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ASYMPTOTIC STABILITY OF A HIGHER ORDER RATIONAL DIFFERENCE EQUATION

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Abstract. In this paper, we study the boundedness nature, the oscillatory character and the global behavior of positive solutions of the difference equation

$$y_{n+1}=\frac{a+b_0y_{n-1}+b_1y_{n-3}}{c+dy_n^{p_0}y_{n-2}^{p_1}},\ n=0,1,...,$$

where a, b_i, $i=0,1$ and the initial conditions are non-negative real numbers, the parameters c, d are positive real numbers and p_i, $i=0,1$ are positive integers.

1. Introduction

Difference equations have many applications in applied sciences such as economy, probability theory, ecology and sociology, and many realistic problems arising from biology can be modelled as difference equations see, for example, [3, 13, 16, 19] and the references cited therein. It is a known fact that the simplest difference equations admit solutions given by explicit formulas, but if the form of the solution is not available then this fact leads to the study of the global behavior of the solutions. There has been a great interest in studying the global asymptotic stability, boundedness and periodicity of solutions for nonlinear difference equations. Recently the study of higher order difference equations attracted a considerable attention see, for example, [1, 2, 5, 7, 9, 10, 11, 14, 17, 18].

This work is motivated by the papers [4, 5], in which Belhannache et al. obtained results concerning the boundedness, oscillation of the positive solutions and the stability of the unique positive equilibrium of the difference equations

$$x_{n+1}=\frac{A+Bx_{n-1}}{C+Dx_n^px_{n-2}^q},\ n=0,1,..., \tag{1}$$

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and

$$x_{n+1} = \frac{A + Bx_{n-2k-1}}{C + D\prod_{i=l}^{k} x_{n-2i}^{m_i}}, \ n = 0, 1, ..., \tag{2}$$

In this paper we generalize the aforementioned works. We consider the following difference equation

$$y_{n+1} = \frac{a + b_0 y_{n-1} + b_1 y_{n-3}}{c + d y_n^{p_0} y_{n-2}^{p_1}}, \ n = 0, 1, ..., \tag{3}$$

where a, b_i, $i = 0, 1$ and the initial conditions are non-negative real numbers, the parameters c, d are positive real numbers and p_i, $i = 0, 1$ are positive integers.
Our aim is to investigate the boundedness, oscillation and global asymptotic stability of the positive solutions of Equation (3).
In order to study Equation (3), we will transform it to a simpler form. We put $y_n = (\frac{c}{d})^{\frac{1}{p}} x_n$. Therefore, we obtain the following equivalent difference equation

$$x_{n+1} = \frac{\alpha + \beta_0 x_{n-1} + \beta_1 x_{n-3}}{1 + x_n^{p_0} x_{n-2}^{p_1}}, \ n = 0, 1, ..., \tag{4}$$

where $\alpha = \frac{a}{c}(\frac{d}{c})^{\frac{1}{p}}$, $\beta_i = \frac{b_i}{c}$, $i = 0, 1$ and $p = p_0 + p_2$ and the initial conditions are arbitrary non-negative real numbers. So, it suffices to study Eq.(4) instead of Eq.(3).
From now on, we let $\beta = \beta_0 + \beta_1$.

2. Linearized stability analysis

In this section we present the local stability of the equilibrium points of Eq.(4). First, we prove the following proposition.

Proposition 2.1. *Let $\alpha > 0$. Then the following statements are true:*

(1) *Assume that $\beta \geq 1$. Then, Eq.(4) has a unique positive equilibrium in $((\frac{\beta-1}{p+1})^{\frac{1}{p}}, +\infty)$.*
(2) *Assume that $\beta < 1$. Then*
   (i): *If $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then Eq.(4) has a unique positive equilibrium in $(0, (\frac{1-\beta}{p-1})^{\frac{1}{p}})$.*
   (ii): *If $\alpha > p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then Eq.(4) has a unique positive equilibrium in $((\frac{1-\beta}{p-1})^{\frac{1}{p}}, +\infty)$.*

Proof. Let us consider the function defined by

$$f(x) = x^{p+1} + (1 - \beta)x - \alpha, \tag{5}$$

whenever $\alpha > 0$, $\beta \geq 0$.
Then, a point $\overline{x}$ is an equilibrium point of Eq.(4) if and only if $\overline{x}$ is a zero of the function f defined by (5). One easily see that

$$f(0) = -\alpha < 0 \text{ and } f'(x) = (p + 1)x^p + (1 - \beta).$$

So, we have two cases:

(1) If $\beta \geq 1$, then f is increasing function on the interval $((\frac{\beta-1}{p+1})^{\frac{1}{p}}, +\infty)$. But

$$f((\frac{\beta-1}{p+1})^{\frac{1}{p}}) = -p(\frac{\beta-1}{p+1})^{\frac{p+1}{p}} - \alpha < 0.$$

Hence, f has a unique zero in $((\frac{\beta-1}{p+1})^{\frac{1}{p}}, +\infty)$.

(2) Assume that $\beta < 1$. Then f is increasing on $(0, +\infty)$. We distinguish also two cases:

(i): If $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then

$$f((\frac{1-\beta}{p-1})^{\frac{1}{p}}) = p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}} - \alpha > 0.$$

Therefore, f has a unique zero in $(0, (\frac{1-\beta}{p-1})^{\frac{1}{p}})$.

(ii): If $\alpha > p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then

$$f((\frac{1-\beta}{p-1})^{\frac{1}{p}}) < 0.$$

Thus, f has a unique zero in $((\frac{1-\beta}{p-1})^{\frac{1}{p}}, +\infty)$. Thus the result.

□

Theorem 2.2. *Assume $\alpha > 0$. Let $\overline{x}$ be the unique positive equilibrium of Eq.(4). Then, the following statements are true:*

(1) *If $\beta \geq 1$, then $\overline{x}$ is unstable.*

(2) *If $\beta < 1$, then*

(i): *$\overline{x}$ is locally asymptotically stable if $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$.*

(ii): *$\overline{x}$ is unstable if $\alpha > p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$.*

Proof. The linearized equation associated with Eq.(4) about $\overline{x}$ is

$$z_{n+1} = -\frac{\overline{x}^p}{1+\overline{x}^p}\sum_{i=0}^{1} p_i z_{n-2i} + \frac{1}{1+\overline{x}^p}\sum_{i=0}^{1}\beta_i z_{n-2i-1}, \text{ for every } n = 0, 1, ...$$

The characteristic equation associated with this equation is

$$\lambda^4 + \frac{\overline{x}^p}{1+\overline{x}^p}\sum_{i=0}^{1} p_i \lambda^{3-2i} - \frac{1}{1+\overline{x}^p}\sum_{i=0}^{1}\beta_i \lambda^{2-2i} = 0. \tag{6}$$

(1) Assume that $\beta \geq 1$ and consider the function g defined by

$$g(\lambda) = \lambda^4 + \frac{\overline{x}^p}{1+\overline{x}^p}\sum_{i=0}^{1} p_i \lambda^{3-2i} - \frac{1}{1+\overline{x}^p}\sum_{i=0}^{1}\beta_i \lambda^{2-2i}.$$

Then, we have

$$\lim_{\lambda\to-\infty} g(\lambda) = +\infty \text{ and } g(-1) = 1 - \frac{\beta + p\overline{x}^p}{1+\overline{x}^p} < 1 - \frac{1+\overline{x}^p}{1+\overline{x}^p} = 0.$$

It follows that, the characteristic equation (6) has a root λ_1 in $(-\infty, -1)$, which completes the proof of the first part of the theorem.

(2) Assume that $\beta < 1$. Thus

(i): If $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then $\overline{x} < (\frac{1-\beta}{p-1})^{\frac{1}{p}}$. Let us consider the functions h_1 and h_2 defined by

$$h_1(\lambda) = \lambda^4;$$

$$h_2(\lambda) = \frac{\overline{x}^p}{1+\overline{x}^p}\sum_{i=0}^{1} p_i\lambda^{3-2i} - \frac{1}{1+\overline{x}^p}\sum_{i=0}^{1}\beta_i\lambda^{2-2i}.$$

Then, we get

$$|h_2(\lambda)| \leq \frac{\beta + p\overline{x}^p}{1+\overline{x}^p} < 1 = |h_1(\lambda)|,$$

for all $\lambda \in \mathbb{C}$ where $|\lambda| = 1$. By Rouché's Theorem, all roots of (6) lie in the open unit disk $|\lambda| < 1$ and, hence, the result follows from Theorem 1.1 [12].

(ii): The proof is similar to that of the case 1 and it will be omitted.

□

Now we study the local stability of the equilibrium points of Eq.(4) in the case when $\alpha = 0$.
It is clear that if $\alpha = 0$ the Eq.(4) becomes

$$x_{n+1} = \frac{\beta_0 x_{n-1} + \beta_1 x_{n-3}}{1 + x_n^{p_0} x_{n-2}^{p_1}},\ n = 0, 1, ... \tag{7}$$

Note that $\overline{x} = 0$ is always an equilibrium point of Eq.(7). When $\beta > 1$, Eq.(7) also possesses the unique positive equilibrium $\overline{x} = (\beta - 1)^{\frac{1}{p}}$.
Some difference equations very close to Eq.(7) have been studied, for example, in [6, 7, 11, 15].

Theorem 2.3. *The following statements are true*

(1) *The equilibrium point $\overline{x} = 0$ of Eq.(7) is locally asymptotically stable if $\beta < 1$ and it is unstable if $\beta > 1$.*
(2) *When $\beta > 1$, then the positive equilibrium $\overline{x} = (\beta-1)^{\frac{1}{p}}$ of Eq.(7) is unstable.*

Proof. (1) The linearized equation associated with Eq.(7) about the equilibrium point $\overline{x} = 0$ is

$$z_{n+1} = \beta_0 z_{n-1} + \beta_1 z_{n-3},\ n = 0, 1, ...$$

Its characteristic equation is

$$\lambda^4 - \beta_0\lambda^2 - \beta_1 = 0. \tag{8}$$

Assume $\beta < 1$. Using theorem of Rouchè, we obtain that all roots of (8) lie in the open unit disk.
Let $\beta > 1$. It is clear that (8) has a root in the interval $(1, +\infty)$. Therefore, the point $\overline{x} = 0$ is locally asymptotically stable if $\beta < 1$ and it is unstable if $\beta > 1$.

(2) Let $\beta > 1$. The linearized equation associated with Eq.(7) about the equilibrium point $\overline{x} = (\beta - 1)^{\frac{1}{p}}$ is

$$z_{n+1} = \frac{1-\beta}{\beta}\sum_{i=0}^{1} p_i z_{n-2i} + \frac{1}{\beta}\sum_{i=0}^{1}\beta_i z_{n-2i-1}\ n = 0, 1, ...$$

Its characteristic equation is

$$\lambda^4 + \frac{\beta - 1}{\beta}\sum_{i=0}^{1} p_i \lambda^{3-2i} - \frac{1}{\beta}\sum_{i=0}^{1} \beta_i \lambda^{2-2i} = 0. \tag{9}$$

We can see that (9) has a root in $(-\infty, -1)$. Therefore, the point $\overline{x} = (\beta - 1)^{\frac{1}{p}}$ is unstable.

□

3. Boundedness and oscillation

Theorem 3.1. *Let $\overline{x}$ be the unique positive equilibrium of Eq.(4) and let $\{x_n\}_{n=-3}^{\infty}$ be a solution of the same equation. If one of the following conditions*
(a_1) $x_{-3}, x_{-1} < \overline{x} \le x_{-2}, x_0,$
(a_2) $x_{-2}, x_0 < \overline{x} \le x_{-3}, x_{-1},$
is satisfied, then the solution $\{x_n\}_{n=-3}^{\infty}$ oscillates about $\overline{x}$ with semicycles of length one.

Proof. Assume that the condition (a_1) is satisfied. Then

$$x_1 = \frac{\alpha + \beta_0 x_{-1} + \beta_1 x_{-3}}{1 + x_0^{p_0} x_{-2}^{p_1}} < \frac{\alpha + \beta\overline{x}}{1 + \overline{x}^p} = \overline{x}$$

and

$$x_2 = \frac{\alpha + \beta_0 x_0 + \beta_1 x_{-2}}{1 + x_1^{p_0} x_{-1}^{p_1}} > \frac{\alpha + \beta\overline{x}}{1 + \overline{x}^p} = \overline{x}.$$

By induction on n, we obtain

$$x_{2n} \ge \overline{x} \text{ and } x_{2n+1} < \overline{x}, \text{ for all } n \ge 0.$$

If the condition (a_2) is satisfied, the proof is similar and it is omitted. □

Theorem 3.2. *Assume that $\beta < 1$ and $\alpha > 0$. Then every solution of Eq.(4) is bounded and persists.*

Proof. Assume $\beta < 1$. let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(4).
From Eq.(4), we have

$$x_{n+1} \le \alpha + \beta_0 x_{n-1} + \beta_1 x_{n-3}, \quad n = 0, 1... \tag{10}$$

Consider the linear difference equation

$$w_{n+1} = \alpha + \beta_0 w_{n-1} + \beta_1 w_{n-3}, \quad n = 0, 1, ... \tag{11}$$

with

$$w_j = x_j, \text{ for } j \in \{-3, -2, -1, 0\}. \tag{12}$$

By induction we get

$$x_n \le w_n, \quad \forall\, n \ge -3 \tag{13}$$

It is clear that $\frac{\alpha}{1-\beta}$ is a particular solution of Eq.(11).
On the other hand the assumption $\beta < 1$ implies that every solution of the homogeneous equation which is associated with Eq.(11) tends to zero as $n \longrightarrow +\infty$.
Hence

$$\lim_{n \longrightarrow +\infty} w_n = \frac{\alpha}{1 - \beta},$$

then $\{x_n\}_{n=-3}^{\infty}$ is bounded by a positive constant, say M. That is

$$x_n \le M, \quad \forall\, n \ge 0. \tag{14}$$

Now we prove that $\{x_n\}_{n=-3}^{\infty}$ persists.
From Eq.(4), we have

$$x_{n+1} > \frac{\alpha}{1 + x_n^{p_0} x_{n-2}^{p_1}}, \ \ \forall\, n \geq 0.$$

Using (14) we obtain

$$x_n > \frac{\alpha}{1 + M^p}, \ \ \forall\, n \geq 3,$$

which completes the proof. □

Lemma 3.3. *Suppose $\beta < 1$ and $\alpha > 0$. let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(4). If $\Lambda = \limsup_{n\to\infty} x_n$ and $\lambda = \liminf_{n\to\infty} x_n$, then Λ and λ satisfy the following inequalities*

$$\frac{\alpha + \beta\lambda}{1 + \Lambda^p} \leq \lambda \leq \Lambda \leq \frac{\alpha + \beta\Lambda}{1 + \lambda^p}.$$

Proof. Assume $\beta < 1$ and $\alpha > 0$. From Theorem 3.2, the solution $\{x_n\}_{n=-3}^{\infty}$ is bounded. Hence, for every $\varepsilon \in (0, \lambda)$ there exists $n_0 \in \mathbb{N}$ such that

$$\lambda - \varepsilon \leq x_n \leq \Lambda + \varepsilon, \text{ for all } n \geq n_0.$$

So, this implies that

$$\frac{\alpha + \beta(\lambda - \varepsilon)}{1 + (\Lambda + \varepsilon)^p} \leq x_{n+1} \leq \frac{\alpha + \beta(\Lambda + \varepsilon)}{1 + (\lambda - \varepsilon)^p}, \text{ for all } n \geq n_0 + 3.$$

Therefore, we obtain

$$\frac{\alpha + \beta\lambda}{1 + \Lambda^p} \leq \lambda \leq \Lambda \leq \frac{\alpha + \beta\Lambda}{1 + \lambda^p}.$$

□

4. Global asymptotic stability

In this section we give global asymptotic stability result for Eq.(4). We show that the unique positive equilibrium is globally asymptotically stable in the subregion of the parametric region of local asymptotic stability when $\alpha > 0$ and that the zero equilibrium is globally asymptotically stable if $\beta < 1$ and $\alpha = 0$.

Theorem 4.1. *Assume that $\alpha > 0$ and $\beta < 1$. If $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then the positive equilibrium $\overline{x} \in (0, (\frac{1-\beta}{p-1})^{\frac{1}{p}})$ of Eq.(4) is globally asymptotically stable.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(4). Since $\beta < 1$, the solution $\{x_n\}_{n=-2k-1}^{\infty}$ is bounded. Let $\Lambda = \limsup_{n\to\infty} x_n$ and $\lambda = \liminf_{n\to\infty} x_n$. Using Lemma 3.3, we have

$$\frac{\alpha + \beta\lambda}{1 + \Lambda^p} \leq \lambda \leq \Lambda \leq \frac{\alpha + \beta\Lambda}{1 + \lambda^p}.$$

This implies that

$$(1 - \beta)\lambda^p - \alpha\lambda^{p-1} \geq (1 - \beta)\Lambda^p - \alpha\Lambda^{p-1}. \tag{15}$$

Now, let us consider the function

$$h(x) = (1 - \beta)x^p - \alpha x^{p-1}.$$

Hence,

$$h'(x) = x^{p-2}(p(1 - \beta)x - (p - 1)\alpha),$$

and the function $h(x)$ is increasing on $(\frac{\alpha(p-1)}{p(1-\beta)},\infty)$. As $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, we get

$$\frac{\alpha(p-1)}{p(1-\beta)} < \overline{x} < (\frac{1-\beta}{p-1})^{\frac{1}{p}}.$$

In view of inequality (15), we have a contradiction. Therefore, $\lambda = \Lambda = \overline{x}$ and so $\overline{x}$ is a global attractor.
The global asymptotically stability of $\overline{x}$ is obtained by combining the global attractivity and the local asymptotic stability of $\overline{x}$ when $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$. □

The following result deals with the global attractivity of the zero equilibrium of Eq.(7).

Theorem 4.2. *Assume $\beta < 1$. Then the equilibrium point $\overline{x} = 0$ of Eq.(7) is globally asymptotically stable.*

Proof. We know by Theorem 2.3 that the equilibrium point $\overline{x} = 0$ of Eq.(7) is locally asymptotically stable.
From Eq.(7) we have

$$x_{n+1} < \beta_0 x_{n-1} + \beta_1 x_{n-3}.$$

By substuting $\alpha = 0$ in Eq.(11), we obtain Eq.(13). Since $\beta < 1$, then $\lim_{n\longrightarrow+\infty} w_n = 0$. Hence

$$\lim_{n\rightarrow+\infty} x_n = 0.$$

This completes the proof.

□

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