Electronic Journal of Mathematical Analysis and Applications
Vol. 7(2) July 2019, pp. 79-88.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

FUNCTIONAL QUADRATIC INTEGRAL EQUATIONS IN $L^1_{loc}(\mathbb{R}^+)$

LATIFA BENHAMOUCHE AND SMAÏL DJEBALI

Abstract. In this work, we study the existence of solutions for a class of functional quadratic integral equations of Volterra type in the space $L^1_{loc}(\mathbb{R}^+)$. The main result of this paper is obtained by applying the Schauder-Tychonov fixed point theorem combined with Ascoli-Arzéla Lemma and Dunford-Pettis compactness criterion. It generalizes some previous results obtained in [12] and [13]. Two illustrative examples are included.

1. Introduction

In this paper, we are concerned with the following equation:

$$x(t) = f\left(t, x(t), g(t, x(t)) \int_0^t k(t,s)h(s,x(s))ds\right),\ t > 0, \tag{1}$$

where f, g, h, and k are Carathéodory functions. We look for a solution in the space $L^1_{loc}(\mathbb{R}^+)$ consisting of all locally integrable real functions on $\mathbb{R}^+$.

In the last couple of years, many authors have considered the solvability of different types of integral equations on the Banach space $BC(\mathbb{R}^+)$ consisting of all real functions defined, bounded, and continuous on $\mathbb{R}^+$, while in some practical situations integral equations are better understood in L^1 settings (see, e.g., [10]).

Recently in [12], a technique using a family of measures of weak noncompactness has been applied to get an existence result for the equation

$$x(t) = f\left(t, \int_0^t v(t,s,x(s))ds\right),\ t > 0,$$

in the space $L^1_{loc}(\mathbb{R}^+)$.

In [13], fixed point theorems were developed in locally convex spaces with the Krein-Šmulian property. As an application, the authors have given an existence result in the space $L^1_{loc}(\mathbb{R}^+)$ for the following Volterra integral equation:

$$x(t) = f\left(t, x(t), \int_0^t k(t,s)v(s,x(s))ds\right),\ t > 0,$$

2010 *Mathematics Subject Classification.* 45D05, 47H08, 47H10.
Key words and phrases. locally convex spaces, fixed point, compactness criterion, quadratic Volterra integral equation.
Submitted October 02, 2017.

where f is lipschitzian in the two last arguments. However the quadratic case is not covered by this work.

In [4]. the authors studied the existence of integrable solutions of the nonlinear quadratic integral equation given by

$$x(t) = u(t, x(t)) + g(t, x(t)) \int_0^{\phi(t)} k(t, s) f(s, x(s)) ds$$

on the bounded interval $[0, 1]$. Other quadratic equations are discussed in [3, 8].

In the present work, we wish to generalize the results obtained in [4] and [13] to the more general equation (1) set on the half axis $\mathbb{R}^+$ and where all the nonlinear functions involved are of Carathéodory type. The strategy we are going to use is a combination of ideas from [2], [4], and [12]. The concept of measure of noncompactness, usually employed in the recent literature, is replaced here by direct arguments from functional analysis that are collected in Section 2. The main existence result is then presented in Section 3 while two examples of application are given in Section 4.

2. Preliminaries

This section is devoted to presenting some definitions and classical results which will be needed in the sequel. Let $m(D)$ denote the Lebesgue measure of a Lebesgue measurable subset $D \subset \mathbb{R}^+$. $L^1[0, T]$ refers to the Banach space of all real functions defined and Lebesgue integrable on the set $[0, T]$ for $T > 0$; it is equipped with the norm

$$\|x\|_T = \int_0^T |x(t)| dt. \tag{2}$$

Denote by $L^1_{loc}(\mathbb{R}^+)$ the space of all real measurable functions $x : \mathbb{R}^+ \to \mathbb{R}$ that are locally Lebesgue integrable on $\mathbb{R}^+$, i.e., $\|x\|_T < \infty$, for all $T > 0$.

The family of semi-norms (2) defines on $L^1_{loc}(\mathbb{R}^+)$ a metrizable topology and so $L^1_{loc}(\mathbb{R}^+)$ can be considered as a Fréchet space with the distance

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|x - y\|_j}{1 + \|x - y\|_j}$$

or equivalently

$$d_1(x, y) = \sup\{2^{-T} \|x - y\|_T : T > 0\}.$$

A subset $X \subset L^1_{loc}(\mathbb{R}^+)$ is said to be bounded if X is bounded for every semi-norm $\|.\|_T$, $T > 0$. Two topologies can be defined on the space $L^1_{loc}(\mathbb{R}^+)$: the Fréchet topology and the weak topology. Recall that the weak topology on a topological space E is the weakest topology (with the fewest open sets) such that all elements of E' (the topological dual of E) remain continuous; it comprises more compact sets. Krein-Šmulian Theorem (see [5]) states that, in a Banach space, the closed convex hull of a weakly compact set is weakly compact.

Let us recall the characterization of the convergence and the relative compactness in the topology of $L^1_{loc}(\mathbb{R}^+)$ as a Fréchet space (see [6]).

Proposition 2.1. (1) *A sequence $(x_n)_{n\in\mathbb{N}} \subset L^1_{loc}(\mathbb{R}^+)$ is convergent to $x \in L^1_{loc}(\mathbb{R}^+)$ if and only if* $\lim_{n\to\infty} \|x_n - x\|_T = 0$, *for $T \geq 0$.*

(2) *A set* $X \subset L^1_{loc}(\mathbb{R}^+)$ *is relatively compact in* $L^1_{loc}(\mathbb{R}^+)$ *if and only if* $\pi_T(X)$ *is relatively compact in the Banach space* $L^1[0,T]$ *for* $T \geq 0$, *where* $\pi_T : L^1_{loc}(\mathbb{R}^+) \to L^1[0,T]$ *refers to the restriction mapping.*

A similar result can be obtained in the weak topology.

Corollary 2.2. [12] *Assume that* M *is a nonempty subset of* $L^1_{loc}(\mathbb{R}^+)$. M *is relatively weakly compact in* $L^1_{loc}(\mathbb{R}^+)$ *if and only if* $\pi_T(M)$ *is relatively weakly compact in Banach space* $L^1[0,T]$, *for each* $T > 0$.

The compactness of $\pi_T(X)$ can be dealt with by the Dunford-Pettis compactness criterion that we state when the image space is reflexive, which is the case of our purpose:

Theorem 2.3. [5] *Let* (Ω, Σ, μ) *be a finite measure space and* X *a reflexive Banach space. A bounded subset* $K \subset L^1(\Omega, X)$ *is relatively weakly compact if and only if* K *is equi-integrable, that is*

$$\lim_{\mu(A)\to 0} \sup_{f\in K} \int_A |f| d\mu = 0.$$

Also we will make use of the classical Scorza Dragoni theorem:

Theorem 2.4. [11] *Let* $J \subset \mathbb{R}$ *be a measurable subset and* $f : J \times \mathbb{R} \to \mathbb{R}$ *a function satisfying Carathéodory conditions. Then for each* $\varepsilon > 0$, *there exists a closed subset* D_ε *of the set* J *such that* $m\left(J \backslash D_\varepsilon\right) < \varepsilon$ *and* $f_{|D_\varepsilon \times \mathbb{R}}$ *is continuous.*

We end these preliminaries with two fixed point results. Let X be a Hausdorff locally convex space with a topology generated by a family of semi-norms $\mathcal{P}$. We have (see, e.g., [9]).

Definition 2.1. *Let* $C \subset X$ *and* $p \in \mathcal{P}$. *A mapping* $A : C \to C$ *is said to be p-contraction if there exists* α_p, $0 \leq \alpha_p < 1$ *such that for all* $x, y \in C$, $p(Ax - Ay) \leq \alpha_p p(x - y)$.

Theorem 2.5. *Suppose* C *is a sequentially complete subset of* X *and the mapping* $A : C \to C$ *is a p-contraction, for every* $p \in \mathcal{P}$. *Then* A *has a unique fixed point* $\bar{x} \in C$ *and, for every* $x \in C$, *the iterate* $A^k x$ *converges to* $\bar{x}$, *as* $k \to \infty$.

The Schauder-Tychonoff fixed point theorem (see e.g., [1]) reads:

Lemma 2.6. *Let* E *be a Hausdorff locally convex linear topological space,* C *a convex subset of* E, *and* $F : C \to E$ *a continuous mapping such that*

$$F(C) \subset A \subset C$$

where A *is compact. Then* F *has at least one fixed point.*

3. Main Existence Result

In this section, we will study the existence of locally integrable solutions of Equation (1), assuming that the following conditions hold true:

(H_1): The function $f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function and there exist two measurable and essentially bounded functions a and $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq a(t)|x_1 - x_2| + b(t)|y_1 - y_2|,$$

for $t \in \mathbb{R}^+$ and $x_i, y_i \in \mathbb{R}$ with $i = 1, 2$. Also $c(t) = |f(t, 0, 0)| \in L^1_{loc}(\mathbb{R}^+)$.

(H_2)**:** The function $g:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and there exists a measurable essentially bounded function $q:\mathbb{R}^+\to\mathbb{R}^+$

$$|g(t,x_1)-g(t,x_2)|\leq q(t)|x_1-x_2|,$$

with $p(t)=|g(t,0)|\in L^1_{loc}(\mathbb{R}^+)$

(H_3)**:** The function $h:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and there exist a measurable essentially bounded function $\beta:\mathbb{R}^+\to\mathbb{R}^+$ and a nonnegative function $\alpha\in L^1_{loc}(\mathbb{R}^+)$ such that

$$|h(t,x)|\leq\alpha(t)+\beta(t)|x|,$$

for $(t,x)\in\mathbb{R}^+\times\mathbb{R}$.

(H_4)**:** The function $k:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a Carathéodory function such that the linear Volterra integral operator K generated by k that is

$$Kx(t)=\int_0^t k(t,s)x(s)ds,\quad t>0$$

transforms $L^1_{loc}(\mathbb{R}^+)$ into $L^\infty_{loc}(\mathbb{R}^+)$ continuously.

(H_5)**:** For all $T>0$

$$\bar a(T)+\bar b(T)\|K\|_T\bar q(T)\|\alpha\|_T+\bar b(T)\|K\|_T\|p\|_T\bar\beta(T)+ 2\sqrt{\left(\|c\|_T+\bar b(T)\|K\|_T\|p\|_T\|\alpha\|_T\right)\bar b(T)\|K\|_T\bar q(T)\bar\beta(T)}<1,$$

where $\bar a(T)=ess\sup\limits_{t\in[0,T]}a(t)$, $\bar b(T)=ess\sup\limits_{t\in[0,T]}b(t)$, $\bar q(T)=ess\sup\limits_{t\in[0,T]}q(t)$, $\bar\beta(T)=ess\sup\limits_{t\in[0,T]}\beta(t)$, and $\|K\|_T$ is the norm of the restriction of the operator K, namely $K_T:L^1[0,T]\to L^\infty[0,T]$, for $T>0$.

Assumption (H_5) allows us to define the following convex closed subset of $L^1_{loc}(\mathbb{R}^+)$:

$$M=\{x\in L^1_{loc}(\mathbb{R}^+):\|x\|_T\leq r(T),\ \forall\,T>0\},\tag{3}$$

where the function $r:\mathbb{R}^+\to\mathbb{R}^+$ is given by:

$$r(T)=\frac{\eta-\sqrt{\eta^2-4\left(\|c\|_T+\bar b(T)\|K\|\|p\|_T\|\alpha\|_T\right)\bar b(T)\|K\|_T\bar q(T)\bar\beta(T)}}{2\bar b(T)\|K\|_T\bar q(T)\bar\beta(T)}\tag{4}$$

with

$$\eta=1-\bar a(T)-\bar b(T)\|K\|_T\bar q(T)\|\alpha\|_T-\bar b(T)\|K\|_T\|p\|_T\bar\beta(T)>0.\tag{5}$$

We start with a technical result:

Lemma 3.1. *Under Assumptions (H_1)-(H_5), for each y in M, there is a unique fixed point $\psi_y\in M$ which verifies:*

$$\psi_y(t)=f\left(t,\psi_y(t),g(t,\psi_y(t))\int_0^t k(t,s)h(s,y(s))ds\right),\ \forall\,t>0.$$

Proof. For $y\in L^1_{loc}(\mathbb{R}^+)$, let $\mathcal{T}_y$ be the mapping defined by

$$\mathcal{T}_y(x)(t)=f\left(t,x(t),g(t,x(t))\int_0^t k(t,s)h(s,y(s))ds\right),\ t>0.$$

Claim 1. $\mathcal{T}_y : M \to M$**.** The invariance of M is in fact ensured by the choice of the function r in (4). Indeed, by (H_1) and (H_2) we have

$$\begin{aligned} |\mathcal{T}_y(x)(t)| &\leq |f(t,0,0)| + a(t)|x(t)| + b(t)|g(t,x(t))| |\int_0^t k(t,s)h(s,y(s))ds| \\ &\leq |f(t,0,0)| + a(t)|x(t)| \\ &\quad + b(t)\left(|g(t,0)| + q(t)|x(t)|\right) |\int_0^t k(t,s)h(s,y(s))ds|. \end{aligned}$$

Hence

$$\begin{aligned} \|\mathcal{T}_y(x)\|_T &\leq \|c\|_T + \bar{a}(T)\|x\|_T \\ &\quad + \bar{b}(T)\left(\|p\|_T + \bar{q}(T)\|x\|_T\right) \|K\|_T \|N_h(y)\|_T, \end{aligned}$$

where N_h denotes the Nemytskii operator associated to h:

$$N_h(y)(t) = h\left(t, y(t)\right). \tag{6}$$

Thus, by (H_3)

$$\begin{aligned} \|\mathcal{T}_y(x)\|_T &\leq \|c\|_T + \bar{a}(T)\|x\|_T \\ &\quad + \bar{b}(T)\|K\|_T \left(\|p\|_T + \bar{q}(T)\|x\|_T\right) \left(\|\alpha\|_T + \bar{\beta}(T)\|y\|_T\right) \end{aligned}$$

because by (H_3), we have that

$$\|N_h(y)\|_T \leq \|\alpha\|_T + \bar{\beta}(T)\|y\|_T.$$

Assuming x and y both in M, we get

$$\begin{aligned} \|\mathcal{T}_y(x)\|_T &\leq \|c\|_T + \bar{a}(T)r(T) \\ &\quad + \bar{b}(T)\|K\|_T \left(\|p\|_T + \bar{q}(T)r(T)\right) \left(\|\alpha\|_T + \bar{\beta}(T)r(T)\right) \\ &\leq r(T). \end{aligned}$$

The last inequality is a quadratic algebraic equation:

$$\|c\|_T + \bar{b}(T)\|K\|_T\|p\|_T\|\alpha\|_T - \eta r(T) + \bar{b}(T)\|K\|_T\bar{q}(T)\bar{\beta}(T)r^2(T) \leq 0,$$

where η is given by (5). Notice that the discriminant

$$\Delta = \eta^2 - 4\left(\|c\|_T + \bar{b}(T)\|K\|_T\|p\|_T\|\alpha\|_T\right)\bar{b}(T)\|K\|_T\bar{q}(T)\bar{\beta}(T)$$

is positive due to (H_5) and thus one positive root is precisely given by (4).

We deduce that $\mathcal{T}_y$ self-maps M, as claimed.

Claim 2. $\mathcal{T}_y$ **is a** T**-contraction for each** $T > 0$**.** Making use of (H_1) and (H_2) for each $x_1, x_2 \in M$ and for all $t \in [0,T]$ $(T > 0)$, we have the estimates:

$$\begin{aligned} |\mathcal{T}_y(x_1)(t) - \mathcal{T}_y(x_2)(t)| &\leq a(t)|x_1(t) - x_2(t)| \\ &\quad + b(t)|g(t,x_1(t)) - g(t,x_2(t))| \left|\int_0^t k(t,s)h(s,y(s))ds\right| \\ &\leq a(t)|x_1(t) - x_2(t)| \\ &\quad + b(t)q(t)|x_1(t) - x_2(t)| \left|\int_0^t k(t,s)h(s,y(s))ds\right|. \end{aligned}$$

By integration, we have

$$\|\mathcal{T}_y(x_1) - \mathcal{T}_y(x_2)\|_T \leq \bar{a}(T)\|x_1 - x_2\|_T + \bar{b}(T)\bar{q}(T)\|K\|_T\|N_h(y)\|_T\|x_1 - x_2\|_T.$$

This means that whenever $y \in M$ we have

$$\|\mathcal{T}_y(x_1) - \mathcal{T}_y(x_2)\|_T \leq \left(\bar{a}(T) + \bar{b}(T)\bar{q}(T)\|K\|_T\left(\|\alpha\|_T + \bar{\beta}(T)r(T)\right)\right)\|x_1 - x_2\|_T.$$

It is easy to check that

$$\nu_T = \bar{a}(T) + \bar{b}(T)\|K\|_T\bar{q}(T)\left(\|\alpha\|_T + \bar{\beta}(T)r(T)\right) < 1$$

which implies that $\mathcal{T}_y$ is a T-contraction for each $T > 0$. Applying Theorem 2.5 we conclude that $\mathcal{T}_y$ has a unique fixed point $\psi_y \in M$ such that:

$$\psi_y(t) = f\left(t, \psi_y(t), g(t, \psi_y(t)) \int_0^t k(t,s)h(s,y(s))ds\right),\ t > 0.$$

$\square$

Now we are able to state and prove our main existence result:

Theorem 3.2. *Under Assumptions (H_1)-(H_5), Equation (1) has at least one solution in the space $L^1_{loc}(\mathbb{R}^+)$.*

Proof. Using Lemma 3.1, we can define the operator $\mathcal{A}$ which associates to each $y \in M$, the unique function $\psi_y \in M$ satisfying

$$\psi_y(t) = f\left(t, \psi_y(t), g(t, \psi_y(t)) \int_0^t k(t,s)h(s,y(s))ds\right),\ t > 0.$$

Claim 1. The operator $\mathcal{A}$ is continuous on M with the respect of the topology of the Fréchet space $L^1_{loc}(\mathbb{R}^+)$. Let $T > 0$ and $x_1, x_2 \in M$. For $t \in [0,T]$, we have the estimates:

$$\begin{array}{rcl} |\mathcal{A}(x_1)(t) - \mathcal{A}(x_2)(t)| & \leq & a(t)|\mathcal{A}(x_1)(t) - \mathcal{A}(x_2)(t)| \\ & & +b(t)\left|g\left(t, \mathcal{A}(x_1)(t)\right) \int_0^t k(t,s)h(s,x_1(s))ds \right. \\ & & \left. -g\left(t, \mathcal{A}(x_2)(t)\right) \int_0^t k(t,s)h(s,x_2(s))ds\right|. \end{array}$$

Then

$$\begin{array}{rcl} |\mathcal{A}(x_1)(t) - \mathcal{A}(x_2)(t)| & \leq & a(t)|\mathcal{A}(x_1)(t) - \mathcal{A}(x_2)(t)| \\ & & +b(t)\left|\int_0^t k(t,s)h(s,x_1(s))ds\right| \\ & & \times \left|g\left(t, \mathcal{A}(x_1)(t)\right) - g\left(t, \mathcal{A}(x_2)(t)\right)\right| \\ & & +b(t)\left|g\left(t, \mathcal{A}(x_2)(t)\right)\right| \\ & & \times \left|\int_0^t k(t,s)\left(h(s,x_1(s) - h(s,x_2(s))\right)ds\right|. \end{array}$$

Hence

$$\begin{array}{rcl} |\mathcal{A}(x_1)(t) - \mathcal{A}(x_2)(t)| & \leq & a(t)|\mathcal{A}(x_1)(t) - \mathcal{A}(x_2)(t)| \\ & & +b(t)\left|\int_0^t k(t,s)h(s,x_1(s))ds\right| \times q(t)|\mathcal{A}(x_1)(t) - \mathcal{A}(x_2)(t)| \\ & & +b(t)\left(p(t) + q(t)|\mathcal{A}(x_2(t))|\right) \\ & & \times \left|\int_0^t k(t,s)\left(h(s,x_1(s) - h(s,x_2(s))\right)ds\right|. \end{array}$$

Integrating both sides from 0 to T, we get

$$\begin{array}{rcl} \|\mathcal{A}(x_1) - \mathcal{A}(x_2)\|_T & \leq & \bar{a}(T)\|\mathcal{A}(x_1) - \mathcal{A}(x_2)\|_T \\ & & +\bar{b}(T)\bar{q}(T)\|K\|_T\|N_h(x_1)\|_T\|\mathcal{A}(x_1) - \mathcal{A}(x_2)\|_T \\ & & +\bar{b}(T)\left(\|p\|_T + \bar{q}(T)\|\mathcal{A}(x_2)\|_T\right)\|K\|_T\|N_h(x_1) - N_h(x_2)\|_T \end{array}$$

i.e.,

$$\begin{array}{rcl} \|\mathcal{A}(x_1) - \mathcal{A}(x_2)\|_T & \leq & \bar{a}(T)\|\mathcal{A}(x_1) - \mathcal{A}(x_2)\|_T \\ & & +\bar{b}(T)\bar{q}(T)\|K\|_T\left(\|\alpha\|_T + \bar{\beta}(T)\|x_1\|_T\right)\|\mathcal{A}(x_1) - \mathcal{A}(x_2)\|_T \\ & & +\bar{b}(T)\left(\|p\|_T + \bar{q}(T)\|\mathcal{A}(x_2)\|_T\right)\|K\|_T\|N_h(x_1) - N_h(x_2)\|_T. \end{array}$$

Therefore

$$\begin{array}{rcl} \|\mathcal{A}(x_1) - \mathcal{A}(x_2)\|_T & \leq & \left(\bar{a}(T) + \bar{b}(T)\bar{q}(T)\|K\|_T(\|\alpha\|_T + \bar{\beta}(T)r(T))\right)\|\mathcal{A}(x_1) - \mathcal{A}(x_2)\|_T \\ & & +\bar{b}(T)\left(\|p\|_T + \bar{q}(T)r(T)\right)\|K\|_T\|N_h(x_1) - N_h(x_2)\|_T. \end{array}$$

Consequently

$$\|\mathcal{A}(x_1)-\mathcal{A}(x_2)\|_T \leq \frac{\bar{b}(T)\left(\|p\|_T+\bar{q}(T)r(T)\right)\|K\|_T}{1-\nu_T}\|N_h(x_1)-N_h(x_2)\|_T,$$

where

$$\nu_T = \bar{a}(T)+\bar{b}(T)\bar{q}(T)\|K\|_T(\|\alpha\|_T+\bar{\beta}(T)r(T)).$$

The continuity of $\mathcal{A}$ is then derived from the continuity of the Nemytskii operator N_h in (6) which is guaranteed by (H_3). Indeed, the superposition operator N_h transforms $L^1[0,T]$ into itself continuously (see [7]). By the way, we point out here that the sub-linear growth condition in (H_3) is optimal.

Claim 2. $\mathcal{A}(M)$ is relatively weakly compact in $L^1_{loc}(\mathbb{R}^+)$.
Let $T>0$, $\varepsilon>0$, and $D_\varepsilon \subset [0,T]$ be such that D_ε is nonempty and measurable with $m(D_\varepsilon)<\varepsilon$. For $x\in M$, we have

$$\mathcal{A}(x)(t) = f\left(t,\mathcal{A}(x)(t),g(t,\mathcal{A}(x)(t))\int_0^t k(t,s)h(s,x(s))ds\right),\ t>0.$$

With $|f(t,0,0)|=c(t)$ and $|g(t,0)|=p(t)$, we get

$$\begin{array}{rcl}|\mathcal{A}(x)(t)| & \leq & c(t)+a(t)|\mathcal{A}(x)(t)| \\ & & +b(t)|g(t,\mathcal{A}(x)(t))||\int_0^t k(t,s)h(s,x(s))ds| \\ & \leq & |c(t)|+a(t)|\mathcal{A}(x)(t)| \\ & & +b(t)\left(p(t)+q(t)|\mathcal{A}(x)(t)|\right)|\int_0^t k(t,s)h(s,x(s))ds|.\end{array}$$

Thus

$$\begin{array}{rcl}\int_{D_\varepsilon}|\mathcal{A}(x)(t)|dt & \leq & \int_{D_\varepsilon}c(t)dt+\bar{b}(T)\|K\|\|N_h(x)\|\int_{D_\varepsilon}p(t)dt \\ & & +\left(\bar{a}(T)+\bar{b}(T)\bar{q}(T)\|K\|(\|\alpha\|_T+\bar{\beta}(T)r(T))\right)\int_{D_\varepsilon}|\mathcal{A}(x)(t)|dt.\end{array}$$

Since by (H_5)

$$\nu_T=\bar{a}(T)+\bar{b}(T)\bar{q}(T)\|K\|(\|\alpha\|_T+\bar{\beta}(T)r(T))<1$$

and by (H_3)

$$\|N_h(x)\|_T \leq \|\alpha\|_T+\bar{\beta}(T)\|x\|_T,$$

we deduce that

$$\int_{D_\varepsilon}|\mathcal{A}(x)(t)|dt \leq \frac{1}{1-\nu_T}\left(\int_{D_\varepsilon}c(t)dt+\bar{b}(T)\|K\|_T(\|\alpha\|_T+\bar{\beta}(T)r(T)))\int_{D_\varepsilon}p(t)dt\right).$$

Taking the supremum over all elements $x\in M$ and all subsets $D_\varepsilon\subset[0,T]$ with $m(D_\varepsilon)<\varepsilon$, passing to the limit when $\varepsilon\to 0$, and using the fact that a set consisting of one element is weakly compact, we derive from the above estimate that $\mathcal{A}(M)$ equi-integrable hence, by Dunford-Pettis Theorem 2.3, $\mathcal{A}(M)$ is relatively weakly compact in $L^1[0,T]$ for all $T>0$. By Corollary 2.2, we conclude that $\mathcal{A}(M)$ is relatively weakly compact in $L^1_{loc}(R^+)$, as desired.

Claim 3. $\mathcal{A}(Y)$ is strongly compact, where $Y=\overline{conv\mathcal{A}(M)}$. First it is easy to show that $\mathcal{A}(Y)\subset Y$ and Y is weakly compact by Krein-Šmulian Theorem. In addition $\mathcal{A}(Y)$ is relatively weakly compact for it is a subset of Y. Denote $\mathcal{H}$ the operator KN_h:

$$\mathcal{H}y(t)=\int_0^t k(t,s)h(s,y(s))ds.$$

Applying the Scorza-Dragoni theorem, for $\varepsilon > 0$, we can find some subset $D_\varepsilon \subset [0,T]$ such that $m(D'_\varepsilon) < \varepsilon$, where $D'_\varepsilon = [0,T]\backslash D_\varepsilon$ such that the function k is continuous on D_ε. Let $y \in Y$ and $t_1, t_2 \in D_\varepsilon$. We have the estimates:

$$\begin{array}{rcl} |\mathcal{H}y(t_1) - \mathcal{H}y(t_2)| & = & |\int_0^{t_1} k(t_1,s)h(s,y(s))ds - \int_0^{t_2} k(t_2,s)h(s,y(s))ds| \\ & \leq & |\int_0^{t_1} (k(t_1,s) - k(t_2,s))h(s,y(s))ds| \\ & & +|\int_{t_1}^{t_2} k(t_2,s)h(s,y(s))ds| \\ & \leq & \omega^T(k,|t_1-t_2|)\int_0^{t_1} \alpha(s) + \beta(s)|y(s)|ds \\ & & +\bar{k}\int_{t_1}^{t_2} (\alpha(s) + \beta(s)|y(s)|)ds \\ & \leq & \omega^T(k,|t_1-t_2|)(\|\alpha\|_T + \bar{\beta}(T)r(T)) \\ & & +\bar{k}\int_{t_1}^{t_2} \alpha(s)ds + \bar{k}\bar{\beta}(T)\int_{t_1}^{t_2} |y(s)|ds, \end{array}$$

where $\omega^T(k,.)$ denotes the modulus of continuity of the function k on the set $D_\varepsilon \times [0,T]$ and $\bar{k} = \max\{k(t,s),\, (t,s) \in D_\varepsilon \times [0,T]\}$.

Taking into account the uniform continuity of k on $D_\varepsilon \times [0,T]$, the weak compactness of the set Y, the equi-integrability of Y, and the fact that a one element set is weakly compact we infer the equi-continuity of the set $\mathcal{H}(Y)$ in the space of continuous functions $C(D_\varepsilon)$.
Moreover for all $y \in Y$ and $t \in D_\varepsilon$, we have

$$\begin{array}{rcl} |\mathcal{H}(y)(t)| & = & |\int_0^t k(t,s)h(s,y(s))ds| \\ & \leq & \|K\|_T \|N_h(y)\|_T \\ & \leq & \|K\|_T(\|\alpha\|_T + \bar{\beta}(T)r(T)) \end{array}$$

which means that the set $\mathcal{H}(Y)$ is equi-bounded in the space $C(D_\varepsilon)$. By Ascoli-Arzela Lemma, we obtain that $\mathcal{H}(Y)$ is relatively compact in $C(D_\varepsilon)$, for each $\varepsilon > 0$.

Claim 4. $\mathcal{A}Y$ is relatively strongly compact in $L^1_{loc}(\mathbb{R}^+)$. For this, consider an arbitrary sequence $\{y_n\} \subset Y$ and let $\varepsilon > 0$ (we can assume that $\{y_n\}$ is a Cauchy sequence $C(D_\varepsilon)$). Using the same arguments as in the first step, we obtain for $n, m \in \mathbb{N}$

$$\begin{array}{rl} & \|\mathcal{A}(y_n) - \mathcal{A}(y_m)\|_T \\ \leq & \bar{a}(T)\|\mathcal{A}(y_n) - \mathcal{A}(y_m)\|_T \\ & +\bar{b}(T)\bar{q}(T)\|K\|_T\|N_h(y_n)\|_T\|\mathcal{A}(y_n) - \mathcal{A}(y_m)\|_T \\ & +\int_0^T b(t)\left(p(t) + q(t)|\mathcal{A}(y_m(t))|\right)|\mathcal{H}(y_n)(t) - \mathcal{H}(y_m)(t)|dt. \end{array}$$

So

$$\begin{array}{rl} & \|\mathcal{A}(y_n) - \mathcal{A}(y_m)\|_T \\ \leq & \frac{1}{1-\nu_T}\int_0^T b(t)\left(p(t) + q(t)|\mathcal{A}(y_m(t))|\right)|\mathcal{H}(y_n)(t) - \mathcal{H}(y_m)(t)|\,dt. \end{array}$$

Hence

$$\begin{array}{rl} & \|\mathcal{A}(y_n) - \mathcal{A}(y_m)\|_T \\ \leq & \frac{1}{1-\nu_T}\int_{D_\varepsilon} b(t)\left(p(t) + q(t)|\mathcal{A}(y_m(t))|\right)|\mathcal{H}(y_n)(t) - \mathcal{H}(y_m)(t)|\,dt \\ & +\frac{1}{1-\nu_T}\int_{[0,T]\backslash D_\varepsilon} b(t)\left(p(t) + q(t)|\mathcal{A}(y_m(t))|\right)|\mathcal{H}(y_n)(t) - \mathcal{H}(y_m)(t)|\,dt. \end{array}$$

As a consequence, we derive

$$\begin{array}{rl} & \|\mathcal{A}(y_n) - \mathcal{A}(y_m)\|_T \\ \leq & \frac{1}{1-\nu_T}\bar{b}(T)(\|p\|_T + \bar{q}(T)r(T))\sup_{t\in D_\varepsilon}|\mathcal{H}(y_n)(t) - \mathcal{H}(y_m)(t)| \\ & +\frac{2}{1-\nu_T}\|K\|_T(\|\alpha\|_T + \bar{\beta}(T)r(T))\int_{[0,T]\backslash D_\varepsilon} b(t)\left(p(t) + q(t)|\mathcal{A}(y_m(t))|\right). \end{array}$$

From the last estimate, it follows that $(\mathcal{A}(y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L^1[0,T]$ and thus it is convergent in this space, for each $T \geq 0$, i.e., $\mathcal{A}Y$ is relatively strongly compact in $L^1_{loc}(\mathbb{R}^+)$ for it is relatively strongly compact in $L^1[0,T]$ for each $T > 0$ (see Proposition 2.1, part 2). Appealing to Schauder-Tychonoff Theorem 2.6, we conclude the existence of at least a fixed point in Y, that is a solution to Equation (1), which completes the proof of Theorem 3.2. $\square$

4. Examples

Example 4.1. *Consider the following quadratic integral equation which generalizes the one discussed in* [4] *for it is set over the real half axis:*

$$x(t) = \varphi(t) + x(t)\int_0^t \frac{t}{t+s}\psi(s)x(s)ds, \quad t > 0, \tag{7}$$

where $\varphi \in L^1_{loc}(\mathbb{R}^+)$ *and* $\psi \in L^\infty_{loc}(\mathbb{R}^+)$.

In order to prove the existence of solutions to Equation (7), which is a particular case of (1), let us put for all $t,s \in \mathbb{R}^+$ *and* $x,y \in \mathbb{R}$:

$$f(t,x,y) = \varphi(t) + y,\ g(t,x) = x,\ h(t,x) = \psi(t)x,\ k(t,s) = \frac{t}{t+s}.$$

Let $a = 0,\ b = 1,\ q = 1,\ p = 0,\ \alpha = 0,\ \beta = \psi,$ *and* $\|K\|_T \leq 1$, *for all* $T > 0$. *Condition* (H_5) *then reduces to* $\|\varphi\|_T\|\psi\|_{L^\infty[0,T]} < \frac{1}{4}$ *which is satisfied for example when* $\varphi(t) = \frac{1}{4\pi(1+t^2)}$ *and* $\psi(s) = \frac{1}{1+s}$. *Since all assumptions in Theorem 3.2 are satisfied, we deduce that Equation (7) has at least one solution in the space* $L^1_{loc}(\mathbb{R}^+)$ *provided that* $\|\varphi\|_T\|\psi\|_{L^\infty[0,T]} < \frac{1}{4}$.

Example 4.2. *Consider the following equation which cannot be covered by the existence result given in* [13] *for it is quadratic and does not verify Hypothesis* $(\mathcal{H}_3)$ *in* [13]:

$$x(t) = \frac{1}{2\pi(1+t^2)} + \zeta(t) + x(t)\int_0^t \frac{ts}{t^2+1}\ln(1+x^2(s))ds, \quad t > 0, \tag{8}$$

where the function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ *is defined by the identity map on the set* $\mathbb{N}$ *of natural numbers and by zero elsewhere. Obviously* ζ *is neither continuous nor bounded. To get the existence of solutions to Equation (8), which is a particular case of (1), we set*

$$f(t,x,y) = \frac{1}{2\pi(1+t^2)} + \zeta(t) + y,\ g(t,x) = x,\ h(t,x) = \ln(1+x^2),\ k(t,s) = \frac{ts}{t^2+1}.$$

All assumptions of Theorem 3.2 are satisfied with $a(t) = 0,\ b(t) = 1,\ p(t) = 0,\ q(t) = 1,\ \alpha(t) = 0,\ \beta(t) = 1,$ *and* $\|K\|_T \leq \frac{T^2}{T^2+1}$ *for all* $T > 0$. *Consequently, Equation (8) has at least one solution in the space* $L^1_{loc}(\mathbb{R}^+)$. *Notice that this solution is neither continuous nor bounded on* $\mathbb{R}^+$.

References

[1] R.P. Agarwal, M. Meehan, and D.O'Regan, *Fixed Point Theory and Applications,* Cambridge Tracts in Mathematics, **141**. Cambridge University Press, Cambridge, 2001.

[2] J. Banaś, A. Chlebowicz, *On existence of integrable solutions of a functional integral equation under Carathódory conditions,* Nonlinear Anal. **70** (2009), no. 9, 3172–3179.

[3] J. Banaś, L. Olszzowy, *On solutions of a quadratic Urysohn integral equation on an unbounded interval,* Dynam. Systems Appl. **17** (2008), no. 2, 255–269.

[4] A. Bellour, D. O'Regan, and M.A. Taoudi, *On the existence of integrable solutions for a nonlinear quadratic integral equation,* J. Appl. Math. Comput. **46** (2014), no. 1-2, 67–77.
[5] N. Dunford, J.T. Schwartz, *Linear Operators. I. General Theory,* Pure and Applied Mathematics, Vol. **7** Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London 1958.
[6] A. Grothendieck, *Topological Vector Spaces,* Translated from the French by Orlando Chaljub. Notes on Mathematics and its Applications. Gordon and Breach Science Publishers, New York-London-Paris, 1973.
[7] M.A. Krasnosel'skii, *On the continuity of the operator $Fu(x) = f(x, u(x))$,* Dokl. Akad. Nauk. SSSR **77** (1951), 185–189.
[8] M.A. Metwali, *Solvability of functional quadratic integral equations with perturbation,* Opuscula Math. **33** (2013), no. 4, 725–739.
[9] G.L. Jr. Cain, M.Z. Nashed *Fixed points and stability for a sum of two operators in locally convex spaces,* Pacific J. Math. **39** (1971), 581–592.
[10] N. Salhi, M.A. Taoudi, *Existence of integrable solutions of an integral equation of Hammerstein type on unbounded interval,* Mediterr. J. Math. **9** (2012), no. 4, 729–739.
[11] G. Scorza Dragoni, *Una theorema sulla funzioni continue rispetto ad una i misurable rispetto at ultra variable,* Rend. Sem. Mat. Univ. Padova, **17** (1948), pp. 102-106
[12] L. Olszowy, *A family of measures of noncompactness in the space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and its application to some nonlinear Volterra integral equation,* Mediterr. J. Math. **11** (2014), no. 2, 687–701.
[13] F. Wang, H. Zhou, *Fixed point theorems and the Krein-Smulian property in locally convex spaces,* Fixed Point Theory Appl. (**2015**), 2015:154.

L. Benhamouche
Department of Mathematics, Faculty of Sciences,
Saad Dahlab University, route de Soumaa,
BP 270. Blida, 09000. Algeria
E-mail address: h_betrouni@yahoo.com

L. Benhamouche
Department of Mathematics, Faculty of Sciences,
Al Imam Mohammad Ibn Saud Islamic University (IMSIU),
PB 90950. Riyadh 11623, Saudi Arabia
E-mail address: ibdjebali@imamu.edu.sa

S. Djebali
Laboratoire "Théorie du Point Fixe et Applications"
ENS, BP 92 Kouba. Algiers, 16006. Algeria
E-mail address: djebali@ens-kouba.dz