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A NUMERICAL METHOD BY REPRODUCING-KERNEL APPROXIMATION FOR THE PANTOGRAPH EQUATION

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Abstract. In this paper, we study the pantograph equation. The reproducing kernel Hilbert space method (RKHSM) is employed to compute an approximation to the solution of this problem. The validity of the RKHSM is ascertained by comparing our results with other methods in the literature. The results reveal that the proposed analytical method can achieve excellent results in predicting the solutions of such problems. The existences of the solution is proved. In addition, the uniform convergent of the proposed method is investigated.

1. Introduction

Delay differential equations (DDEs) have several applications in various fields of physics, biology, chemistry, economics, and engineering such as heat exchanges, urban traffics, control theory, conveyor belts, robotics, mathematical biology, chatter, and age-structured population growth. DDEs involve past values of the state variables. The solution of the DDEs requires knowledge of not only the current state, but also of the state of a certain time previously. Recently, DDEs have received considerable attention and have proven to model many real life problems accurately. Researchers used several numerical methods to solve such problems such as Runge-Kutta methods [3, 4], linear multi-step methods [21], Adomian decomposition method [8], perturbation-iteration algorithms [15], homotopy analysis method [1], homotopy perturbation methods [17], iterative decomposition method [14], power series [12], block methods [13] and variational iteration method [18].

The pantograph equation is a special case of the DDEs. It has several applications in several fields of applied mathematics such as electrodynamics, control systems, number theory, probability, and quantum mechanics. Researchers used several numerical methods to solve this type of problems [10]. A pantograph is a device that collects electronic current from overhead lines for electric trains or trams.

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The theory of reproducing kernel Hilbert space method was used for the first time at the beginning of the 21-st century. This method is used to provide the series solution which rapidly convergent. Cui and Lin [6] presented an overview of the existing reproducing kernel methods for solving integral, integro-differntial, and differential equations. Serval researchers used this method to solve different mathematical models. For example, Geng and Cui [9, 24] used it to solve second order boundary value problem, Wang et al. [19, 23] used it to solve singular of boundary value problems, Wang and Chao [20] used it to solve variable coefficient partial differential equations,Yang et al. [22, 25] used it to solve system of linear Volterra integral equation with variable coefficients, and Du and Shen [7] used it to solve singular integral equations. Reproducing kernel Hilbert space method is a useful numerical technique to solve nonlinear problems [2].

Definition 1.1. Let E be a nonempty abstract set. A function $M : E \times E \to C$ is a reproducing kernel of the Hilbert space H if and only if

- $M(.,x) \in H$ for all $x \in E$,
- $(\phi(.), M(.,x)) = \phi(x)$ for all $x \in E$ and $\phi \in H$.

The second condition is called the reproducing property and a Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space.

We organize this paper as follows. In Section 2, we present the mathematical formulation of the problem. In Section 3, we present a numerical technique for solving second order linear boundary value problem using the RKHSM. Convergence of the presented method is given in this section. The nonlinear boundary value problem is considered in Section 4. Some numerical results are presented in Section 5 to illustrate the efficiency of the presented method. Comparison with other methods are presented. Finally, we conclude with some comments and conclusions in Section 6.

2. Mathematical formulations

Consider the second order pantograph delay differential equation

$$u''(x) = F(x, u'(x), u(qx)),\ 0 < x < X \tag{1}$$

subject to

$$u(0) = a, u(X) = b \tag{2}$$

where $F(x, u'(x), u(qx))$ is analytic function, $0 < q < 1$, and a and b are constants. It is convenient to translate the domain $[0, X]$ into $[0, 1]$ by

$$s = \frac{x}{X}. \tag{3}$$

In addition, we homogenize the boundary condition using the transformation

$$f(s) = u(sX) - a + (a - b)s. \tag{4}$$

Substitute (2.3) and (2.4) in (2.1) and (2.2) to get

$$f''(s) = X^2 F(Xs, \frac{f'(s) + b - a}{X}, f(qs) + a + (b - a)qs) = G(s, f'(s), f(qs)) \tag{5}$$

subject to

$$f(0) = f(1) = 0. \tag{6}$$

In the next section, we present the RKHSM to solve the linear case of Problem (2.5)-(2.6).

3. Analysis of RKHSM for second-order linear delay boundary value problem

In this section, we discuss how to solve the following second-order linear boundary value problem using RKHSM:

$$Lf(s) = f''(s) + a_1(s)f'(s) + a_2(s)f(qs) = h(s),\ 0 < s < 1, \tag{7}$$

subject to

$$f(0) = f(1) = 0. \tag{8}$$

In order to solve Problem (3.1)-(3.2), we construct the kernel Hilbert spaces $W_2^1[0,1]$ and $W_2^3[0,1]$ in which every function satisfies the boundary conditions (3.2). Let

$$W_2^1[0,1] = \{u(s) :\ u \text{ is absolutely continuous real value function, } u' \in L^2[0,1]\}.$$

The inner product in $W_2^1[0,1]$ is defined as

$$(u(y), v(y))_{W_2^1[0,1]} = u(0)v(0)\ + \sum_0^1 u'(y)v'(y)dy,$$

and the norm $\|u\|_{W_2^1[0,1]}$ is given by

$$\|u\|_{W_2^1[0,1]} = \sqrt{(u(y), u(y))_{W_2^1[0,1]}}$$

where $u, v \in W_2^1[0,1]$.

Theorem 3.1. The space $W_2^1[0,1]$ is a reproducing kernel Hilbert space, i.e.; there exists $R(s,y) \in W_2^1[0,1]$ and its second partial derivative with respect to y exists such that for any $u \in W_2^1[0,1]$ and each fixed $y, s \in [0,1]$, we have

$$(u(y), R(s,y))_{W_2^1[0,1]} = u(s).$$

In this case, $R(s,y)$ is given by

$$R(s,y) = \left\{ \begin{array}{cc} 1+y, & y \le s \\ 1+s\ , & y > s \end{array} \right\}.$$

Proof: Using integration by parts, one can get

$$\begin{aligned} (u(y), R(s,y))_{W_2^1[0,1]} &= u(0)R(s,0)\ + \sum_0^1 u'(y)\frac{\partial R}{\partial y}(s,y)dy \\ &= u(0)R(s,0) + u(1)\frac{\partial R}{\partial y}(s,1) - u(0)\frac{\partial R}{\partial y}(s,0) - \int_0^1 u(y)\frac{\partial^2 R}{\partial y^2}(s,y)dy. \end{aligned}$$

Since $R(s,y)$ is a reproducing kernel of $W_2^1[0,1]$,

$$(u(y), R(s,y))_{W_2^1[0,1]} = u(s)$$

which implies that

$$-\frac{\partial^2 R}{\partial y^2}(s,y) = \delta(y - s\ ), \tag{9}$$

$$R(s,0)-\frac{\partial R}{\partial y}(s,0)=0, \tag{10}$$

and

$$\frac{\partial R}{\partial y}(s,1)=0. \tag{11}$$

Since the characteristic equation of $-\frac{\partial^2 R}{\partial y^2}(s,y) = \delta(y-s\ )$ is $\lambda^2 = 0$ and its characteristic value is $\lambda\ = 0$ with 2 multiplicity roots, we write $R(s,y)$ as

$$R(s,y)=\left\{\begin{array}{ll} c_0(s)+c_1(s)y, & y\leq s \\ d_0(s)+d_1(s)y, & y>s \end{array}\right. .$$

Since $\frac{\partial^2 R}{\partial y^2}(s,y) = -\delta(y-s\ )$, we have

$$R(s,s+0)-R(s,s+0) \quad = \quad 0\ , \tag{12}$$

$$\frac{\partial R}{\partial y}(s,s+0)-\frac{\partial R}{\partial y}(s,s+0) \quad = \quad -1. \tag{13}$$

Using the conditions (3.4)-(3.7), we get the following system of equations

$$\begin{aligned} c_0(s)-c_1(s) &= 0, \\ d_1(s) &= 0, \\ c_0(s)+c_1(s)\ s &= d_0(s)+d_1(s)\ s, \\ d_1(s)-\ c_1(s) &= -1, \end{aligned} \tag{14}$$

which implies that

$$c_0(s)=1,\ \ c_1(s)=1, d_0(s)=1+s,\ \ d_1(s)=0.$$

which completes the proof of the theorem. Next, we study the space $W_2^3[0,1]$. Let

$$\begin{aligned} W_2^3[0,1] \quad &= \quad \{f(s):\ f\ , f', \text{and } f'' \text{ are absolutely continuous real value functions,} \\ f''' \quad &\in \quad L^2[0,1], f(0)=f(1)=0\}. \end{aligned}$$

The inner product in $W_2^3[0,1]$ is defined as

$$(u(y),v(y))_{W_2^3[0,1]} = u(0)v(0)+u'(0)v'(0)+u'(1)v'(1)+u'(1)v'(1)+\int_0^1 u^{(3)}(y)v^{(3)}(y)dy,$$

and the norm $\|u\|_{W_2^3[0,1]}$ is given by

$$\|u\|_{W_2^3[0,1]}=\sqrt{(u(y),u(y))_{W_2^3[0,1]}}$$

where $u,v\in W_2^3[0,1]$.

It is worth mention that there are several inner products can be defined on the space $W_2^3[0,1]$ such as

$$(u(y),v(y))_{W_2^3[0,1]} = u(0)v(0)+u'(1)v'(1)+\int_0^1 u^{(3)}(y)v^{(3)}(y)dy.$$

However, we chose the previous definition to be able to system of one solution which produce all unknowns in K.

Theorem 3.2. The space $W_2^3[0,1]$ is a reproducing kernel Hilbert space, i.e.; there exists $K(s,y)\in W_2^3[0,1]$ which has its six partial derivative with respect to y such that for any $u\in W_2^3[0,1]$ and each fixed $y,s\in[0,1]$, we have

$$(u(y),K(s,y))_{W_2^3[0,1]}=u(s).$$

In this case, $K(s,y)$ is given by

$$K(s,y)=\left\{\begin{array}{ll}\sum_{i=0}^{5}c_i(s)y^i, & y\le s\\ \sum_{i=0}^{5}d_i(s)y^i, & y>s\end{array}\right\}$$

where

$$\begin{aligned}
c_0 &= 0,\ c_1=\frac{1}{4}(-124y+127y^2-5y^4+2y^5),\ c_2=\frac{1}{96}(3048y-3117y^2-8y^3+127y^4-50y^5),\\
c_3 &= 0,\ c_4=\frac{1}{96}(-124y+127y^2-5y^4+2y^5),\ c_5=\frac{1}{240}(2+120y-125y^2+5y^4-2y^5),\\
d_0 &= \frac{y^5}{120},\ d_1=\frac{1}{24}(-744y+762y^2-31y^4+12y^5),\ d_2=\frac{1}{96}(3048y-3117y^2+127y^4-50y^5),\\
d_3 &= -\frac{y^2}{12},\ d_4=\frac{1}{96}(-120y+127y^2-5y^4+2y^5),\ d_5=\frac{1}{240}(120y-125y^2+5y^4-2y^5).
\end{aligned}$$

Proof: Using integration by parts, one can get

$$\begin{aligned}
(u(y),K(s,y))_{W_2^3[0,1]} &= u(0)K(s,0)+u(1)K(s,1)+u'(0)K_y(s,0)+u'(1)K_y(s,1)\\
&\quad +u''(1)\frac{\partial^3K}{\partial y^3}(s,1)-u''(0)\frac{\partial^3K}{\partial y^3}(s,0)
\end{aligned}$$

$$-u'(1)\frac{\partial^4K}{\partial y^4}(s,1)+u'(0)\frac{\partial^4K}{\partial y^4}(s,0)+u(1)\frac{\partial^5K}{\partial y^5}(s,1)-u(0)\frac{\partial^5K}{\partial y^5}(s,0)-\int_0^1 u(y)\frac{\partial^6K}{\partial y^6}(s,y)dy.$$

Since $u(y)$ and $K(s,y)\in W_2^3[0,1]$,

$$u(0)=u(1)=0$$

and

$$K(s,0)=K(s,1)=0. \tag{15}$$

Thus,

$$\begin{aligned}
(u(y),K(s,y))_{W_2^3[0,1]} &= u'(0)K_y(s,0)+u'(1)K_y(s,1)+\ u''(1)\frac{\partial^3K}{\partial y^3}(s,1)-u''(0)\frac{\partial^3K}{\partial y^3}(s,0)\\
&\quad -u'(1)\frac{\partial^4K}{\partial y^4}(s,1)+u'(0)\frac{\partial^4K}{\partial y^4}(s,0)-\int_0^1 u(y)\frac{\partial^6K}{\partial y^6}(s,y)dy.
\end{aligned}$$

Since $K(s,y)$ is a reproducing kernel of $W_2^3[0,1]$,

$$(u(y),K(s,y))_{W_2^3[0,1]}=u(s)$$

which implies that

$$\frac{\partial^6K}{\partial y^6}(s,y)=\delta(y-s) \tag{16}$$

where δ is the dirac-delta function and

$$\frac{\partial^3K}{\partial y^3}(s,1) = 0, \tag{17}$$

$$\frac{\partial^3K}{\partial y^3}(s,0) = 0, \tag{18}$$

$$K_y(s,1)-\frac{\partial^4K}{\partial y^4}(s,1) = 0, \tag{19}$$

$$K_y(s,0)+\frac{\partial^4K}{\partial y^4}(s,0) = 0. \tag{20}$$

Since the characteristic equation of $\frac{\partial^6 K}{\partial y^6}(s,y) = \delta(s-y)$ is $\lambda^6 = 0$ and its characteristic value is $\lambda\ = 0$ with 6 multiplicity roots, we write $K(s,y)$ as

$$K(s,y) = \left\{ \begin{array}{ll} \sum_{i=0}^{5} c_i(s)y^i, & y \leq s \\ \sum_{i=0}^{5} d_i(s)y^i, & y > s \end{array} \right\}.$$

Since $\frac{\partial^6 K}{\partial y^6}(s,y) = \delta(s-y)$, we have

$$\frac{\partial^m K}{\partial y^m}(s,s+0) = \frac{\partial^m K}{\partial y^m}(s,s-0),\ m = 0,1,...,4. \tag{21}$$

On the other hand, integrating $\frac{\partial^6 K}{\partial y^s}(s,y) = \delta(s-y)$ from $s-\epsilon$ to $s+\epsilon$ with respect to y and letting $\epsilon \to 0$ to get

$$\frac{\partial^5 K}{\partial y^5}(s,s+0) - \frac{\partial^5 K}{\partial y^5}(s,s-0) = -1. \tag{22}$$

Using the conditions (3.9) and (3.11)-(3.16), we get the following system of equations

$$\begin{aligned} c_0(s) &= 0,\ \sum_{i=0}^{5} d_i(s) = 0, \\ 6c_3(s) &= 0,\ 6d_3(s) + 24d_4(s) + 60d_6(s) = 0, \\ \sum_{i=i}^{5} id_i(s) - 24d_4(s) - 120d_5(s) &= 0,\ c_1(s) - 24c_4(s) = 0, \end{aligned}$$

$$\sum_{i=0}^{5} c_i(s)s^i = \sum_{i=0}^{5} d_i(s)s^i,$$

$$\begin{aligned} \sum_{i=1}^{5} ic_i(s)s^{i-1} &= \sum_{i=i}^{5} id_i(s)s^{i-1}, \\ \sum_{i=2}^{5} i(i-1)c_i(s)s^{i-2} &= \sum_{i=1}^{5} i(i-1)d_i(s)s^{i-2}, \\ \sum_{i=3}^{5} i(i-1)(i-2)c_i(s)s^{i-3} &= \sum_{i=3}^{5} i(i-1)(i-2)d_i(s)s^{i-3}, \end{aligned}$$

$$\begin{aligned} \sum_{i=4}^{5} i(i-1)(i-2)(i-3)c_i(s)s^{i-4} &= \sum_{i=4}^{5} i(i-1)(i-2)(i-3)d_i(s)s^{i-4}, \\ 5!d_5(s) - 5!c_5(s) &= -1. \end{aligned}$$

We solved the last system using Mathematica to get

$$
\begin{aligned}
c_0 &= 0,\ c_1 = \frac{1}{4}(-124y+127y^2-5y^4+2y^5),\ c_2 = \frac{1}{96}(3048y-3117y^2-8y^3+127y^4-50y^5),\\
c_3 &= 0,\ c_4 = \frac{1}{96}(-124y+127y^2-5y^4+2y^5),\ c_5 = \frac{1}{240}(2+120y-125y^2+5y^4-2y^5),\\
d_0 &= \frac{y^5}{120},\ d_1 = \frac{1}{24}(-744y+762y^2-31y^4+12y^5),\ d_2 = \frac{1}{96}(3048y-3117y^2+127y^4-50y^5),\\
d_3 &= -\frac{y^2}{12},\ d_4 = \frac{1}{96}(-120y+127y^2-5y^4+2y^5),\ d_5 = \frac{1}{240}(120y-125y^2+5y^4-2y^5),
\end{aligned}
$$

which completes the proof of the theorem.

Now, we present how to solve Problem (3.1)-(3.2) using the reproducing kernel method. Let

$$\sigma_i(s) = R(s_i, s)$$

for $i = 1, 2, ...$ where $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$. It is clear that $L : W_2^3[0,1] \to W_2^1[0,1]$ is bounded linear operator. Let

$$\psi_i(s) = L^*\sigma_i(s)$$

where $L(\sigma_i(s)) = \frac{\partial^3\sigma_i(s)}{\partial y^3} + a_1(s)\frac{\partial\sigma_i(s)}{\partial y} + a_2(s)\sigma_i(s)$ and L^* is the adjoint operator of L. Using Gram-Schmidt orthonormalization to generate orthonormal set of functions $\left\{\bar{\psi}_i(s)\right\}_{i=1}^{\infty}$ where

$$\bar{\psi}_i(s) = \sum_{j=1}^{i} \alpha_{ij}\psi_j(s) \tag{23}$$

and α_{ij} are coefficients of Gram-Schmidt orthonormalization. In the next theorem, we show the existence of the solution of Problem (3.1)-(3.2).

Theorem 3.3. If $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then

$$f(s) = \sum_{i=1}^{\infty}\sum_{j=1}^{i} \alpha_{ij}h(s_j)\bar{\psi}_i(s). \tag{24}$$

Proof: First, we want to prove that $\{\psi_i(s)\}_{i=1}^{\infty}$ is complete system of $W_2^3[0,1]$ and $\psi_i(s) = L(K(s, s_i))$. It is clear that $\psi_i(s) \in W_2^3[0,1]$ for $i = 1, 2, ...$ Simple calculations imply that

$$
\begin{aligned}
\psi_i(s) &= L^*\sigma_i(s) = (L^*\sigma_i(s), K(s,y))_{W_2^3[0,1]}\\
&= (\sigma_i(s), L(K(s,y)))_{W_2^3[0,1]} = L(K(s,s_i)).
\end{aligned}
$$

For each fixed $f(s) \in W_2^3[0,1]$, let

$$(f(s), \psi_i(s))_{W_2^3[0,1]} = 0,\ i = 1, 2, ...$$

Then

$$
\begin{aligned}
(f(s), \psi_i(s))_{W_2^3[0,1]} &= (f(s), L^*\sigma_i(s))_{W_2^3[0,1]}\\
&= (Lf(s), \sigma_i(s))_{W_2^3[0,1]}\\
&= Lf(s_i) = 0.
\end{aligned}
$$

Since $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, $Lf(s)=0$. Since L^{-1} exists, $u(s)=0$. Thus, $\{\psi_i(s)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$.

Second, we prove Equation (3.18). Simple calculations implies that

$$\begin{aligned} f(s) &= \sum_{i=1}^{\infty}(f(s),\bar{\psi}_i(s))_{W_2^3[0,1]}\bar{\psi}_i(s) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}(f(s),L^*(K(s,s_j)))_{W_2^3[0,1]}\bar{\psi}_i(s) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}(Lf(s),K(s,s_j))_{W_2^3[0,1]}\bar{\psi}_i(s) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}(h(s),K(s,s_j))_{W_2^3[0,1]}\bar{\psi}_i(s) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}h(s_j)\bar{\psi}_i(s) \end{aligned}$$

and the proof is complete.

Let the approximate solution of Problem (3.1)-(3.2) be given by

$$f_N(s)=\sum_{i=1}^{N}\sum_{j=1}^{i}\alpha_{ij}h(s_j)\bar{\psi}_i(s). \tag{25}$$

In the next theorem, we show the uniformly convergence of the $\left\{\frac{d^m f_N(s)}{ds^m}\right\}_{N=1}^{\infty}$ to $\frac{df(s)}{ds}$ for $m=0,1,2$.

Theorem 3.4. If $f(s)$ and $f_N(s)$ are given as in (3.18) and (3.19), then $\left\{\frac{d^m f_N(s)}{ds^m}\right\}_{N=1}^{\infty}$ converges uniformly to $\frac{d^m f(s)}{ds^m}$ for $m=0,1,2$.

Proof: First, we prove the theorem for $m=0$. For any $s\in[0,1]$,

$$\begin{aligned} \|f(s)-f_N(s)\|^2_{W_2^3[0,1]} &= (f(s)-f_N(s),f(s)-f_N(s))_{W_2^3[0,1]} \\ &= \sum_{i=N+1}^{\infty}((f(s),\bar{\psi}_i(s))_{W_2^3[0,1]}\bar{\psi}_i(s),(f(s),\bar{\psi}_i(s))_{W_2^3[0,1]}\bar{\psi}_i(s))_{W_2^3[0,1]} \\ &= \sum_{i=N+1}^{\infty}(f(s),\bar{\psi}_i(s))^2_{W_2^3[0,1]}. \end{aligned}$$

Thus,

$$\underset{s\in[0,1]}{Sup}\|f(s)-f_N(s)\|^2_{W_2^3[0,1]}=\underset{s\in[0,1]}{Sup}\sum_{i=N+1}^{\infty}(f(s),\bar{\psi}_i(s))^2_{W_2^3[0,1]}.$$

From Theorem (3.3), one can see that $\sum_{i=1}^{\infty}(f(s),\bar{\psi}_i(s))_{W_2^3[0,1]}\bar{\psi}_i(s)$ converges uniformly to $f(s)$. Thus,

$$\underset{N\to\infty}{Lim}\ \underset{s\in[0,1]}{Sup}\|f(s)-f_N(s)\|_{W_2^3[0,1]}=0$$

which implies that $\{f_N(s)\}_{N=1}^{\infty}$ converges uniformly to $f(s)$.

Second, we prove the uniformly convergence for $m = 1$ and 2. Since $\frac{d^m K(s,y)}{ds^m}$ is bounded function on $[0,1] \times [0,1]$,

$$\left\| \frac{d^m K(s,y)}{ds^m} \right\|_{W_2^3[0,1]} \leq \chi_m, \; m = 1, 2.$$

Thus, for any $s \in [0,1]$,

$$\begin{aligned} \left| f^{(m)}(s) - f_N^{(m)}(s) \right| &= \left| (f(s) - f_N(s), \frac{d^m K(s,y)}{ds^m})_{W_2^3[0,1]} \right| \\ &\leq \| f(s) - f_N(s) \|_{W_2^3[0,1]} \left\| \frac{d^m K(s,y)}{ds^m} \right\|_{W_2^3[0,1]} \\ &\leq \chi_m \| f(s) - f_N(s) \|_{W_2^3[0,1]} \\ &\leq \chi_m \underset{s\in[0,1]}{Sup} \| f(s) - f_N(s) \|_{W_2^3[0,1]} . \end{aligned}$$

Hence,

$$\underset{s\in[0,1]}{Sup} \left\| f^{(m)}(s) - f_N^{(m)}(s) \right\|_{W_2^3[0,1]} \leq \chi_m \underset{s\in[0,1]}{Sup} \| f(s) - f_N(s) \|_{W_2^3[0,1]}$$

which implies that

$$\underset{N\to\infty}{Lim} \; \underset{s\in[0,1]}{Sup} \left\| f^{(m)}(s) - f_N^{(m)}(s) \right\|_{W_2^3[0,1]} = 0.$$

Therefore, $\left\{ \frac{d^m f_N(s)}{ds^m} \right\}_{N=1}^{\infty}$ converges uniformly to $\frac{d^m f(s)}{ds^m}$ for $m = 1, 2$.

4. Analysis of RKHSM for second-order nonlinear delay boundary value problem

In this section, we discuss how to solve the following second-order nonlinear boundary value problem using RKHSM:

$$f''(s) = G(s, f'(s), f(qs)) \tag{26}$$

subject to

$$f(0) = f(1) = 0. \tag{27}$$

Let

$$G(s, f'(s), f(qs)) = r(s) + \ell(f'(s), f(qs)) + N(s, f'(s), f(qs))$$

where $\ell(f'(s), f(qs))$ and $N(s, f'(s), f(qs))$ are the linear and nonlinear parts of $G(s, f'(s), f(qs))$, respectively. We construct the homotopy as follows:

$$H(f, \lambda) = f''(s) - (r(s) + \ell(f'(s), f(qs))) - \lambda N(s, f'(s), f(qs)) = 0 \tag{28}$$

where $\lambda \in [0,1]$ is an embedding parameter. If $\lambda = 0$, we get a linear equation

$$f''(s) - r(s) - \ell(f'(s), f(qs)) = 0$$

which can be solved by using RKHSM as we described in the pervious section. If $\lambda = 1$, we turns out to be Problem (4.1). Following the Homotopy Perturbation method [3], we expand the solution in terms of the Homotopy parameter λ as

$$f = f_0 + \lambda f_1 + \lambda^2 f_2 + \lambda^3 f_3 + ... \tag{29}$$

Substitute Equation (4.4) into Equation (4.3) and equating the coefficients of the identical powers of λ to get the following system

$$\begin{aligned}
\lambda^0 &: f_0''(s) - \ell(f_0'(s), f_0(qs) = r(s)\ ,\ f_0(0) = f_0(1) = 0,\\
\lambda^1 &: f_1''(s) - \ell(f_1'(s), f_1(qs) = r(s) + N(s, \sum_{i=0}^{\infty} \lambda^i f_i'(s), \sum_{i=0}^{\infty} \lambda^i f_i(s)) \mid_{\lambda=0}\ ,\ f_1(0) = f_1(1) = 0,\\
\lambda^2 &: f_2''(s) - \ell(f_2'(s), f_2(qs) = r(s) + \frac{dN(s, \sum_{i=0}^{\infty} \lambda^i f_i'(s), \sum_{i=0}^{\infty} \lambda^i f_i(s))}{d\lambda} \mid_{\lambda=0}\ ,\ f_2(0) = f_2(1) = 0,\\
\lambda^3 &: f_3''(s) - \ell(f_3'(s), f_3(qs) = r(s) + \frac{d^2N(s, \sum_{i=0}^{\infty} \lambda^i f_i'(s), \sum_{i=0}^{\infty} \lambda^i f_i(s))}{d\lambda^2} \mid_{\lambda=0}\ ,\ f_3(0) = f_3(1) = 0,\\
&\vdots\\
\lambda^k &: f_k''(s) - \ell(f_k'(s), f_k(qs) = r(s) + \frac{d^{k-1}G(s, \sum_{i=0}^{\infty} \lambda^i f_i'(s), \sum_{i=0}^{\infty} \lambda^i f_i(s))}{d\lambda^{k-1}} \mid_{\lambda=0}\ ,\ f_k(0) = f_k(1) = 0.
\end{aligned}$$

To solve the above equations, we use the RKHSM which is described in the previous section and we obtain

$$f_k(s) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} h_k(s_j) \bar{\psi}_i(s)\ , k = 0, 1, ... \tag{30}$$

where

$$\begin{aligned}
h_0(s) &= r(s)\\
h_1(s) &= r(s) + N(s, \sum_{i=0}^{\infty} \lambda^i f_i'(s), \sum_{i=0}^{\infty} \lambda^i f_i(s)) \mid_{\lambda=0}\\
&\vdots\\
h_k(s) &= r(s) + \frac{d^{k-1}G(s, \sum_{i=0}^{\infty} \lambda^i f_i'(s), \sum_{i=0}^{\infty} \lambda^i f_i(s))}{d\lambda^{k-1}} \mid_{\lambda=0}, k > 1.
\end{aligned}$$

From Equation (4.5), it is easy to see the solution to Problem (4.1)-(4.2) is given by

$$f(s) = \sum_{k=0}^{\infty} f_k(s) = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} h_k(s_j) \bar{\psi}_i(s) \right). \tag{31}$$

We approximate the solution of Problem (4.1)-(4.2) by

$$f_{n,m}(s) = \sum_{k=0}^{m} \left(\sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} h_k(s_j) \bar{\psi}_i(s) \right). \tag{32}$$

5. Results and discussion

In order to illustrate the accuracy and applicability of the presented method, the propose method is applied to pantograph-type delay differential equations. For comparison purposes, the solution intervals of problems are chosen generally the same as those in the references.

Example 5.1. Consider the following two-points BVP [11]

$$y''(t) = -2e^{-t} + \frac{y(t)}{2} + e^{-t/2}y(t/2), \quad 0 \leq t \leq 1,$$

subject to

$$y(0) = 0, \; y(1) = e^{-1}.$$

The exact solution is

$$y(t) = te^{-t}.$$

The figures of the approximate and exact solutions are given in Figure 1. The absolute error obtained by the presented method, GA-ASA method [11], and Bica's results [5] are shown in Table 1.

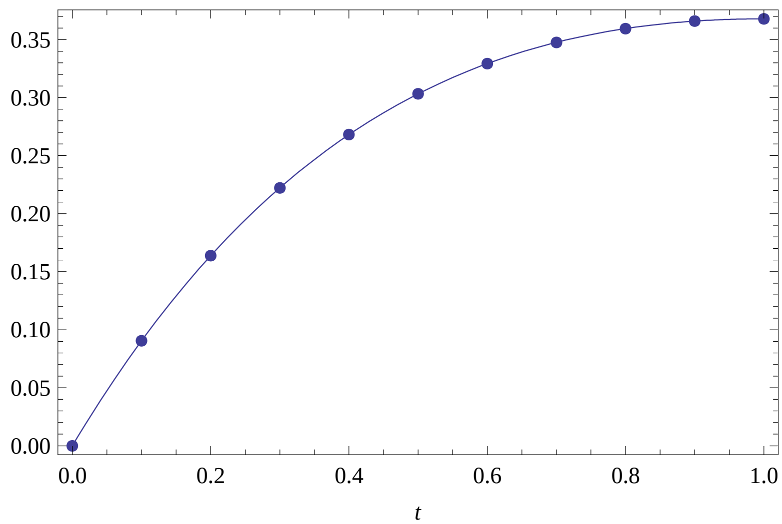

FIGURE 1. The exact and approximate solutions of Example (5.1).

t	Our results	GA-ASA	Bica's results
0.0	0.0	$4.91 * 10^{-10}$	0.0
0.1	$2.31 * 10^{-14}$	$1.06 * 10^{-10}$	$5.11 * 10^{-5}$
0.2	$2.98 * 10^{-14}$	$3.01 * 10^{-9}$	$9.51 * 10^{-5}$
0.3	$3.21 * 10^{-14}$	$3.30 * 10^{-9}$	$1.14 * 10^{-4}$
0.4	$3.62 * 10^{-14}$	$1.33 * 10^{-9}$	$1.28 * 10^{-4}$
0.5	$3.99 * 10^{-14}$	$2.37 * 10^{-9}$	$1.24 * 10^{-4}$
0.6	$3.87 * 10^{-14}$	$2.17 * 10^{-9}$	$1.16 * 10^{-4}$
0.7	$3.11 * 10^{-14}$	$5.20 * 10^{9}$	$9.48 * 10^{-5}$
0.8	$2.78 * 10^{-14}$	$2.56 * 10^{-9}$	$7.10 * 10^{-5}$
0.9	$2.22 * 10^{-14}$	$3.56 * 10^{10}$	$3.63 * 10^{-5}$
1.0	0.0	$2.10 * 10^{-9}$	0.0

Table 1: Absolute error $= |y(x) - y_{app}(x)|$

Example 5.2. Consider the following two-points BVP [11]

$$y''(t) = 1 + 2(1 + \frac{t^2}{8})\cos(\frac{t}{2}) - 2\cos(\frac{t}{2})y(\frac{t}{2}), \quad 0 \leq t \leq \frac{\pi}{4},$$

subject to

$$y(0) = 1, \; y(\frac{\pi}{4}) = 1 + \frac{\sqrt{2}}{2} + \frac{\pi^2}{32}.$$

The exact solution is

$$y(t) = \frac{t^2}{2} + \sin(t) + 1.$$

The figures of the approximate and exact solutions are given in Figure 2. The absolute error obtained by the presented method, GA-ASA method [11], and Bica's results [5] are shown in Table 2.

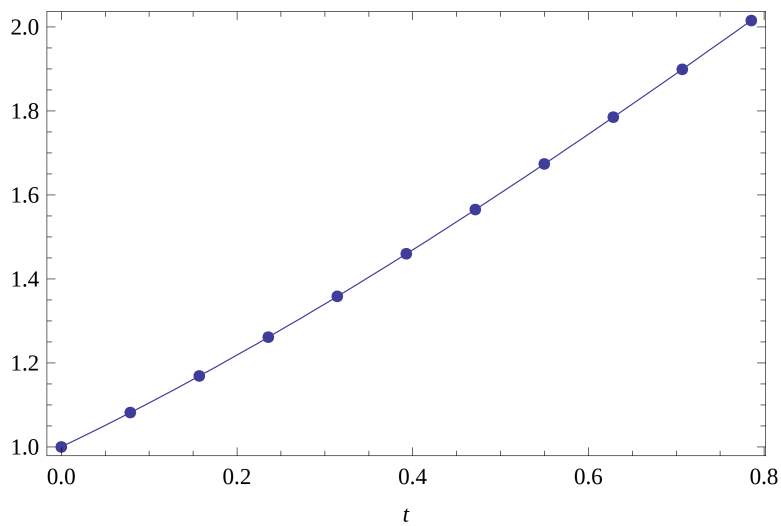

FIGURE 2. The exact and approximate solutions of Example (5.2).

t	Our results	GA-ASA	Bica's results
0.0	0.0	$3.97 * 10^{-9}$	$2.63 * 10^{-5}$
$\pi/40$	$4.76 * 10^{-12}$	$3.29 * 10^{-9}$	$4.32 * 10^{-5}$
$3\pi/40$	$4.87 * 10^{-12}$	$1.46 * 10^{-8}$	$5.92 * 10^{-5}$
$\pi/10$	$5.21 * 10^{-12}$	$5.61 * 10^{-9}$	$6.57 * 10^{-5}$
$\pi/8$	$5.70 * 10^{-12}$	$1.49 * 10^{-8}$	$7.08 * 10^{-5}$
$3\pi/20$	$4.88 * 10^{-12}$	$1.81 * 10^{-8}$	$6.65 * 10^{-5}$
$7\pi/40$	$4.65 * 10^{-12}$	$1.59 * 10^{-9}$	$6.02 * 10^{-5}$
$\pi/5$	$4.31 * 10^{-12}$	$2.51 * 10^{-9}$	$4.47 * 10^{-5}$
$9\pi/40$	$4.10 * 10^{-12}$	$1.61 * 10^{-8}$	$2.67 * 10^{-5}$
$\pi/4$	0.0	$9.75 * 10^{-9}$	0.0

Table 2: Absolute error $= |y(x) - y_{app}(x)|$

Example 5.3. Consider the following two-points BVP [11]

$$y''(t) = 4e^{-t/2} \sin(\frac{t}{2}) y(\frac{t}{2}), \quad 0 \le t \le \frac{\pi}{4},$$

subject to

$$y(0) = 1,\ y(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}e^{-\pi/4}.$$

The exact solution is

$$y(t) = e^{-t}\cos t.$$

The figures of the approximate and exact solutions are given in Figure 3. The absolute error obtained by the presented method, GA-ASA method [11], and Bica's results [5] are shown in Table 3.

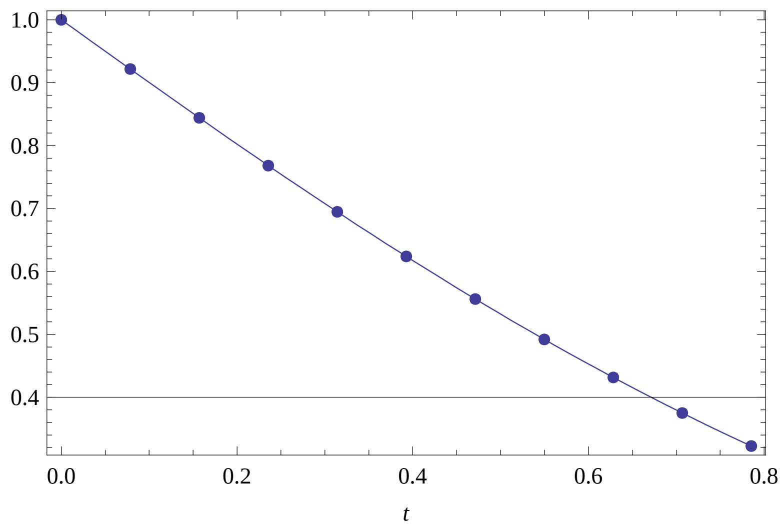

FIGURE 3. The exact and approximate solutions of Example (5.3).

t	Our results	GA-ASA	Bica's results
$\pi/40$	$1.10 * 10^{-13}$	$8.85 * 10^{-9}$	$4.20 * 10^{-7}$
$\pi/20$	$1.89 * 10^{-13}$	$2.91 * 10^{-9}$	$7.22 * 10^{-7}$
$3\pi/40$	$2.42 * 10^{-13}$	$1.05 * 10^{-9}$	$9.17 * 10^{-7}$
$\pi/10$	$2.53 * 10^{-13}$	$1.22 * 10^{-8}$	$1.01 * 10^{-7}$
$\pi/8$	$3.19 * 10^{-13}$	$1.14 * 10^{-8}$	$1.02 * 10^{-7}$
$3\pi/20$	$2.99 * 10^{-13}$	$5.98 * 10^{-10}$	$9.50 * 10^{-7}$
$7\pi/40$	$2.31 * 10^{-13}$	$4.27 * 10^{-9}$	$8.04 * 10^{-7}$
$\pi/5$	$1.80 * 10^{-13}$	$3.89 * 10^{-9}$	$5.93 * 10^{-7}$
$9\pi/40$	$1.31 * 10^{-13}$	$1.18 * 10^{-8}$	$3.23 * 10^{-7}$
$\pi/4$	0.0	$4.03 * 10^{-9}$	0.0

Table 3: Absolute error $= |y(x) - y_{app}(x)|$

Example 5.4: Consider the following two-points BVP

$$y''(t) = (y'(t))^2 - y(\frac{t}{2}) + e^t - e^{2t} + e^{\frac{t}{2}},\quad 0 \le t \le 1,$$

subject to

$$y(0) = 1,\ y(1) = e.$$

The exact solution is

$$y(t) = e^t.$$

The figures of the approximate and exact solutions are given in Figure 4. The absolute error obtained by the presented method is shown in Table 4.

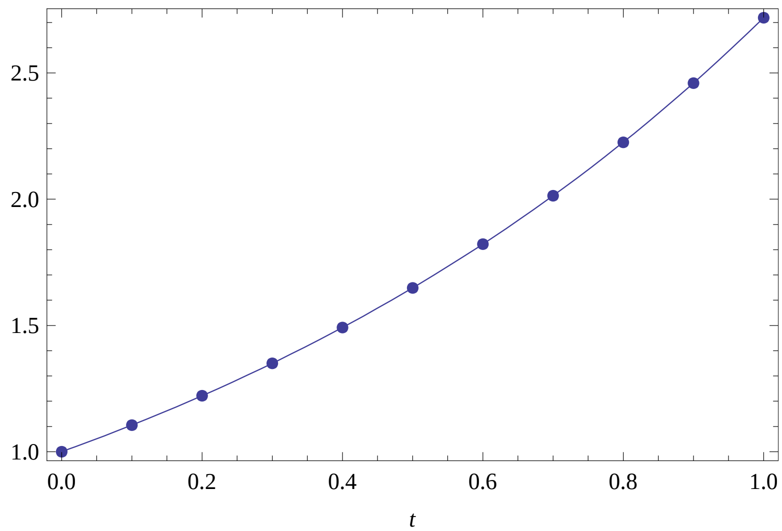

FIGURE 4. The exact and approximate solutions of Example (5.4).

t	Propose method
0	0
0.1	$1.99 * 10^{-13}$
0.2	$2.52 * 10^{-13}$
0.3	$2.78 * 10^{-13}$
0.4	$3.56 * 10^{-13}$
0.5	$4.29 * 10^{-13}$
0.6	$3.38 * 10^{-13}$
0.7	$2.84 * 10^{-13}$
0.8	$2.22 * 10^{-13}$
0.9	$1.93 * 10^{-13}$
1	0.0

Table 4: Absolute error $= |y(x) - y_{app}(x)|$

Example 5.5: Consider the following two-points BVP

$$y''(t) = y'(t)y(\frac{t}{2}) - 8t^2y(\frac{t}{2}) - 1 + \frac{11t}{2} + 5t^2 + \frac{19t^3}{8} + \frac{5t^5}{8}, \quad 0 \le t \le 1,$$

subject to

$$y(0) = 1, \; y(1) = 3.$$

The exact solution is

$$y(t) = 1 + t + t^3.$$

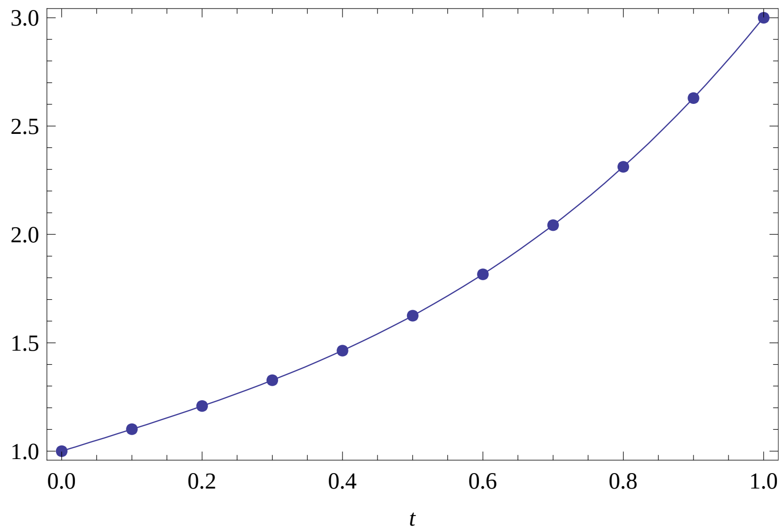

FIGURE 5. The exact and approximate solutions of Example (5.5).

The figures of the approximate and exact solutions are given in Figure 5.

Example 6: Consider the following two-points BVP

$$y''(t) = y'(t)\ y(\frac{t}{4}) - t\ y(\frac{t}{4}) - \frac{1}{4}\left(2t\sqrt{4+t} - \frac{1+(1+t)\sqrt{4+t}}{\sqrt{(1+t)^3}}\right), \quad 0 \le t \le 1,$$

subject to

$$y(0) = 1,\ y(1) = \sqrt{2}.$$

The exact solution is

$$y(t) = \sqrt{1+t}.$$

The figures of the approximate and exact solutions are given in Figure 6. The absolute error obtained by the presented method is shown in Table 5.

t	Propose method
0	0
0.1	$3.11 * 10^{-9}$
0.2	$3.39 * 10^{-9}$
0.3	$3.45 * 10^{-9}$
0.4	$4.12 * 10^{-9}$
0.5	$4.53 * 10^{-9}$
0.6	$4.11 * 10^{-9}$
0.7	$3.98 * 10^{-9}$
0.8	$3.62 * 10^{-9}$
0.9	$3.28 * 10^{-9}$
1	0.0

Table 5 : Absolute error $= |y(x) - y_{app}(x)|$

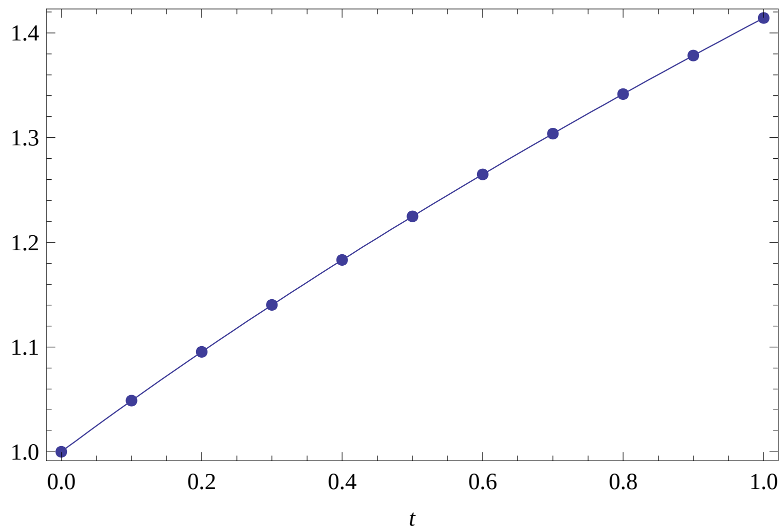

FIGURE 6. The exact and approximate solutions of Example (5.6).

6. Conclusions

In this paper, we studied the Pantograph equation using the reproducing kernel Hilbert space method. It is employed to compute an approximation to the solution of this problem. The validity of the RKHSM is ascertained by comparing our results with other methods in the literature. The results reveal that the proposed analytical method can achieve excellent results in predicting the solutions of such problems. The existences of the solution is proved in Theorem 3.4. In addition, the uniformly convergence of the proposed method is investigated in Theorem 3.5. In Examples (5.1)-(5.3), we studied linear Pantograph equation. In Tables 1, 2, and 3, we compared our results with GA-ASA [11] and Bica's results [5] for three different examples. In Examples (5.4) and (5.5), we studied the nonlinear Pantograph equations. In Figures 1-5, we compared between the exact solutions and our solutions. To show the efficiency of the proposed method, we compared between the computational cost of the proposed method, GA-ASA [11], and Bica's results [5] in Examples (5.1)-(5.3). Let $\Psi(s, f'(s), f(qs)) = a_1(s)f'(s) + a_2(s)f(qs) - h(s)$. These costs are given in Table 6 and 7.

Operation/Method	Proposed Method	GA-ASA	Bica's Method
Addition/Subtraction	n^2	$(n+2)(7n+23)$	$(3n+7)(n-1)$
Multiplication/Division	$5(n+n^2)/2$	$(n+2)(9n+29)$	$(n-1)(4n+4)$
Evaluation of $\Psi(s, f'(s), f(qs))$	$2n$	$6(n+2)(n+3)$	$2n(n-1)$
Evaluation of e^t	0	$3(n+2)(n+3)$	0
Value of n	15	30	20

Table 6: The approximate cost for the proposed method, GA-ASA, and Bica's results.

The approximate cost for the proposed method, GA-ASA, and Bica's results when we substitute in the values of n are reported in Table 7.

Operation/Method	Proposed Method	GA-ASA	Bica's Method
Addition/Subtraction	225	7456	1273
Multiplication/Division	600	9668	1596
Evaluation of $\Psi(s, f'(s), f(qs))$	30	6336	1740
Evaluation of e^t	0	3168	0
Value of n	15	30	20

Table 7: The approximate cost when we substitute in the values of n.

From the previous section, we can conclude the following:

- From Tables 1, 2, and 3, we see that our results agree exceptionally well with the exact solution and give results more accurate than GA-ASA [11] and Bica's results [5].
- From Tables 4 and 5, we see that our results agree exceptionally well with the exact solution for the nonlinear Pantograph equation.
- From Table 4, we see that we get the exact solution. For this example, we took $n = 8$.
- Tables 6 and 7 show that the cost of the proposed method is cheaper than GA-ASA [11] and Bica's results [5]
- Figures 1-6 show the comparison between the current method and exact solutions for five different examples. We see that there is agreement between the exact solutions and our results.
- It is worth mentioning that the truncation error $|y - y_n|$ using the proposed method is smaller than the GA-ASA [11] and Bica's results [5].
- If the domain is $[0, X]$ where $X > 1$, then we transform the problem, as described in section 2, into a problem with domain $[0, 1]$. for this reason, we considered only the case when $X = 1$.
- RKHSM is excellent tool due to rapid convergent.
- The results in this paper confirm that the proposed method is a powerful and efficient method for solving delay differential equations in different fields of sciences and engineering.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of the paper.

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