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COEFFICIENT INEQUALITIES FOR BOUNDED TURNING FUNCTIONS ASSOCIATED WITH CONIC DOMAINS

J. O. HAMZAT

ABSTRACT. Suppose that $RT_k[A,B]$, $k \geq 0$, $-1 \leq B < A \leq 1$ denotes the class of analytic functions defined in the open unit disk $E = \{z : |z| < 1\}$ satisfying the condition that

$$\Re\left(\frac{(B-1)f'(z)-(A-1)}{(B+1)f'(z)-(A+1)}\right) > k\left|\frac{(B-1)f'(z)-(A-1)}{(B+1)f'(z)-(A+1)} - 1\right|.$$

The author studied coefficient inequalities for bounded turning functions associated with conic domain in the open unit disk E.

1. Introduction

Denote by A the class of all functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_k z^k \tag{1}$$

normalizeed with $f(0) = f'(0) - 1 = 0$ that are analytic in the open unit disk $E = \{z : |z| < 1\}$ and by Ψ the class of univalent functions $f \in A$. Noor and Malik [10] had earlier studied the classes $k - ST[A,B]$ and $k - UCV[A,B]$ defined as follow:
A function $f(z) \in A$ is said to be in the class $k-ST[A,B]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)} - 1\right|.$$

A function $f(z) \in A$ is said to be in the class $k - UCV[A,B]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)} - 1\right|.$$

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It is not difficult to verify that

$$f(z) \in k - UCV[A,B] \Leftrightarrow zf'(z) \in k - ST[A,B].$$

Special cases of the above definitions can be found in [7, 8, 9, 11]. Recently, Vamshee et al [13] studied third Hankel determinant for bounded turning functions of order alpha while several other researchers (see[1, 2, 3, 5, 6]) examined some properties of bounded turning from different perspective and their results litters everywhere . However, the study of functions of bounded turning with respect to conic domain were not famous in literatures. Consequently, the present work aim at investigating certain properties of bounded turning functions associated with conic domain and for the purpose of this investigation, the following definition shall be necessary. A function $f(z) \in A$ is said to be in the class $RT_k[A,B]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)f'(z)-(A-1)}{(B+1)f'(z)-(A+1)}\right) > k\left|\frac{(B-1)f'(z)-(A-1)}{(B+1)f'(z)-(A+1)} - 1\right|. \tag{2}$$

Geometrically, if a function $f(z) \in RT_k[A,B]$ then $\omega = \frac{(B-1)f'(z)-(A-1)}{(B+1)f'(z)-(A+1)}$ takes all values from the domain Ω_k, $k \geq 0$ as $\Omega_k = \{\omega : \Re(\omega) > k|\omega - 1|\}$ or equivalently $\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}$. The domain Ω_k represents the right half plane for $k \geq 0$, a hyperbola for $0 < k < 1$, a parabola for $k = 1$ and an ellipse for $k > 1$, (see[12, 14] for more details).

Let B denote the class of function $p(z)$ having the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{3}$$

which are analytic in E. Following Hayami and Srivastava technique [4] the author wishes to present the main results.

2. Coefficient inequalities

The following Lemmas shall be necessary for the purpose of our present investigation.

Lemma 1. A function $p(z) \in B$ satisfies the following condition:

$$\Re p(z) > 0 \tag{4}$$

if and only if

$$p(z) \neq \frac{\psi - 1}{\psi + 1} \quad (z \in E; \psi \in C; |\psi| = 1)[4]. \tag{5}$$

Proof. Let

$$\omega = \frac{z-1}{z+1} \tag{6}$$

maps the unit circle ∂E onto the imaginary axis $\Re(\omega) = 0$ and for all ψ such that $|\psi| = 1$ $(\psi \in C)$, let

$$\omega = \frac{\psi - 1}{\psi + 1} \quad (z \in E; \psi \in C; |\psi| = 1).$$

Then,

$$\left|\psi\right| = \left|\frac{1+\omega}{1-\omega}\right| = 1, \tag{7}$$

which shows that

$$\Re(\omega) = \Re\left(\frac{\psi-1}{\psi+1}\right) = 0 \quad (\psi \in C; |\psi| = 1).$$

Since $p(0) = 1$ for $p(z) \in B$, then

$$p(z) \neq \frac{\psi-1}{\psi+1} \quad (z \in E; \psi \in C; |\psi| = 1) \tag{8}$$

and this obviously completes the proof.

Lemma 2. A function $f(z) \in A$ is in the class $RT_k[A,B]$ if and only if

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \tag{9}$$

where

$$A_n = \frac{n\Big[\psi\big[(1-2k)\big(B-(B-A)(B+1)\big)-1\big]+(1-2k)\big(B+(B-A)(B+1)\big)-1\Big]}{(1-2k)(B-A)\Big[\psi(1+A-B)+(1-A+B)\Big]} \tag{10}$$

Proof. Upon setting

$$p(z) = \frac{(1-k)\left(\frac{(B-1)f'(z)-(A-1)}{(B+1)f'(z)-(A+1)}\right)-k}{1-2k} \tag{11}$$

for $f(z) \in RT_k[A,B]$, one observes that

$$p(z) \in B \quad and \quad \Re p(z) > 0, \quad (z \in E). \tag{12}$$

Then by Lemma 1,

$$\frac{(1-k)\left(\frac{(B-1)f'(z)-(A-1)}{(B+1)f'(z)-(A+1)}\right)-k}{1-2k} \neq \frac{\psi-1}{\psi+1} \quad (z \in E; \psi \in C; |\psi| = 1). \tag{13}$$

It implies that

$$\Big[(1-k)\big((B-1)f'(z)-(A-1)\big)-k\big((B+1)f'(z)-(A+1)\big)\Big](\psi+1)$$

$$-(\psi-1)(B-A)(1-2k)\Big[(B+1)f'(z)-(A+1)\Big] \neq 0 \tag{14}$$

and

$$(1-2k)(B-A)\Big[\psi(1+A-B)+(1-A+B)\Big]$$

$$+\sum_{n=2}^{\infty} n\Big[\psi\big[(1-2k)\big(B-(B-A)(B+1)\big)-1\big]+(1-2k)\big(B-(B-A)(B+1)\big)-1\Big]a_n z^{n-1}$$

$$\neq 0. \tag{15}$$

Therefore,

$$1+\sum_{n=2}^{\infty}\frac{n\Big[\psi\big[(1-2k)\big(B-(B-A)(B+1)\big)-1\big]+(1-2k)\big(B+(B-A)(B+1)\big)-1\Big]}{(1-2k)(B-A)\Big[\psi(1+A-B)+(1-A+B)\Big]}\neq 0$$

and this compltes the proof of Lemma 2.

Corollary 1. A function $f(z)\in A$ is in the class $RT_k[1,-1]$ if and only if

$$1+\sum_{n=2}^{\infty}A_n z^{n-1}\neq 0 \tag{16}$$

where

$$A_n=\frac{n(\psi-1)}{3\psi-1}. \tag{17}$$

Theorem 1. If $f(z)\in A$ satisfies the condition that

$$\sum_{n=2}^{\infty}\left(\left|\sum_{m=1}^{n}\left[\sum_{j=1}^{m}(-1)^{m-j}j\Big[(1-2k)\big(B+(B-A)(B+1)\big)-1\Big]a_j\binom{\beta}{n-j}\right]\binom{\gamma}{n-m}\right|\right.$$
$$\left.+\left|\sum_{m=1}^{n}\left[\sum_{j-1}^{m}(-1)^{m-j}j\Big[(1-2k)\big(B+(B-A)(B+1)\big)-1\Big]a_j\binom{\beta}{m-j}\right]\binom{\gamma}{n-m}\right|\right)$$
$$\leq 2(B-A)(1-2k) \tag{18}$$

then $f(z)\in RT_k[A,B]$.

Proof. Here, we note that

$$(1-z)^{\beta}\neq 0 \ \ and \ \ (1+z)^{\gamma}\neq 0. \tag{19}$$

Hence, if the following inequality

$$\Big(1+\sum_{n=2}^{\infty}A_n z^{n-1}\Big)(1-z)^{\beta}(1+z)^{\gamma}\neq 0 \tag{20}$$

holds true, then

$$1+\sum_{n=2}^{\infty}A_n z^{n-1}\neq 0,$$

which is the expression earlier given in (9) of Lemma (2.2). This implies that

$$\Big(1+\sum_{n=2}^{\infty}A_n z^{n-1}\Big)\Big(\sum_{n=0}^{\infty}(-1)^n b_n z^n\Big)\Big(\sum_{n=0}^{\infty}c_n z^n\Big)\neq 0 \tag{21}$$

where

$$b_n=\binom{\beta}{n} \ \ and \ \ c_n=\binom{\gamma}{n}.$$

By Cauchy product of the first two factors in (21), we obtain

$$\Big(1+\sum_{n=2}^{\infty}B_n z^{n-1}\Big)\Big(\sum_{n=0}^{\infty}c_n z^n\Big)\neq 0, \tag{22}$$

where

$$B_n=\sum_{j=1}^{n}(-1)^{n-j}A_j b_{n-j}.$$

Likewise, following the same process, (22) can be expressed as

$$1+\sum_{n=2}^{\infty}\Big(\sum_{m=1}^{n}B_m c_{n-m}\Big)z^{n-1}\neq 0 \quad (z\in E). \tag{23}$$

Obviously, (23) may be expressed as

$$1+\sum_{n=2}^{\infty}\Big[\sum_{m=1}^{n}\Big(\sum_{j=1}^{m}(-1)^{m-j}A_j b_{m-j}\Big)c_{n-j}\Big]z^{n-1}\neq 0, \quad (z\in E).$$

Now, if $f(z)\in A$ satisfies the condition that

$$\sum_{n=2}^{\infty}\Big|\sum_{m=1}^{n}\Big(\sum_{j=1}^{m}(-1)^{m-j}A_j b_{m-j}\Big)c_{n-j}\Big|\leq 1,$$

that is if

$$L\sum_{n=2}^{\infty}\Big|\sum_{m=1}^{n}\Big(\sum_{j=1}^{m}(-1)^{m-j}j\Big[(1-2k)\big(B+(B-A)(B+1)\big)-1+\psi\big[(1-2k)\big(B-(B-A)(B+1)\big)-1\big]\Big]a_j b_{m-j}\Big)c_{n-m}\Big|$$

$$\leq 2L\sum_{n=2}^{\infty}\Bigg(\Big|\sum_{m=1}^{n}\Big[\sum_{j=1}^{m}(-1)^{m-j}j\Big[(1-2k)\big(B+(B-A)(B+1)\big)-1\Big]a_j b_{m-j}\Big]c_{n-m}\Big|$$

$$+\Big|\sum_{m=1}^{n}\Big[\sum_{j-1}^{m}(-1)^{m-j}j\Big[(1-2k)\big(B+(B-A)(B+1)\big)-1\Big]a_j b_{m-j}\Big]c_{n-m}\Big|\Bigg)$$

$$\leq 1,$$

where

$$L=\frac{1}{2(1-2k)(B-A)}$$

then $f(z)\in RT_k[A,B]$ and hence the proof.
Supposing $k=0$, the the following corollary is immediate.
Corollary 2.

$$\sum_{n=2}^{\infty}\Bigg(\Big|\sum_{m=1}^{n}\Big[\sum_{j=1}^{m}(-1)^{m-j}j\Big[\big(B+(B-A)(B+1)\big)-1\Big]a_j\binom{\beta}{n-j}\Big]\binom{\gamma}{n-m}\Big|$$

$$+\Big|\sum_{m=1}^{n}\Big[\sum_{j-1}^{m}(-1)^{m-j}j\Big[\big(B+(B-A)(B+1)\big)-1\Big]a_j\binom{\beta}{m-j}\Big]\binom{\gamma}{n-m}\Big|\Bigg)$$

$$\leq 2(B-A)$$

then $f(z)\in RT[A,B]$.
Corollary 3. If $f(z)\in A$ satisfies the following condition

$$\sum_{n=2}^{\infty}\Big|\sum_{m=1}^{n}\Big(\sum_{j=1}^{m}(-1)^{m-j}j(1-k)a_j\binom{\beta}{m-j}\Big)\binom{\gamma}{n-m}\Big|\leq 1-2k$$

then $f(z)\in RT_k[1,-1]$
Suppose that $\beta=\gamma=0$, then we have the following corollary.
Corollary 4. If $f(z)\in A$ satisfies the following condition

$$\sum_{n=2}^{\infty}n|a_n|\leq\Big|\frac{B-A}{B-1}\Big|$$

then $f(z) \in RT_k[A, B]$.
Also, if $\beta = \gamma = 0$, $\beta = -1$ and $A = 1$. The following corollary follows:
Corollary 5. If $f(z) \in A$ satisfies the following condition that

$$\sum_{n=2}^{\infty} n|a_n| \leq 1$$

then $f(z) \in RT_k[1, -1]$.
If $k = 0$, $\beta = -1$ and $A = 1$. Then, we have the following corollary.
Corollary 6. Let $f(z) \in A$ satisfies the following condition that

$$\sum_{n=2}^{\infty} \Big| \sum_{m=1}^{n} \Big(\sum_{j=1}^{m} (-1)^{m-j} j a_j \binom{\beta}{m-j} \Big) \binom{\gamma}{n-m} \Big| \leq 1$$

then $f(z) \in RT[1, -1]$.

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Hamzat J. O.
Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria
E-mail address: emmanther2012@gmail.com, emmanuelhamzat@yahoo.com