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PROPERTIES AND CHARACTERISTICS OF A FAMILY CONSISTING OF BAZILEVIĆ (TYPE) FUNCTIONS SPECIFIED BY CERTAIN LINEAR OPERATORS

A.R.S. JUMA, S.N. AL-KHAFAJI, H. IRMAK

Abstract. In this paper, through the instrument of the well-known Gauss hypergeometric function and Hadamard product, a linear operator is firstly defined and a family consisting of Bazilević (type) function is then described by using the related operator. Several systematic investigations of the various properties and characteristics of the concerned family are also presented.

1. Introduction, Definitions and Preliminaries

Let $\mathbb{R}^+ = (0,\infty)$ be the set of positive real numbers, $\mathbb{N} = \{1,2,3,\cdots\}$ be the set of positive integers, $\mathbb{Z}^- = \{-1,-2,-3,\cdots\}$ be the set of negative integers, $\mathbb{C}$ be the set of complex numbers and also let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$ and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

We let $\mathcal{A}$ denote the family of functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C} \text{ and } |z| < 1\}$$

and normalized with the following conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Clearly, these functions $f(z)$ have the following Taylor-Mclaurin series form:

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots + a_kz^k + \cdots \quad \big(a_k \geq 0; z \in \mathbb{U}\big). \tag{1}$$

In the light of well-known binomial expansion, we can now introduce a novel operator denoted by

$$\mathcal{D}_{l,\alpha}^{n,\lambda}[f] \quad \text{or} \quad \mathcal{D}_{l,\alpha}^{n,\lambda}[f](z)$$

also defined by

$$\mathcal{D}_{l,\alpha}^{n,\lambda}[f] := z^\alpha + \sum_{k=2}^{\infty} c_k z^{k+\alpha-1}, \tag{2}$$

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where

$$c_k := c_k\big(n, l, \alpha, \lambda\big)a_k := \Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^n a_k$$

and

$$f \equiv f(z) \in \mathcal{A}\ ,\ n \in \mathbb{N}_0\ ,\ \lambda \in \mathbb{R}_0^+\ ,\ \alpha \in \mathbb{R}^+\ ,\quad \text{and}\quad z \in \mathbb{U}.$$

As a natural consequence of the operator given by (), it can be easily determined by the following relationships:

$$\mathcal{D}_{l,\alpha}^{0,\lambda}[f] = z^\alpha + \sum_{k=2}^{\infty} a_k z^{k+\alpha-1} \tag{3}$$

$$\begin{aligned}\mathcal{D}_{l,\alpha}^{1,\lambda}[f] &= \Big(\frac{l-\lambda\alpha+\alpha}{l+\alpha}\Big)\mathcal{D}_{l,\alpha}^{0,\lambda}[f] + \Big(\mathcal{D}_{l,\alpha}^{0,\lambda}[f]\Big)'\frac{\lambda z}{l+\alpha} \\ &= z^\alpha + \sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)a_k z^{k+\alpha-1}\end{aligned} \tag{4}$$

and also

$$\mathcal{D}_{l,\alpha}^{2,\lambda}[f] = z^\alpha + \sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^2 a_k z^{k+\alpha-1}, \tag{5}$$

where $f \equiv f(z) \in \mathcal{A}$, $\lambda \in \mathbb{R}_0^+$, $\alpha \in \mathbb{R}^+$ and $z \in \mathbb{U}$.

For our main results, we need also to recall the Gauss hypergeometric function defined by

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k(1)_k}z^k \quad (z \in \mathbb{U}), \tag{6}$$

where $a, b \in \mathbb{C}$, $c \notin \mathbb{Z}_0^-$ and $(k)_n$ denotes Pochhammer symbol (*or* the shifted factorial) given, in terms of the Gamma function Γ, by

$$(k)_n = \frac{\Gamma(k+n)}{\Gamma(k)}\begin{cases} k(k+1)(k+2)\cdots(k+n-1) & if\quad n \in \mathbb{N} \\ 1 & if\quad n = 0.\end{cases} \tag{7}$$

Next, corresponding to the function ${}_2F_1(a, b, c; z)$, we introduce a linear operator:

$$\mathcal{H}_{l,\alpha}^{n,\lambda}[f](z) \quad \text{or} \quad \mathcal{H}_{l,\alpha}^{n,\lambda}[f]$$

which is defined by means of the following Hadamard product (or convolution):

$$\mathcal{D}_{l,\alpha}^{n,\lambda}[f](z)\ *\ {}_2F_1(a, b, c; z)$$

of the functions

$$\mathcal{D}_{l,\alpha}^{n,\lambda}[f](z) \quad \text{and} \quad {}_2F_1(a, b, c; z)$$

or, equaivalently, by

$$\begin{aligned}\mathcal{H}_{l,\alpha}^{n,\lambda}[f] &:= \mathcal{D}_{l,\alpha}^{n,\lambda}[f](z)\ *\ {}_2F_1(a, b, c; z) \\ &= z^\alpha + \sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^n\ \frac{(a)_k(b)_k}{(c)_k(1)_k}a_k z^{k+\alpha-1},\end{aligned} \tag{8}$$

where $n \in \mathbb{N}_0$, $\lambda \in \mathbb{R}_0^+$, $\alpha \in \mathbb{R}^+$, $c \notin \mathbb{Z}_0^-$, $f(z) \in \mathcal{A}$ and $z \in \mathbb{U}$.

Finally, under the conditions of the definitions in (1), (6), and (7) and also by making use of the above-linear operator $\mathcal{H}_{l,\alpha}^{n,\lambda}[f]$, we say that a function $f := f(z) \in$

$\mathcal{A}$, given by (1), is in the aforementioned family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$, which also consists of Bazilević (type) functions, if it satisfies the following inequality:

$$\left|\frac{z^{1+\alpha}\Big(\mathcal{H}_{l,\alpha}^{n,\lambda}[f]\Big)' - \alpha z^{2\alpha}}{(\delta-1)z^{1+\alpha}\Big(\mathcal{H}_{l,\alpha}^{n,\lambda}[f]\Big)' - \big(\delta\gamma - \alpha z^{2\alpha}\big)}\right| < \beta \quad \big(z \in \mathbb{U}\big) \tag{9}$$

for all $n \in \mathbb{N}_0$, $\lambda \in \mathbb{R}_0^+$, $\alpha \in \mathbb{R}^+$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$ and $0 \leq \delta \leq 1$.

In the literature, from time to time, we encounter several works associated with a great number of operators (linear *or* nonlinear) constituted by certain complex functions (with one variable *or* several variables) which are analytic *or* univalent in certain domains of the complex plane $\mathbb{C}$. As examples, one can see those given by the papers in [3], [4], [7], [8], [9], [10], [11], [12], and also [13]. Moreover, since some of them are very important for the theory of functions, intrinsically, they or some of them have also an important roles for analytic or geometric function theory, which is a special field of complex analysis and includes the study of the relations between the analytic properties of a function $f(z)$ and the geometric properties of the image domain $f(\mathbb{U})$. For these properties of certain complex functions, one may also refer to the works in [1], [2], [5], [6] and also others.

By virtue of the reasons indicated by above, a function $f(z) \in \mathcal{A}$ in the family (9) defined by the operator in (8) is of significance for its analytic and geometric properties. For this, firstly, the authors introduced a new and also general family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$ of functions which are Bazilević (type-univalent) functions in the open unit disk $\mathbb{U}$ and then obtained a necessary and sufficient condition for a function to be in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$. They also presented several basic results, derived here for functions in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$, in relation with some growth and distortion theorems, the radii of the Bazilević (type-univalent) functions and certain extremal functions. In addition, they also pointed some interesting and possible other results of the basic results out.

2. Basic Properties and Characteristics of the Family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$ and Some of its Consequences

We begin by giving and then proving a characterization property of the above-defined family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$, which is contained in the following theorem.

Theorem 1. Let the function $f(z) \in \mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$ be in the form (1). Then, $f(z)$ is in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$ if and only if

$$\sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^n \frac{(a)_k(b)_k}{(c)_k(1)_k}(k+\alpha-1)a_k \leq \frac{\beta\delta(\alpha-\gamma)}{1-\beta\delta+\beta}, \tag{10}$$

where $a > 0$, $b > 0$, $c > 0$, $\lambda > 0$, $\alpha > 0$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $\alpha - \gamma \neq 0$, and $n \in \mathbb{N}_0$.

The result is sharp for the extremal function given by

$$f(z) = z + \frac{\beta\delta(\alpha-\gamma)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(a)_2(b)_2}\Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^{-n} z^2, \tag{11}$$

where $a>0$, $b>0$, $c>0$, $\lambda>0$, $\alpha>0$, $0<\beta\leq 1$, $0\leq\gamma\leq 1$, $0\leq\delta\leq 1$, $\alpha-\gamma\neq 0$, $n\in\mathbb{N}_0$, and $z\in\mathbb{U}$

Proof. Firstly, assume that the function $f(z)\in\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$ is defined by (1) and the inequality (10) holds true. Then, for $z\in\partial(\mathbb{U})$, we calculate that

$$\begin{aligned}
&\left|z^{\alpha+1}\Big(\mathcal{H}_{l,\alpha}^{n,\lambda}[f]\Big)'-\alpha z^{2\alpha}\right|-\beta\left|(\delta-1)z^{\alpha+1}\Big(\mathcal{H}_{l,\alpha}^{n,\lambda}[f]\Big)'-(\delta\gamma-\alpha z^{2\alpha})\right|\\
&\quad=\left|\sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^{n}\frac{(a)_k(b)_k}{(c)_k(1)_k}(k+\alpha-1)a_kz^{2\alpha+k-1}\right|\\
&\qquad\qquad-\beta\Big|(\delta-1)\Big[\alpha z^{2\alpha}+\sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^{n}\frac{(a)_k(b)_k}{(c)_k(1)_k}\\
&\qquad\qquad\qquad\times(k+\alpha-1)a_kz^{2\alpha+k-1}\Big]-\big(\delta\gamma-\alpha z^{2\alpha}\big)\Big|\\
&\quad\leq\sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^{n}\frac{(a)_k(b)_k}{(c)_k(1)_k}(k+\alpha-1)a_k-\frac{\beta\delta(\alpha-\gamma)}{(1-\beta\delta+\beta)}\leq 0,
\end{aligned}$$

by virtue of the desired inequality in (10). Hence, by maximum modulus theorem, the function $f(z)$ belongs to the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$.

To prove the converse, suppose that the function $f(z)$ is defined by (1) and is in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$. So that the condition (9) readily yields that

$$\begin{aligned}
&\left|\frac{z^{\alpha+1}\Big(\mathcal{H}_{l,\alpha}^{n,\lambda}[f]\Big)'-\alpha z^{2\alpha}}{(\delta-1)z^{\alpha+1}\Big(\mathcal{H}_{l,\alpha}^{n,\lambda}[f]\Big)'-(\delta\gamma-\alpha z^{2\alpha})}\right|\\
&\quad=\left|\frac{z^{\alpha+1}[\alpha z^{\alpha-1}+\sum_{k=2}^{\infty}(\frac{l+\alpha+\lambda(k-1)}{l+\alpha})^{n}\frac{(a)_k(b)_k}{(c)_k(1)_k}(\alpha+k-1)a_kz^{\alpha+k-2}]-\alpha z^{2\alpha}}{(\delta-1)z^{\alpha+1}[\alpha z^{\alpha-1}+\sum_{k=2}^{\infty}(\frac{l+\alpha+\lambda(k-1)}{l+\alpha})^{n}\frac{(a)_k(b)_k}{(c)_k(1)_k}(\alpha+k-1)a_kz^{\alpha+k-2}]-\delta\gamma+\alpha z^{2\alpha}}\right|\\
&\quad<\beta. \qquad (12)
\end{aligned}$$

Since $\big|\Re e(z)\big|\leq|z|$ for any $z\in\mathbb{U}$, if we choose z to be real and let $z\to 1-$, we shall find from (12) that

$$\begin{aligned}
&\sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^{n}\frac{(a)_k(b)_k}{(c)_k(1)_k}(\alpha+k-1)a_k(\alpha)\\
&\qquad\leq\beta\left(\sum_{k=2}^{\infty}(\delta-1)\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^{n}\frac{(a)_k(b)_k}{(c)_k(1)_k}(\alpha+k-1)a_k(\alpha)+\delta(\alpha-\gamma)\right),
\end{aligned}$$

which readily yields that the desired assertion (10).

Finally, by observing that the function $f(z)$ defined by (1) is indeed an extremal function for the assertion (10), the proof of Theorem 1 is therefore completed.

The next theorem is also contained in the following form.

Theorem 2. The family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$ is closed under convex linear combination.

Proof. Let the functions

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k\quad\big(a_{k,j}\geq 0;j=1,2\big),\tag{13}$$

be in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$. Then, we have to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad \big(0 \leq \lambda \leq 1\big), \tag{14}$$

is in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$. If one takes cognizance of the assertion (10) for the definition of functions $f_j(z)$ (j=1,2) defined by (13) and also for the function $h(z)$ defined by (14), or, equivalently, in the form:

$$h(z) = z + \sum_{k=2}^{\infty} \big[\lambda a_{k,1} z^k + (1-\lambda) a_{k,2} z^k\big],$$

the inequality:

$$\sum_{k=2}^{\infty} \Big\{ \Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^n \frac{(a)_k (b)_k}{(c)_k (1)_k} (k+\alpha-1) \\ \times \big[\lambda a_{k,1}(\alpha) + (1-\lambda) a_{k,2}(\alpha)\big] \Big\} \leq \frac{\beta\delta(\alpha-\gamma)}{(1-\beta\delta+\beta)},$$

is easily obtained. Thereby, that function $h(z)$ belongs to $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$ and the desired proof is here completed.

The following result is more generalization of Theorem 2 and it can be similarly proven, which is contained in the following theorem.

Theorem 3. Let each of the functions $f_i(z)$ be defined by

$$f_j(z) = z + a_{1,j} z + a_{2,j} z^2 + \cdots + a_{k,j} z^k + \cdots$$

and also be in the same family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$, where

$$a_k, j \geq 0 \ ; \ j = 1, 2, \cdots, m \ ; \ m \in \mathbb{N} \text{ and } z \in \mathbb{U}.$$

Then, the function $h(z)$ defined by

$$h(z) = c_1 f_1(z) + c_2 f_2(z) + \cdots + c_m f_m(z),$$

where

$$c_j \geq 0 \ \big(j = 1, 2, \cdots, m\big) \quad \text{and} \quad c_1 + c_2 + \cdots + c_m = 1,$$

is also in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$.

The growth and distortion theorems for the function $f(z)$, defined by (1), in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$ are also presented by the following theorems, which are Theorems 4 and 5 below.

Theorem 4. Let the function $f(z)$ be in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$. Then, the following inequalities:

$$\big|f(z)\big| \leq |z| + \frac{\beta\delta(\alpha-\gamma)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(a)_2(b)_2} \Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^{-n} |z|^2 \tag{15}$$

and

$$\big|f(z)\big| \geq |z| - \frac{\beta\delta(\alpha-\gamma)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(a)_2(b)_2} \Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^{-n} |z|^2 \tag{16}$$

are satisfied, where $a > 0$, $b > 0$, $c > 0$, $\lambda > 0$, $\alpha > 0$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $\alpha - \gamma \neq 0$, and $n \in \mathbb{N}_0$ and $z \in \mathbb{U}$. The bounds in (15) and (16) are attained for the function $f(z)$ given by

$$f(z) = z + \frac{\beta\delta(\alpha-\gamma)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(a)_2(b)_2}\Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^{-n} z^2 \quad (z \in \mathbb{U}). \tag{17}$$

Proof. Let $f(z)$, defined by (1), be in the family $\mathcal{B}^{n,\lambda}_{l,\alpha}(\beta,\gamma,\delta)$. By using (10), we then obtain

$$\begin{aligned}(1+\alpha)\Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^n \frac{(a)_2(b)_2}{(c)_2(1)_2}\sum_{k=2}^{\infty} a_k \\ &\leq \sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^n \frac{(a)_k(b)_k}{(c)_k(1)_k}(k+\alpha-1)a_k \\ &\leq \frac{\beta\delta(\alpha-\gamma)}{1-\beta\delta+\beta},\end{aligned}$$

which yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{\beta\delta(\alpha-\gamma)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(a)_2(b)_2}\Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^{-n} \quad \big(n \in \mathbb{N}_0\big).$$

By the help of the inequality just above, we get that

$$\begin{aligned}\big|f(z)\big| = \Big|z + \sum_{k=2}^{\infty} a_k z^k\Big| &\leq |z| + |z|^2\sum_{k=2}^{\infty} a_k \\ &\leq |z| + \frac{\beta\delta(\alpha-\gamma)}{\alpha(\frac{l+\alpha+\lambda}{l+\alpha})^n \frac{(a)_2(b)_2}{(c)_2(1)_2}(1-\beta\delta+\beta)}|z|^2\end{aligned}$$

and

$$\begin{aligned}\big|f(z)\big| = \Big|z - \sum_{k=2}^{\infty} a_k z^k\Big| &\geq |z| - |z|^2\sum_{k=2}^{\infty} a_k \\ &\geq |z| - \frac{\beta\delta(\alpha-\gamma)}{\alpha(\frac{l+\alpha+\lambda}{l+\alpha})^n \frac{(a)_2(b)_2}{(c)_2(1)_2}(1-\beta\delta+\beta)}|z|^2,\end{aligned}$$

where $a > 0$, $b > 0$, $c > 0$, $\lambda > 0$, $\alpha > 0$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $\alpha - \gamma \neq 0$, and $n \in \mathbb{N}_0$ and $z \in \mathbb{U}$. Thus, the desired prof completes.

Theorem 5. Let the function $f(z)$ be in the family $\mathcal{B}^{n,\lambda}_{l,\alpha}(\beta,\gamma,\delta)$. Then, the following inequalities:

$$\big|f'(z)\big| \leq 1 + \frac{2\beta\delta(\alpha-\gamma)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(a)_2(b)_2}\Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^{-n}|z| \tag{18}$$

and

$$\big|f'(z)\big| \geq 1 - \frac{2\beta\delta(\alpha-\gamma)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(a)_2(b)_2}\Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^{-n}|z| \tag{19}$$

are also satisfied, *where* $a > 0$, $b > 0$, $c > 0$, $\lambda > 0$, $\alpha > 0$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $\alpha - \gamma \neq 0$, and $n \in \mathbb{N}_0$ and $z \in \mathbb{U}$. The bounds in (18) and (19) are attained for the function $f(z)$ given by (17).

Proof. Let $f(z)$, defined by (1), be in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$. By using (10), we then obtain

$$\begin{aligned}\frac{1+\alpha}{2}\Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^n\frac{(a)_2(b)_2}{(c)_2(1)_2}\sum_{k=2}^{\infty}ka_k&\\ &\leq \sum_{k=2}^{\infty}\frac{k}{k}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^n\frac{(a)_k(b)_k}{(c)_k(1)_k}(k+\alpha-1)a_k\\ &=\sum_{k=2}^{\infty}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^n\frac{(a)_k(b)_k}{(c)_k(1)_k}(k+\alpha-1)a_k\\ &\leq\frac{\beta\delta(\alpha-\gamma)}{1-\beta\delta+\beta},\end{aligned}$$

which yields

$$\sum_{k=2}^{\infty}ka_k\leq\frac{2\beta\delta(\alpha-\gamma)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(a)_2(b)_2}\Big(\frac{l+\alpha+\lambda}{l+\alpha}\Big)^{-n}\quad\big(n\in\mathbb{N}_0\big).\tag{20}$$

By means of the inequality given by (20), the assertions in (19) and (20) can be easily obtained. The details are here omitted.

The last result in regard to the radii of Bazilević (type-univalent) function of order ω $(0\leq\omega<1)$ is contained in the following theorem.

Theorem 6. Let $f(z)$, defined by (1), be in the family $\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$. Then, $f(z)$ is Bazilevic (univalent) function of order ω $(0\leq\omega<1)$ in disk $|z|<r$, where

$$\begin{aligned}r&:=r(n,l,\lambda,n,\alpha,\beta,\gamma,\delta,\omega)\\ &=\inf_k\left(\frac{(1-\omega)(a)_k(b)_k(k+\alpha-1)(1-\beta\delta+\beta)}{\beta\delta(\alpha-\gamma)k(k-\omega-2)(c)_k(1)_k}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^n\right)^{\frac{1}{k-1}},\end{aligned}\tag{21}$$

where $a>0$, $b>0$, $c>0$, $\lambda>0$, $\alpha>0$, $0<\beta\leq1$, $0\leq\gamma\leq1$, $0\leq\delta\leq1$, $\alpha-\gamma\neq0$, and $n\in\mathbb{N}_0$. The result is sharp for the extremal function given by

$$f(z)=z+\frac{\beta\delta(\alpha-\gamma)(c)_k(1)_k}{(k+\alpha-1)(1-\beta\delta+\beta)(a)_k(b)_k}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^{-n}z^k\tag{22}$$

for all $k=3,4,\cdots$ and $z\in\mathbb{U}$.

Proof. Let $f(z)\in\mathcal{B}_{l,\alpha}^{n,\lambda}(\beta,\gamma,\delta)$ be in the form (**??**). Then, the function $f(z)$ is Bazilević (type-univalent) function of order ω $(0\leq\omega<1)$ in $|z|<r$, provided that

$$\begin{aligned}\left|\frac{\big[zf'(z)\big]'}{f'(z)}-1\right|&=\left|\frac{\sum_{k=2}^{\infty}k(k-1)a_kz^{k-1}}{1+\sum_{k=2}^{\infty}ka_kz^{k-1}}\right|\\ &\leq\frac{\sum_{k=2}^{\infty}k(k-1)a_k|z|^{k-1}}{1+\sum_{k=2}^{\infty}ka_k|z|^{k-1}}\\ &\leq1-\omega\quad\big(|z|<r;0\leq\omega<1\big),\end{aligned}\tag{23}$$

where r is given by (21).

As is known, the last inequality in (23) holds true if

$$\frac{k(k-\omega-2)}{1-\omega}|z|^{k-1}\leq\frac{(1-\beta\delta+\beta)(\alpha+k-1)(a)_k(b)_k}{\beta\delta(\alpha-\gamma)(c)_k(1)_k}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^n,$$

which yields the expression in (21), when it is solved for $|z|$. This completes the proof of Theorem 6.

It is easy to see that all theorems, which are Theorems 1-6, include several special *or* general results. For all of them, it is enough to choose the values of the parameters which are available in all theorems. As certain consequences of our results, we want to present only four.

Under the conditions of Theorem 1, an immediate consequence of Theorem 1 may be given by the following corollary, which is Corollary 1.

Corollary 1. If the function $f(z)$ given by (1) is in $\mathcal{B}^{n,\lambda}_{l,\alpha}(\beta,\gamma,\delta)$, then

$$a_k \le \frac{\beta\delta(\alpha-\gamma)(c)_k(1)_k}{(k+\alpha-1)(1-\beta\delta+\beta)(a)_k(b)_k}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^{-n}$$

for all $k = 2, 3, \cdots$.

Under the conditions of Theorem 1 and as a special application of the operator (1), by setting $n := 0$ in Theorem 1, the next consequence can be presented by the following corollary.

Corollary 2. If the function $f(z)$ given by (1) is in $\mathcal{B}^{0,\lambda}_{l,\alpha}(\beta,\gamma,\delta)$ if and only if

$$\sum_{k=2}^{\infty}\frac{(a)_k(b)_k}{(c)_k(1)_k}(k+\alpha-1)a_k \le \frac{\beta\delta(\alpha-\gamma)}{1-\beta\delta+\beta}.$$

The result is sharp for the extremal function given by

$$f(z) = z + \frac{\beta\delta(\alpha-\gamma)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(a)_2(b)_2}z^2 \quad \big(z \in \mathbb{U}\big).$$

Under the conditions of Theorem 4 and as a special application of the operator (5), by setting $n := 1$ in Theorem 4, the third consequence can be also given by the following corollary.

Corollary 3. If the function $f(z)$ given by (1) is in $\mathcal{B}^{1,\lambda}_{l,\alpha}(\beta,\gamma,\delta)$, then

$$\Big||f(z)| - |z|\Big| \le \frac{\beta\delta(\alpha-\gamma)(l+\alpha)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(l+\alpha+\lambda)(a)_2(b)_2}|z|^2 \quad \big(z \in \mathbb{U}.\big) \tag{24}$$

The above bounds are attained for the function $f(z)$ given by

$$f(z) = z + \frac{\beta\delta(\alpha-\gamma)(l+\alpha)(c)_2(1)_2}{(1+\alpha)(1-\beta\delta+\beta)(l+\alpha+\lambda)(a)_2(b)_2}z^2 \quad (z \in \mathbb{U}). \tag{25}$$

Under the conditions of Theorem 6 and as a special application of the operator (5), by letting $n := 2$ in Theorem 6, the last consequence of our results can be given by the following corollary.

Corollary 4. If the function $f(z)$ given by (1) is in $\mathcal{B}^{1,\lambda}_{l,\alpha}(\beta,\gamma,\delta)$, then $f(z)$ is Bazilevic (type-univalent) function of order ω $(0 \le \omega < 1)$ in disk $|z| < r^*$, where

$$\begin{aligned} r^* &:= r^*(l,\lambda,n,\alpha,\beta,\gamma,\delta,\omega) \\ &= \inf_k \left(\frac{(1-\omega)(a)_k(b)_k(k+\alpha-1)(1-\beta\delta+\beta)}{\beta\delta(\alpha-\gamma)k(k-\omega-2)(c)_k(1)_k}\Big(\frac{l+\alpha+\lambda(k-1)}{l+\alpha}\Big)^2\right)^{\frac{1}{k-1}}. \end{aligned}$$

The result is sharp for the extremal function given by

$$f(z) = z + \frac{\beta\delta(\alpha-\gamma)(c)_k(1)_k}{(k+\alpha-1)(1-\beta\delta+\beta)(a)_k(b)_k}\Big(\frac{l+\alpha}{l+\alpha+\lambda(k-1)}\Big)^2 z^k$$

for all $k = 3, 4, \cdots$ and $z \in \mathbb{U}$.

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Abdul Rahman S. Juma

Department of Mathematics, University of Anbar, Ramadi, IRAQ

E-mail address: dr_juma@hotmail.com *or* eps.abdulrahman.juma@uoanbar.edu.iq

Saba Nazar Al-khafaji

Department of Mathematics, AL-Mustansiriyah University, IRAQ

E-mail address: sabanf.mc11p@uokufa.edu.iq

Hüseyin Irmak

Department of Mathematics, Faculty of Sciences, Çankiri Karatekin University, Tr 18100, Çankiri, TURKEY

E-mail address: hisimya@yahoo.com *or* hirmak@karatekin.edu.tr