Electronic Journal of Mathematical Analysis and Applications
Vol. 7(1) Jan. 2019, pp. 341-350.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

GROUND STATE SOLUTION FOR NONHOMOGENEOUS ELLIPTIC EQUATIONS INVOLVING CRITICAL CAFFARELLI-KOHN-NIRENBERG EXPONENT

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Abstract. In this paper, we consider a nonhomogeneous singular elliptic equation involving a critical Caffarelli-Kohn-Nirenberg exponent. By using the Nehari manifold we establish the existence of a ground state solution.

1. Introduction

This work deals with the existence of a ground state solution to the following nonhomogeneous problem

$$(\mathcal{P})\begin{cases} -\operatorname{div}\left(\frac{|\nabla u|^{p-2}}{|x|^{pa}}\nabla u\right)-\mu\frac{|u|^{p-2}}{|x|^{p(a+1)}}u=\frac{|u|^{p_*-2}}{|x|^{p_*b}}u+f(x) \text{ in } \Omega, \\ u=0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\geq 3)$ containing 0 in its interior, $1<p<N$, $-\infty<a<(N-p)/p$, $a\leq b<a+1$, $p_*=pN/(N-pa-p+pb)$ is the critical Caffarelli-Kohn-Nirenberg exponent, $-\infty<\mu<\overline{\mu}$, $\overline{\mu}:=[(N-pa-p)/p]^p$ and f is a given measurable function different than 0.
This problem is related to the following well known Caffarelli-Kohn-Nirenberg inequality [4]:

$$\left(\int_\Omega \frac{|u|^{p_*}}{|x|^{p_*b}}dx\right)^{1/p_*}\leq C_{a,b}\left(\int_\Omega \frac{|\nabla u|^p}{|x|^{pa}}dx\right)^{1/p} \text{ for all } u\in C_0^\infty(\Omega), \tag{1.1}$$

for some positive constant $C_{a,b}$. For sharp constants and extremal functions associated to (1.1), see [5, 8, 11]. If $b=a+1$ in (1.1), then $p_*=p$, $C_{a,b}=1/\overline{\mu}$ and we have the following weighted Hardy inequality [4, 6, 10]:

$$\int_\Omega \frac{|u|^p}{|x|^{pa+p}}dx\leq\frac{1}{\overline{\mu}}\int_\Omega\frac{|\nabla u|^p}{|x|^{pa}}dx, \text{ for all } u\in C_0^\infty(\Omega). \tag{1.2}$$

1991 *Mathematics Subject Classification.* 35A15, 35B25, 35B33, 35J60.
Key words and phrases. Variational methods, critical Caffarelli-Kohn-Nirenberg exponent, singular weights, Nehari manifold, Palais-Smale condition.
Submitted Mach 10, 2018. Revised April 15, 2018 .

We shall work with the space $W_{\mu}^{1,p} := W_{\mu}^{1,p}(\Omega)$ for $-\infty < \mu < \overline{\mu}$ endowed with the norm

$$\|u\|_{\mu}^{p} := \int_{\Omega}\left(\frac{|\nabla u|^{p}}{|x|^{pa}} - \mu\frac{|u|^{p}}{|x|^{pa+p}}\right)dx,$$

which is equivalent to the norm $\|.\|_0$.

By the so called Pohozaev's identity, if Ω is a star-shaped domain in $\mathbb{R}^N$, then the problem $(\mathcal{P})$ has no nontrivial solution for $f \equiv 0$. When the problem $(\mathcal{P})$ has no singular term ($a = b = \mu = c = 0$), Tarantello [12] proved the existence of two nontrivial solutions for it with $p = 2$ and $f \in H^{-1}$ (the dual of H_0^1) such that

$$\int_{\Omega} fu\ dx < \frac{4}{N-2}\left[\frac{N-2}{N+2}\int_{\Omega}|\nabla u|^{2}\,dx\right]^{(N+2)/4}.$$

Elliptic problems with singular terms has been studied by some authors in either bounded domain or in the whole space $\mathbb{R}^N$, see [1,2,3,7,9] and references therein.

For $f \equiv 0$ and $\Omega = \mathbb{R}^N$, Kang in [9] proved that the problem

$$\begin{cases} -\mathrm{div}(\frac{|\nabla u|^{p-2}}{|x|^{pa}}\nabla u) - \mu\frac{|u|^{p-2}}{|x|^{p(a+1)}}u = \frac{|u|^{p_*-2}}{|x|^{p_* b}}u \text{ in } \mathbb{R}^N, \\ u \in W_{\mu}^{1,p}\left(\mathbb{R}^N\right) \end{cases}$$

has radial ground state solution.

Benmansour et.al. in [2] studied the existence of solutions for the elliptic problem

$$\begin{cases} -div(\frac{\nabla u}{|x|^{2a}}) - \mu\frac{u}{|x|^{2a+2}} = \frac{|u|^{2_*-2}}{|x|^{2_* b}}u + \lambda\frac{u}{|x|^{c}} + f(x) \text{ in } \Omega, \\ u = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing 0 in its interior, $-\infty < a < (N-2)/2$, $a \leq b < a+1$, $c < 2a+2$, and $2_* = 2N/(N-2a-2+2b)$ is the critical Caffarelli-Kohn-Nirenberg exponent, λ and μ are two non-negative parameters and f is a given measurable nonzero function.

In the case when $p \neq 2$, problem $(\mathcal{P})$ becomes much more complicated. We can not prove the existence of two solutions by using the same method as in [2]. However, in this paper, we prove the existence of a ground state solution for all $\mu < \overline{\mu}$ without the perturbation $\lambda|x|^{-c}u$.

To state our result, let set for all $u \in W_{\mu}^{1,p}$ and $f \in W_{\mu}^{*}$ (the dual of $W_{\mu}^{1,p}$)

$$\|u\|_{p_*} := \left(\int_{\Omega}\frac{|u|^{p_*}}{|x|^{p_* b}}dx\right)^{\frac{1}{p_*}},$$

$$I_f(u) := \int_{\Omega} fu\ dx,\ S_{\mu} := \inf_{\|u\|_{p_*}=1}\|u\|_{\mu}^{p},$$

$$\gamma_f =: \inf_{\|u\|_{p_*}=1}\left\{(p_*-p)\left[\frac{1}{p_*-1}\|u\|_{\mu}^{p}\right]^{\frac{p_*-1}{p_*-p}} - I_f(u)\right\}$$

and

$$\mathcal{D} := \left\{g \in W_{\mu}^{*},\ g \not\equiv 0;\ \gamma_g > 0\right\}.$$

Note $\mathcal{D} \neq \varnothing$; notice that if $f \in L^p(\Omega)$ then

$$\int_\Omega |f|^p \, dx < (p_* - p)^p \left[\frac{1}{p_* - 1}\right]^{\frac{p(p_*-1)}{p_*-p}} S_\mu^{p_*/(p_*-p)},$$

which implies that $f \in \mathcal{D}$.

The purpose of this paper is to investigate the existence of a ground state solution for the problem $(\mathcal{P})$ by a "smallness" condition on f. The main result is concluded as the following theorem, which is new for the singular case when $p \neq 2$.

Theorem 1 Let $-\infty < a < (N-p)/p$, $a \leq b < a+1$ and $-\infty < \mu < \bar{\mu}$. Assume that $f \in \mathcal{D}$, then $(\mathcal{P})$ has a ground state solution u_1.

This paper is organized as follows. In Section 2, we give some preliminaries about Nehari manifold. Section 3 is devoted to the proof of Theorem 1.

2. Preliminaries

In this section, we give some preliminary results which will be used later.
First, we know by [9] that $S_\mu > 0$ and is attained when $\Omega = \mathbb{R}^N$.
Since $f \in W_\mu^*$ then the Euler-Lagrange functional I associated to the problem $(\mathcal{P})$ is given by:

$$I(u) = \frac{1}{p}\|u\|_\mu^p - \frac{1}{p_*}\|u\|_{p_*}^{p_*} - I_f(u) \quad \text{for all } u \in W_\mu^{1,p},$$

it's clear that $I \in C^1\left(W_\mu^{1,p}, \mathbb{R}\right)$ and satisfies

$$\langle I'(u), v\rangle = \int_\Omega \left(\frac{|\nabla u|^{p-2}}{|x|^{pa}}\nabla u \nabla v - \mu \frac{|u|^{p-2}}{|x|^{p(a+1)}} uv - \frac{|u|^{p_*-2}}{|x|^{p_* b}} uv - fv\right) dx$$

for all $u, v \in W_\mu^{1,p}$.
Hence, weak solutions of $(\mathcal{P})$ are critical points of the functional I.

We denote the Nehari manifold by

$$\mathcal{N} = \left\{u \in W_\mu^{1,p} \setminus \{0\}, \ \langle I'(u), u\rangle = 0\right\}.$$

It is easy to see that $u \in \mathcal{N}$ if and only if

$$J(u) = \|u\|_\mu^p - \|u\|_{p_*}^{p_*} - I_f(u) = 0.$$

Lemma 1 The function I is coercive and bounded from below in $\mathcal{N}$.
Proof. Let $u \in \mathcal{N}$, by Holder and Young inequalities we have

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_\mu^p - \frac{1}{p_*}\|u\|_{p_*}^{p_*} - I_f(u) \\ &\geq \frac{1}{p}\|u\|_\mu^p - \frac{1}{p_*}\|u\|_{p_*}^{p_*} + \|u\|_{p_*}^{p_*} - \|u\|_\mu^p \\ &\geq -\left(\frac{p-1}{p}\right)\|u\|_\mu^p + \left(\frac{p_*-1}{p_*}\right) S_\mu^{-p_*/p}\|u\|_\mu^{p_*}. \end{aligned}$$

Let $\rho = \|u\|_\mu^p$ and

$$h(\rho) = -\left(\frac{p-1}{p}\right)\rho^p + \left(\frac{p_*-1}{p_*}\right) S_\mu^{-p_*/p}\rho^{p_*}.$$

Direct calculations show that h is convex and achieves its minimum at

$$\rho_0 = \left[\frac{p-1}{p_*-1} S_\mu^{p_*/p}\right]^{\frac{1}{p_*-p}},$$

so

$$I(u) \geq h(\rho_0) = -\frac{(p-1)(p_*-p)}{p\ p_*}\left[\frac{p-1}{p_*-1} S_\mu^{p_*/p}\right]^{\frac{p}{p_*-p}}.$$

Then I is coercive and bounded from below in $\mathcal{N}$. □

The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of the function of the form $\Phi_u(t) : t \to I(tu)$, which for $t>0$ is defined by

$$\Phi_u(t) = \frac{t^p}{p}\|u\|_\mu^p - \frac{t^{p_*}}{p_*}\|u\|_{p_*}^{p_*} - tI_f(u).$$

Lemma 2 Let $u \in W_\mu^{1,p}$, then $tu \in \mathcal{N}$ if and only if $\Phi_u'(t) = 0$.
Proof. We have

$$\begin{aligned}\Phi_u'(t) &= \langle I'(tu), u\rangle \\ &= \frac{1}{t}\langle I'(tu), tu\rangle.\end{aligned}$$

Then the conclusion holds. □

The elements in $\mathcal{N}$ correspond to stationary points of the maps Φ_u. We note that

$$\Phi_u'(t) = t^{p-1}\|u\|_\mu^p - t^{p_*-1}\|u\|_{p_*}^{p_*} - I_f(u),$$

and

$$\Phi_u''(t) = (p-1)t^{p-2}\|u\|_\mu^p - (p_*-1)t^{p_*-2}\|u\|_{p_*}^{p_*}.$$

By Lemma 2 we have $u \in \mathcal{N}$ if and only if $\Phi_u'(1) = 0$. Hence

$$\Phi_u''(1) = (p-1)\|u\|_\mu^p - (p_*-1)\|u\|_{p_*}^{p_*}$$

Then, it is natural to split $\mathcal{N}$ into three subsets corresponding to local minima, local maxima, and point of inflexion, i.e.,

$$\mathcal{N}^+ = \{u \in \mathcal{N} : \Phi_u''(1) > 0\},$$
$$\mathcal{N}^- = \{u \in \mathcal{N} : \Phi_u''(1) < 0\},$$

and

$$\mathcal{N}^0 = \{u \in \mathcal{N} : \Phi_u''(1) = 0\}.$$

First, we prove that $\Phi_u''(1) \neq 0$ for all $u \in \mathcal{N}\setminus\{0\}$.
Lemma 3 Assume that $f \in \mathcal{D}$. Then $\mathcal{N}^0 = \varnothing$.
Proof. Suppose that $\mathcal{N}^0 \neq \varnothing$. For $u \in \mathcal{N}^0$, we have

$$\begin{aligned}(p-1)\|u\|_\mu^p &= (p_*-1)\|u\|_{p_*}^{p_*}, \\ (p-1)I_f(u) &= (p_*-p)\|u\|_{p_*}^{p_*},\end{aligned}$$

and

$$(p_*-1)I_f(u) = (p_*-p)\|u\|_\mu^p.$$

Using the definition of S_μ we get

$$\begin{aligned}\|u\|_{p_*}^{p_*} &= (p-1)\|u\|_\mu^p/(p_*-1) \\ &\geq \left[\frac{p-1}{p_*-1}S_\mu\right]^{p_*/(p_*-p)}.\end{aligned}$$

Thus

$$\frac{\|u\|_{\mu}^{p}}{\|u\|_{p_*}^{p_*}}=\frac{p_*-1}{p-1}.$$

Therefore,

$$\begin{aligned}
0 &= \frac{p_*-p}{p_*-1}\|u\|_{\mu}^{p}-I_f(u)\\
&= \|u\|_{p_*}^{p_*}\left[\frac{p_*-p}{p_*-1}\frac{\|u\|_{\mu}^{p}}{\|u\|_{p_*}^{p_*}}-\frac{I_f(u)}{\|u\|_{p_*}^{p_*}}\right]\\
&\geq \|u\|_{p_*}^{p_*}\left[(p_*-p)\left[\frac{\|u\|_{\mu}^{p}}{(p_*-1)\|u\|_{p_*}^{p_*}}\right]^{(p_*-1)/(p_*-p)}-\frac{I_f(u)}{\|u\|_{p_*}^{p_*}}\right]\\
&> 0,
\end{aligned}$$

which is impossible. □

Define for all $u\in W_{\mu}^{1,p}\backslash\{0\}$

$$t_u^{\max}:=\left[\|u\|_{\mu}^{p}(p-1)/(p_*-1)\|u\|_{p_*}^{p_*}\right]^{\frac{1}{p_*-p}}.$$

Lemma 4 Assume that $f\in\mathcal{D}$. Then for any $u\in W_{\mu}^{1,p}\backslash\{0\}$, there exists a unique positive value t_u^{+} such that

$$t_u^{+}>t_u^{\max},\ t_u^{+}u\in\mathcal{N}^{-}\text{ and }I\left(t_u^{+}u\right)=\max_{t\geq t_u^{\max}}I(tu).$$

Moreover, if $I_f(u)>0$, then there exists a unique positive value t_u^{-} such that

$$0<t_u^{-}<t_u^{\max},\ t_u^{-}u\in\mathcal{N}^{+}\text{ and }I\left(t_u^{-}u\right)=\inf_{0\leq t\leq t_u^{\max}}I(tu).$$

Proof. Set

$$\Psi_u(t)=t^{p-1}\|u\|_{\mu}^{p}-t^{p_*-1}\|u\|_{p_*}^{p_*}$$

for $u\in W_{\mu}^{1,p}\backslash\{0\}$, then

$$\Phi_u'(t)=\Psi_u(t)-I_f(u).$$

Easy computations show that Ψ_u is concave and achieves its maximum at $t_u^{\max}$, also

$$\Psi_u(t_u^{\max})=(p_*-p)\left(\frac{\|u\|_{\mu}^{p}}{p_*-1}\right)^{\frac{p_*-1}{p_*-p}}\left(\frac{p-1}{\|u\|_{p_*}^{p_*}}\right)^{\frac{p-1}{p_*-p}}.$$

Then we can get easily the conclusion of our lemma. □

By the previous lemma we know that $\mathcal{N}^{+}$ and $\mathcal{N}^{-}$ are not empty, so we can define

$$\theta^{+}:=\inf_{u\in\mathcal{N}^{+}}I(u)\text{ and }\theta^{-}:=\inf_{u\in\mathcal{N}^{-}}I(u).$$

Lemma 5 Assume that $f\in\mathcal{D}$. Then for any $u\in\mathcal{N}^{\pm}$, there exist $\varepsilon>0$ and a differentiable function $\zeta = \zeta(v)$, $v \in W_{\mu}^{1,p}$, $\|v\|_{\mu} < \varepsilon$, such that $\xi(0) = 1$, $\zeta(v)(u-v) \in \mathcal{N}^{\pm}$ and

$$\left(\zeta'(0), v\right) = \frac{\displaystyle\int_{\Omega}\left[p\left(\frac{|\nabla u|^{p-2}\nabla u \nabla v}{|x|^{pa}} - \mu \frac{|u|^{p-2}uv}{|x|^{p(a+1)}}\right) - p_* \frac{|u|^{p_*-2}uv}{|x|^{2_* b}} - fv\right]dx}{(p-1)\|u\|_{\mu}^{p} - (p_*-1)\|u\|_{p_*}^{p_*}}.$$

Proof. Define $\varphi : \mathbb{R} \times W_{\mu}^{1,p} \longrightarrow \mathbb{R}$ such that

$$\varphi(\zeta, v) = \zeta^{p-1}\|u-v\|_{\mu}^{p} - \zeta^{p_*-1}\|u-v\|_{p_*}^{p_*} - \int_{\Omega} f(u-v)\ dx.$$

As $u \in \mathcal{N}$ and $\mathcal{N}^0 = \varnothing$, we have

$$\varphi(1,0) = 0,\ \frac{\partial \varphi}{\partial \zeta}(1,0) = (p-1)\|u\|_{\mu}^{p} - (p_*-1)\|u\|_{p_*}^{p_*} \neq 0.$$

Then by the implicit function Theorem, we get our result. □

Lemma 6 Let $f \in \mathcal{D}$, then there exist $\theta_0^+ < 0$ and $\theta_0^- > 0$ such that $\theta^+ \leq \theta_0^+$ and $\theta^- \geq \theta_0^-$.

Proof. Let $v \in W_{\mu}^{1,p}$ be the unique solution of the following problem

$$\begin{cases} -div\left(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{pa}}\right) - \mu \dfrac{|u|^{p-2}u}{|x|^{p(a+1)}} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, as $f \not\equiv 0$ we have $I_f(v) = \|v\|_{\mu}^{p} > 0$ and $\|v\|_{\mu}^{p} = \|f\|_{-}^{p}$ where $\|.\|_{-} = \|.\|_{W_{\mu}^{*}}$. Moreover, from Lemma 4, there exists $t_v^- > 0$ such that $t_v^- v \in \mathcal{N}^+$. This implies that

$$\begin{aligned} \theta^+ &\leq I\left(t_v^- v\right) \\ &= \frac{(1-p)\left(t_v^-\right)^p}{p}\|v\|_{\mu}^{p} + \frac{1-p_*}{p_*}\left(t_v^-\right)^{p_*}\|v\|_{p_*}^{p_*} \\ &\leq \frac{(1-p)\left(t_v^-\right)^p}{p}\|v\|_{\mu}^{p} \\ &\leq \frac{1-p}{p}\left(t_v^-\right)^p\|f\|_{-}^{p}. \end{aligned}$$

Thus $\theta^+ \leq \theta_0^+ < 0$ where $\theta_0^+ = \dfrac{1-p}{p}\left(t_v^-\right)^p\|f\|_{-}^{p}$.

On the other hand, there exists $t_v^+ > 0$ such that $t_v^+ v \in \mathcal{N}^-$ which yields

$$\begin{aligned} \theta^- &\geq I\left(t_v^+ v\right) \\ &= \left(t_v^+\right)^p\|v\|_{\mu}^{p} - \frac{p_*-1}{p-1}\left(t_v^+\right)^{p_*}\|v\|_{p_*}^{p_*} \\ &\geq \left(t_v^+\right)^p\left[\frac{p-1}{p_*-1}S_{\mu}\right]^{\frac{p_*}{p_*-p}}. \end{aligned}$$

Therefore, $\theta^- \geq \theta_0^- > 0$ where

$$\theta_0^- = \left(t_v^+\right)^p\left[\frac{p-1}{p_*-1}S_{\mu}\right]^{\frac{p_*}{p_*-p}}.$$

The proof is complete. $\square$

Lemma 7 Assume that $f \in \mathcal{D}$. Then, there exists a minimizing sequence (u_n) such that

$$I(u_n) \longrightarrow \theta^+ \text{ and } I'(u_n) \to 0 \text{ in } W_\mu^*.$$

Proof. It is easy to prove that I is bounded in $\mathcal{N}^+$. Then by applying Ekeland's variational principle, we show that there exists a minimizing sequence $(u_n) \subset \mathcal{N}^+$ satisfying

$$\theta^+ \le I(u_n) \le \theta^+ + \frac{1}{n} \text{ and } I(u) \ge I(u_n) - \frac{1}{n}\|u - u_n\|_\mu \text{ for all } u \in \mathcal{N}^+.$$

From the preceding lemma we have $\theta^+ \le \theta_0$. So that

$$\left(\frac{1}{p} - \frac{1}{p_*}\right)\|u_n\|_\mu^p < \left(\frac{1}{p} - \frac{1}{p_*}\right)\frac{1-p}{p}\left(t_v^-\right)^p \|f\|_-^p + \frac{p_*-1}{p_*}\|f\|_-^{p-1}\|u_n\|_\mu,$$

and

$$\frac{p_*(p-1)}{p(p_*-1)}\left(t_v^-\right)^p\|f\|_-^p \le I_f(u_n) \le \|f\|_-^{p-1}\|u_n\|_\mu,$$

for n large. This implies that $C_1 \le \|u_n\|_\mu \le C_2$ with

$$C_1 = \frac{p_*(p-1)}{p(p_*-1)}\left(t_v^-\right)^p\|f\|_-$$

and

$$C_2 = \frac{p(p_*-1)}{(p-1)(p_*-p)}\|f\|_-.$$

Now, we show that $I'(u_n) \to 0$ in W_μ^*. For that, fix n such that $\|I'(u_n)\|_- \neq 0$. Then by Lemma 5 there exist $\varepsilon > 0$ and a function $\zeta_n : B_\varepsilon \longrightarrow \mathbb{R}$ such that $w_n = \zeta_n(v_n)(u_n - v_n) \in \mathcal{N}^+$ with

$$v_n = \delta\frac{I'(u_n)}{\|I'(u_n)\|_-} \text{ and } 0 < \delta < \varepsilon.$$

Let $A_n = \|w_n - u_n\|_\mu$. By the Taylor expansion of I, we obtain

$$\begin{aligned}
-\frac{1}{n}A_n &\le I(w_n) - I(u_n)\\
&\le \langle I'(u_n),\ w_n - u_n\rangle + o(A_n)\\
&= (\zeta_n(v_n) - 1)\langle I'(u_n),\ u_n\rangle - \delta\zeta_n(v_n)\left\langle I'(u_n),\ \frac{I'(u_n)}{\|I'(u_n)\|_-}\right\rangle +\\
&\quad o(A_n).
\end{aligned}$$

Then

$$\zeta_n(v_n)\|I'(u_n)\|_- \le \frac{\zeta_n(v_n) - 1}{\delta}\langle I'(u_n), u_n\rangle + \frac{A_n}{n\delta} + \frac{o(A_n)}{\delta}. \tag{2.1}$$

We have

$$\lim_{\delta\to 0}\zeta_n(v_n) = 1, \lim_{\delta\to 0}\frac{|\zeta_n(v_n) - 1|}{\delta} = \lim_{\delta\to 0}\frac{|\zeta_n(v_n) - \zeta_n(0)|}{\delta} \le \left\|\zeta_n'(0)\right\|_-,$$

and

$$\begin{aligned}\lim_{\delta\to 0}\frac{A_n}{n\delta} &= \lim_{\delta\to 0}\frac{1}{n\delta}\left\|(\zeta_n(v_n)-1)u_n-\zeta_n(v_n)v_n\right\|_\mu \\ &\leq \frac{1}{n}\left(\left\|\zeta_n'(0)\right\|_-\left\|u_n\right\|_\mu+1\right).\end{aligned}$$

Taking $\delta\to 0$ in (2.1) and since (u_n) is a bounded sequence we get

$$\left\|I'(u_n)\right\|_\mu \leq \frac{C_3}{n}\left(\left\|\zeta_n'(0)\right\|_-+1\right),$$

for a suitable constant $C_3>0$. Now, we must show that $\left\|\zeta_n'(0)\right\|_-$ is uniformly bounded in n.
From the boundedness of (u_n) we have by Lemma 5

$$\left\langle\zeta_n'(0),\ v\right\rangle \leq \frac{C_4\left\|v\right\|_\mu}{\left|(p-1)\left\|u_n\right\|_\mu^p-(p_*-1)\left\|u_n\right\|_{p_*}^{p_*}\right|},$$

for all $v\in W_\mu^{1,p}$ and some constant $C_4>0$. We only need to show that for any sequence $(u_n)\subset\mathcal{N}^+$

$$\left|(p-1)\left\|u_n\right\|_\mu^p-(p_*-1)\left\|u_n\right\|_{p_*}^{p_*}\right|>C_5,$$

for some constant $C_5>0$.
Assume by contradiction that there exists $(u_n)\subset\mathcal{N}^+$ such that

$$\lim_{n\to\infty}\left[(p-1)\left\|u_n\right\|_\mu^p-(p_*-1)\left\|u_n\right\|_{p_*}^{p_*}\right]=0.$$

Then as $\left\|u_n\right\|_\mu\geq C_1>0$, we get

$$\frac{\left\|u_n\right\|_{p_*}^{p_*}}{\left\|u_n\right\|_\mu^p}=\frac{(p-1)}{p_*-1}+o_n(1)\ \text{ and } (p-1)I_f(u_n)=(p_*-p)\left\|u_n\right\|_{p_*}^{p_*}+o_n(1),$$

where $o_n(1)\to 0$ as $n\to\infty$. But this is impossible since, as in the proof of Lemma 3 we have

$$\begin{aligned}o_n(1) &= (p-1)\left\|u_n\right\|_\mu^p-(2_*-1)\left\|u_n\right\|_{p_*}^{p_*} \\ &= (p_*-p)\left\|u_n\right\|_{p_*}^{p_*}-(p-1)I_f(u_n) \\ &= \left\|u_n\right\|_{p_*}\left[(p_*-p)\left(\frac{\left\|u_n\right\|_\mu^p}{(p_*-1)\left\|u_n\right\|_{p_*}^p}\right)^{(p_*-1)/(p_*-p)}-\frac{I_f(u_n)}{\left\|u_n\right\|_{p_*}}\right] \\ &> 0.\end{aligned}$$

At this point we conclude that $I'(u_n)\to 0$ in W_μ^*.□

3. Proof of Theorem 1

First, we prove that I can achieve a local minimum on $\mathcal{N}^+$.
According to the proof of Lemma 7, there exists a minimizing sequence $(u_n)\subset\mathcal{N}^+$ such that $C_1\leq\left\|u_n\right\|_\mu\leq C_2$. Up to a subsequence if necessary, we have

$$\begin{aligned} u_n &\rightharpoonup u_1 \text{ in } W_\mu^{1,p} \\ u_n &\rightharpoonup u_1 \text{ in } L_{p*}\left(\Omega, |x|^{-p*b}\right) \\ u_n &\to u_1 \text{ a.e.in } \Omega \end{aligned}$$

for some $u_1 \in W_\mu^{1,p}$. As $\theta^+ < 0$ then $u_1 \not\equiv 0$.
Now we show that $u_n \to u_1$ in $W_\mu^{1,p}$. Suppose otherwise, so $\|u_1\|_\mu < \underline{\lim}_{n\to\infty} \|u_n\|_\mu$, which implies that

$$\begin{aligned} \theta^+ &\leq I(u_1) \\ &= \left(\frac{1}{p} - \frac{1}{p_*}\right) \|u_1\|_\mu^p - \left(1 - \frac{1}{p_*}\right) I_f(u_1) \\ &< \underline{\lim}_{n\to\to\infty} \left(\frac{p_* - p}{p\ p_*} \|u_n\|_\mu^p - \frac{p_* - 1}{p_*} I_f(u_n)\right) \\ &= \theta^+. \end{aligned}$$

This is a contradiction, which leads to conclude that $u_n \to u_1$ in $W_\mu^{1,p}$ and $I(u_1) = \theta^+$.
Moreover, we have $u_1 \in \mathcal{N}^+$. In fact, if $u_1 \in \mathcal{N}^-$ then by Lemma 4, $t_{u_1}^+ = 1$ and there exists an unique $t_{u_1}^- > 0$ such that $t_{u_1}^- u_1 \in \mathcal{N}^+$.
Since

$$\left.\frac{dI(tu_1)}{dt}\right|_{t=t_{u_1}^-} = 0, \quad \left.\frac{d^2 I(tu_1)}{dt}\right|_{t=t_{u_1}^-} > 0,$$

there exists $t_{u_1}^- < t_{u_1}^0 < t_{u_1}^+$ such that $I\left(t_{u_1}^- u_1\right) < I\left(t_{u_1}^0 u_1\right) \leq I\left(t_{u_1}^+ u_1\right) = I(u_1)$, which is a contradiction.
Hence $u_1 \in \mathcal{N}^+$ and

$$\theta^+ = \inf_{u\in\mathcal{N}^+} I(u) = \inf_{u\in\mathcal{N}} I(u).$$

By the Lagrange multiplier rule, there exists $\lambda \in \mathbb{R}$ such that

$$\Phi'_{u_1}(1) = I'(u_1) = \lambda \Phi''_{u_1}(1),$$

which implies that

$$0 = \langle I'(u_1), u_1\rangle = \lambda \langle J'(u_1), u_1\rangle,$$

we have $\langle J'(u_1), u_1\rangle \neq 0$, so $\lambda = 0$ and $I'(u_1) = 0$. Thus u_1 is a ground state solution of problem $(\mathcal{P})$.

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