon$ for all $k \geq k_0$. In this case we write $(\mu, \nu) - \lim x = L$.

Definition 1.6 Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_l, t) < \epsilon$ and $\nu(x_k - x_l, t) < \epsilon$ for all $k, l \geq k_0$.

Definition 1.7 Let K be the subset of natural numbers $\mathbb{N}$. Then the asymptotic density of K, denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bars denotes the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be statistically convergent to a number ℓ if, for each $\epsilon > 0$, the set $K(\epsilon) = \{k \leq n :\mid x_k - \ell \mid > \epsilon\}$ has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n} |\{k \leq n :\mid x_k - \ell \mid > \epsilon\}| = 0.$$

In this case we write $st - \lim x = \ell$.

Definition 1.8 A number sequence $x = (x_k)$ is said to be statistically Cauchy sequence if, for every $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that

$$\lim_n \frac{1}{n} |\{j \leq n :\mid x_j - x_N \mid \geq \epsilon\}| = 0.$$

The concepts of statistical convergence and statistical Cauchy for double sequences in intuitionistic fuzzy normed spaces have been studied by Mursaleen and Mohiuddine[28].

Definition 1.9 Let $I \subset 2^{\mathbb{N}}$ be a non trivial ideal and $(X, \mu, \nu, *, \diamond)$ be an IFNS. A sequence $x = (x_k)$ of elements of X is said to be I-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$, the set

$$\{k \in \mathbb{N} : \mu(x_k - L, t) \geq 1 - \epsilon \text{ or } \nu(x_k - L, t) \leq \epsilon\} \in I.$$

In this case L is called the I-limit of the sequence (x_k) with respect to the intuitionistic fuzzy norm (μ, ν) and we write $I_{(\mu,\nu)} - \lim x_k = L$.

Definition 1.10(See [14]) Let X and Y be two normed linear spaces and $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D} \subset X$. Then, the operator T is said to be bounded, if there exists a positive real k such that

$$\|Tx\| \leq k\|x\|, \text{ for all } x \in \mathcal{D}(T).$$

The set of all bounded linear operators $\mathcal{B}(X, Y)$[17] is a normed linear spaces normed by

$$\|T\| = \sup_{x \in X, \|x\|=1} \|Tx\|$$

and $\mathcal{B}(X, Y)$ is a Banach space if Y is a Banach space.

Definition 1.11(see [14]) Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator (or completely continuous linear operator), if

(1) T is linear,
(2) T maps every bounded sequence (x_k) in X on to a sequence $(T(x_k))$ in Y which has a convergent subsequence.

The set of all compact linear operators $\mathcal{C}(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$ and $\mathcal{C}(X, Y)$ is Banach space, if Y is a Banach space.

2. I_2-Convergence in an IFNS

Definition 2.1 Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{ij})$ is said to be statistically convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$,

$$\delta(\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - L, t) \leq 1 - \epsilon \text{ or } \nu(x_{ij} - L, t) \geq \epsilon\}) = 0.$$

or equivalently

$$\lim_{mn} \frac{1}{mn} |\{i \leq m, j \leq n, : \mu(x_{ij} - L, t) \leq 1 - \epsilon \text{ or } \nu(x_{ij} - L, t) \geq \epsilon\}| = 0.$$

In this case we write $st^2_{(\mu,\nu)} - \lim x = L$.

Definition 2.2 Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{ij})$ is said to be statistically Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exist $N = N(\epsilon)$ and $M = M(\epsilon)$ such that for all $i, p \geq N$ and $j, q \geq M$,

$$\delta(\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - x_{pq}, t) \leq 1 - \epsilon \text{ or } \nu(x_{ij} - x_{pq}, t) \geq \epsilon\}) = 0.$$

Definition 2.3 Let I_2 be a non trivial ideal of $\mathbb{N} \times \mathbb{N}$ and $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. A double sequence $x = (x_{ij})$ of elements of X is said to be I_2 convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for each $\epsilon > 0$ and $t > 0$,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - L, t) \leq 1 - \epsilon \text{ or } \nu(x_{ij} - L, t) \geq \epsilon\} \in I_2.$$

In this case we write $I_2^{(\mu,\nu)} - \lim x = L$.

In this article we introduce the following sequence spaces:

$_2S^I_{(\mu,\nu)}(T) = \{(x_{ij}) \in {}_2\ell_\infty : \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{ij}) - L, t) \leq 1 - \epsilon \text{ or } \nu(T(x_{ij}) - L, t) \geq \epsilon\} \in I_2\};$

$_2S^I_{0(\mu,\nu)}(T) = \{(x_{ij}) \in {}_2\ell_\infty : \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{ij}), t) \leq 1 - \epsilon \text{ or } \nu(T(x_{ij}), t) \geq \epsilon\} \in I_2\}.$

We also define an open ball with centre x and radius r with respect to t as follows:

$$_2B_x(r,t)(T) = \{(y_{ij}) \in {}_2\ell_\infty : \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{ij}) - T(y_{ij}), t) \leq 1 - \epsilon$$
$$\text{or } \nu(T(x_{ij}) - T(y_{ij}), t) \geq \epsilon\} \in I_2\}.$$

3. Main results

Theorem 3.1 $_2S^I_{(\mu,\nu)}(T)$ and $_2S^I_{0(\mu,\nu)}(T)$ are linear spaces.

Proof. We shall prove the result for $_2S^I_{(\mu,\nu)}(T)$. The proof for the other space will follow similarly. Let $x = (x_{ij}), y = (y_{ij}) \in {}_2S^I_{(\mu,\nu)}(T)$ and α, β be scalars. Then for a given $\epsilon > 0$, we have

$$A_1 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu\left(T(x_{ij}) - L_1, \frac{t}{2|\alpha|}\right) \leq 1 - \epsilon \text{ or } \nu\left(T(x_{ij}) - L_1, \frac{t}{2|\alpha|}\right) \geq \epsilon\right\} \in I_2;$$

$$A_2 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu\left(T(y_{ij}) - L_2, \frac{t}{2|\beta|}\right) \leq 1 - \epsilon \text{ or } \nu\left(T(y_{ij}) - L_2, \frac{t}{2|\beta|}\right) \geq \epsilon\right\} \in I_2.$$

$$A_1^c = \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mu\left(T(x_{ij}) - L_1, \tfrac{t}{2|\alpha|}\right) > 1 - \epsilon \text{ or } \nu\left(T(x_{ij}) - L_1, \tfrac{t}{2|\alpha|}\right) < \epsilon \right\} \in \mathcal{F}(I_2);$$

$$A_2^c = \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mu\left(T(y_{ij}) - L_2, \tfrac{t}{2|\beta|}\right) > 1 - \epsilon \text{ or } \nu\left(T(y_{ij}) - L_2, \tfrac{t}{2|\beta|}\right) < \epsilon \right\} \in \mathcal{F}(I_2).$$

Define the set $A_3 = A_1 \cup A_2$, so that $A_3 \in I_2$. It follows that A_3^c is a non-empty set in $\mathcal{F}(I_2)$. We shall show that for each $(x_{ij}), (y_{ij}) \in {}_2S_{(\mu,\nu)}^I(T)$.

$A_3^c \subset \{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mu((\alpha T(x_{ij}) + \beta T(y_{ij}) - (\alpha L_1 + \beta L_2), t) > 1 - \epsilon$
or $\nu((\alpha T(x_{ij}) + \beta T(y_{ij})) - (\alpha L_1 + \beta L_2), t) < \epsilon\}$.

Let $(m,n) \in A_3^c$. In this case

$$\mu\left(T(x_{mn}) - L_1, \frac{t}{2 \mid \alpha \mid}\right) > 1 - \epsilon \text{ or } \nu\left(T(x_{mn}) - L_1, \frac{t}{2 \mid \alpha \mid}\right) < \epsilon$$

and

$$\mu\left(T(y_{mn}) - L_2, \frac{t}{2 \mid \beta \mid}\right) > 1 - \epsilon \text{ or } \nu\left(T(y_{mn}) - L_2, \frac{t}{2 \mid \beta \mid}\right) < \epsilon.$$

We have

$$\mu\left((\alpha T(x_{mn}) + \beta T(y_{mn})) - (\alpha L_1 + \beta L_2), t\right)$$

$$\geq \mu\left(\alpha T(x_{mn}) - \alpha L_1, \frac{t}{2}\right) * \mu\left(\beta T(x_{mn}) - \beta L_2, \frac{t}{2}\right)$$

$$= \mu\left(T(x_{mn}) - L_1, \frac{t}{2 \mid \alpha \mid}\right) * \mu\left(T(x_{mn}) - L_2, \frac{t}{2 \mid \beta \mid}\right)$$

$$> (1 - \epsilon) * (1 - \epsilon) = 1 - \epsilon.$$

and

$$\nu\left((\alpha T(x_{mn}) + \beta T(y_{mn})) - (\alpha L_1 + \beta L_2), t\right)$$

$$\leq \nu\left(\alpha T(x_{mn}) - \alpha L_1, \frac{t}{2}\right) \diamond \nu\left(\beta T(x_{mn}) - \beta L_2, \frac{t}{2}\right)$$

$$= \mu\left(T(x_{mn}) - L_1, \frac{t}{2 \mid \alpha \mid}\right) \diamond \mu\left(T(x_{mn}) - L_2, \frac{t}{2 \mid \beta \mid}\right)$$

$$< \epsilon \diamond \epsilon = \epsilon.$$

This implies that

$A_3^c \subset \{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mu((\alpha T(x_{ij}) + \beta T(y_{ij})) - (\alpha L_1 + \beta L_2), t) > 1 - \epsilon$
or $\nu((\alpha T(x_{ij}) + \beta T(y_{ij})) - (\alpha L_1 + \beta L_2), t) < \epsilon\}$. Hence ${}_2S_{(\mu,\nu)}^I(T)$ is a linear space.

Theorem 3.2 Every open ball ${}_2B_x(r,t)(T)$ is an open set in ${}_2S_{(\mu,\nu)}^I(T)$.

Proof. Let ${}_2B_x(r,t)(T)$ be an open ball with centre x and radius r with respect to t. That is

${}_2B_x(r,t)(T) = \{ y = (y_{ij}) \in {}_2\ell_\infty : \{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{ij}) - T(y_{ij}), t) \leq 1 - r$
or $\nu(T(x_{ij}) - T(x_{ij}), t) \geq r \} \in I_2 \}$.

Let $y \in {}_2B_x^c(r,t)(T)$. Then $\mu(T(x_{ij}) - T(y_{ij}), t) > 1 - r$ and $\nu(T(x_{ij}) - T(y_{ij}), t) < r$. Since $\mu(T(x_{ij}) - T(y_{ij}), t) > 1 - r$, there exists $t_0 \in (0,t)$ such that $\mu(T(x_{ij}) - T(y_{ij}), t_0) > 1 - r$ and $\nu(T(x_{ij}) - T(y_{ij}), t_0) < r$. Putting $r_0 = \mu(T(x_{ij}) - T(y_{ij}), t_0)$, we have $r_0 > 1 - r$, there exists $s \in (0,1)$ such that $r_0 > 1 - s > 1 - r$. For $r_0 > 1 - s$, we have $r_1, r_2 \in (0,1)$ such that $r_0 * r_1 > 1 - s$ and $(1 - r_0) \diamond (1 - r_0) \leq s$. Putting $r_3 = \max\{r_1, r_2\}$. Consider the ball ${}_2B_y^c(1 - r_3, t - t_0)(T)$. We prove that

$_2B_y^c(1-r_3, t-t_0)(T) \subset {}_2B_x^c(r,t)(T)$. Let $z = (z_{ij}) \in {}_2B_y^c(1-r_3, t-t_0)(T)$, then $\mu(T(y_{ij}) - T(z_{ij}), t-t_0) > r_3$ and $\nu(T(y_{ij}) - T(z_{ij}), t-t_0) < 1-r_3$. Therefore

$$\mu(T(x_{ij}) - T(z_{ij}), t) \geq \mu(T(x_{ij}) - T(y_{ij}), t_0) * \mu(T(y_{ij}) - T(z_{ij}), t-t_0)$$

$$\geq (r_0 * r_3) \geq (r_0 * r_1) \geq (1-s) \geq (1-r)$$

and

$$\nu(T(x_{ij}) - T(z_{ij}), t) \leq \nu(T(x_{ij}) - T(y_{ij}), t_0) \diamond \nu(T(y_{ij}) - T(z_{ij}), t-t_0)$$

$$\leq (1-r_0) \diamond (1-r_3) \leq (1-r_0) \diamond (1-r_2) \leq s \leq r.$$

Thus $z \in {}_2B_x^c(r,t)(T)$ and hence

$$_2B_y^c(1-r_3, t-t_0)(T) \subset {}_2B_x^c(r,t)(T).$$

Remark 3.3 $_2S_{(\mu,\nu)}^I(T)$ is an IFNS.
Define $_2\tau_{(\mu,\nu)}^I(T) = \{A \subset {}_2S_{(\mu,\nu)}^I(T) : \text{ for each } x \in A \text{ there exists } t > 0$
 and $r \in (0,1)$ such that $_2B_x(r,t)(T) \subset A\}$.
 Then $_2\tau_{(\mu,\nu)}^I(T)$ is a topology on $_2S_{(\mu,\nu)}^I(T)$.

Theorem 3.4 The topology $_2\tau_{(\mu,\nu)}^I(T)$ on $_2S_{0(\mu,\nu)}^I(T)$ is first countable.

Proof. $\{_2B_x(\frac{1}{n}, \frac{1}{n})(T) : n = 1,2,3, ...\}$ is a local base at x, the topology $_2\tau_{(\mu,\nu)}^I(T)$ on $_2S_{0(\mu,\nu)}^I(T)$ is first countable.

Theorem 3.5 $_2S_{(\mu,\nu)}^I(T)$ and $_2S_{0(\mu,\nu)}^I(T)$ are Hausdorff spaces.

Proof. We prove the result for $_2S_{(\mu,\nu)}^I(T)$. Similarly the proof follows for $_2S_{0(\mu,\nu)}^I(T)$. Let $x,y \in {}_2S_{(\mu,\nu)}^I(T)$ such that $x \neq y$. Then $0 < \mu(T(x) - T(y), t) < 1$ and $0 < \nu(T(x) - T(y), t) < 1$. Putting $r_1 = \mu(T(x) - T(y), t)$, $r_2 = \nu(T(x) - T(y), t)$ and $r = \max\{r_1, 1-r_2\}$. For each $r_0 \in (r,1)$ there exists r_3 and r_4 such that $r_3 * r_4 \geq r_0$ and $(1-r_3) \diamond (1-r_4) \leq (1-r_0)$. Putting $r_5 \max\{r_3, 1-r_4\}$ and consider the open balls $_2B_x(1-r_5, \frac{t}{2})$ and $_2B_y(1-r_5, \frac{t}{2})$. Then clearly $_2B_x^c(1-r_5, \frac{t}{2}) \cap {}_2B_y^c(1-r_5, \frac{t}{2}) = \phi$. For if there exists $z \in {}_2B_x^c(1-r_5, \frac{t}{2}) \cap {}_2B_y^c(1-r_5, \frac{t}{2})$, then

$$r_1 = \mu(T(x) - T(y), t) \geq \mu(T(x) - T(z), \frac{t}{2}) * \mu(T(z) - T(y), \frac{t}{2}) \geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1$$

and $r_2 = \nu(T(x) - T(y), t) \leq \nu(T(x) - T(z), \frac{t}{2}) \diamond \nu(T(z) - T(y), \frac{t}{2})$
 $\leq (1-r_5) \diamond (1-r_5) \leq (1-r_4) \diamond (1-r_4) \leq (1-r_0) < r_2$
which is a contradiction. Hence $_2S_{(\mu,\nu)}^I(T)$ is Hausdorff.

Theorem 3.6 $_2S_{(\mu,\nu)}^I(T)$ is an IFNS and $_2\tau_{(\mu,\nu)}^I(T)$ is a topology on $_2S_{(\mu,\nu)}^I(T)$. Then a sequence $(x_{ij}) \in {}_2S_{(\mu,\nu)}^I(T), x_{ij} \to x$ if and only if $\mu(T(x_{ij}) - T(x), t) \to 1$ and $\nu(T(x_{ij}) - T(x), t) \to 0$ as $k \to \infty$.

Proof. Fix $t_0 > 0$. Suppose $x_{ij} \to x$. Then for $r \in (0,1)$, there exists $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ such that $(x_{ij}) \in {}_2B_x(r,t)(T)$ for all $(i,j) \geq (m_0, n_0)$,

$$_2B_x(r,t)(T) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{ij}) - T(x), t) \leq 1 - r \text{ or } \nu(T(x_{ij}) - T(x), t) \geq r\} \in I_2,$$

such that $_2B_x^c(r,t)(T) \in \mathcal{F}(I_2)$. Then $1 - \mu(T(x_{ij}) - T(x),t) < r$ and $\nu(T(x_{ij}) - T(x),t) < r$. Hence $\mu(T(x_{ij}) - T(x),t) \to 1$ and $\nu(T(x_{ij}) - T(x),t) \to 0$ as $i \to \infty, j \to \infty$.

Conversely, if for each $t > 0, \mu(T(x_{ij}) - T(x),t) \to 1$ and $\nu(T(x_{ij}) - T(x),t) \to 0$ as $i \to \infty, j \to \infty$, then for $r \in (0,1)$, there exists $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ such that $1 - \mu(T(x_{ij}) - T(x),t) < r$ and $\nu(T(x_{ij}) - T(x),t) < r$, for all $i \geq m_0, j \geq n_0$. It follows that $\mu(T(x_{ij}) - T(x),t) > 1 - r$ and $\nu(T(x_{ij}) - T(x),t) < r$ for all $i \geq m_0, j \geq n_0$. Thus $(x_{ij}) \in {_2B_x^c}(r,t)(T)$ for all $i \geq m_0, j \geq n_0$ and hence $x_{ij} \to x$.

Theorem 3.7 A sequence $x = (x_{ij}) \in {_2S_{(\mu,\nu)}^I}(T)$ is I_2-convergent if and only if for every $\epsilon > 0$ and $t > 0$ there exists numbers $M = M(x,\epsilon,t)$ and $N = N(x,\epsilon,t)$ such that

$$\{(M,N) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{MN}) - L, \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(T(x_{MN}) - L, \frac{t}{2}) < \epsilon\} \in \mathcal{F}(I_2).$$

Proof. Suppose that $I_2^{(\mu,\nu)} - \lim x = L$ and let $\epsilon > 0$ and $t > 0$. For a given $\epsilon > 0$, choose $s > 0$ such that $(1 - \epsilon) * (1 - \epsilon) > 1 - s$ and $\epsilon \diamond \epsilon < s$. Then for each $x \in {_2S_{(\mu,\nu)}^I}(T)$,

$$A = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{ij}) - L, \frac{t}{2}) \leq 1 - \epsilon \text{ or } \nu(T(x_{ij}) - L, \frac{t}{2}) \geq \epsilon\} \in I_2,$$

which implies that

$$A^c = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{ij}) - L, \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(T(x_{ij}) - L, \frac{t}{2}) < \epsilon\} \in \mathcal{F}(I_2).$$

Conversely let us choose $(M,N) \in A$. Then

$$\mu(T(x_{MN}) - L, \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(T(x_{MN}) - L, \frac{t}{2}) < \epsilon.$$

Now we want to show that there exists a number $M = M(x,\epsilon), N = N(x,\epsilon,t)$ such that

$$\{(M,N) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{ij}) - T(x_{MN}),t) \leq 1 - s \text{ or } \nu(T(x_{ij}) - T(x_{MN}),t) \geq s\} \in I_2.$$

For this, define for each $x \in {_2S_{(\mu,\nu)}^I}(T)$

$$B = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{ij}) - T(x_{MN}),t) \leq 1 - s \text{ or } \nu(T(x_{ij}) - T(x_{MN}),t) \geq s\} \in I_2.$$

Now we have to show that $B \subset A$. Suppose that BA. Then there exists $(m,n) \in B$ and $n \notin A$. Therefore we have

$$\mu(T(x_{mn}) - T(x_{MN}),t) \leq 1 - s \text{ or } \mu(T(x_{mn}) - L, \frac{t}{2}) > 1 - \epsilon.$$

In particular $\mu(T(x_{MN}) - L, \frac{t}{2}) > 1 - \epsilon$. Therefore we have

$$1 - s \geq \mu(T(x_{mn}) - T(x_{MN}),t) \geq \mu(T(x_{mn}) - L, \frac{t}{2}) * \mu(T(x_{MN}) - L, \frac{t}{2}) \geq (1 - \epsilon) * (1 - \epsilon) > 1 - s,$$

which is not possible. On the other hand

$$\nu(T(x_{mn}) - T(x_{MN}),t) \geq s \text{ or } \nu(T(x_{mn}) - L, \frac{t}{2}) < \epsilon$$

In particular $\nu(T(x_{MN}) - L, \frac{t}{2}) < \epsilon$. Therefore we have

$$s \leq \nu(T(x_{mn}) - T(x_{MN}),t) \leq \nu(T(x_{mn}) - L, \frac{t}{2}) \diamond \nu(T(x_{MN}) - L, \frac{t}{2}) \leq \epsilon \diamond \epsilon < s,$$

which is not possible. Hence $B \subset A$. $A \in I_2$ implies $B \in I_2$.

4. CONCLUSION:

In the present paper we have studied the concept of ideal convergence of double sequence spaces in intuitionistic fuzzy sequence spaces defined by compact operator and studied the fuzzy topology on the said spaces.

5. AUTHORS CONTRIBUTIONS

All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

6. AUTHOR DETAILS

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7. COMPETING INTERESTS

The authors declare that they have no competing interests.

8. ACKNOWLEDGMENT

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