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FIVE VALUE THEOREM APPLIED TO DERIVATIVES ON ANNULI

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Abstract. The purpose of this paper is to investigate the problems on the derivatives of two meromorphic functions partially sharing five or more values on annuli and obtain results that improve and generalize the previous results given by Cao and Yi [4] , H. Y. Xu and H. Wang [9].

1. Introduction

For a meromorphic function f in the complex plane $\mathcal{C}$, we assume that the reader is familiar with the standard notations such as $T(r, f), m(r, f), N(r, f)$, and so on (see [8, 18]).
In 1929, R. Nevanlinna (see[17]) established the well known uniqueness theorem : The five IM theorem.

Theorem A.([17]) *If f and g are two non-constant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, a_5 IM in C, then $f(z) \equiv g(z)$.*

After this many researchers shown interest in proving uniqueness theorems of meromorphic functions sharing sets and sharing values. From the several decades many mathematicians such as I. Lahari [14], A. Banerjee [2], S. S. Bhoosnurmath and R. S. Dyavanal [3], X. M. Li and H. X. Yi [15] studied the uniqueness theorems of (entire) meromorphic functions.

In 2009, Z. Q. Mao and H. F. Liu [16] gave different approach in establishing the uniqueness of meromorphic functions in the unit disc and in the same year T. B. Cao, H. X. Yi and J. H. Zhang [7] proved the uniqueness theorems of two meromorphic functions sharing five values in an angular domain. As we all know that the unit disc and the angular domain are called as the simply connected domains in the whole complex plane. But there exists many other sub-regions, such as the annuli, the m-punctured complex plane, *etc.*

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In Recent years, Nevanlinna theory of meromorphic functions on the annulus(doubly-connected region) became the hot topic of research(see [4, 5, 6, 9, 10, 11, 12, 13]).

From [1], we get that each doubly-connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \le r \le R \le +\infty$. For two cases: $r = 0, R = +\infty$, simultaneously, and $0 < r < R < +\infty$; the latter case, the homothety $z \to \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$ in two cases.

In 2005, Khrystiyanyn and Kondratyuk [10, 11] introduced the Nevanlinna theory for meromorphic functions on annuli.
In 2009 and 2011, Cao [4, 5, 6] established the uniqueness of meromorphic functions on annuli sharing some values and some sets and obtained an analog of Nevanlinna's five-value theorem.

Theorem B.(See [4], Corollary 3.4). *Let f_1 and f_2 are two transcendental or admissible meromorphic functions on the annulus $A = \{z : \frac{1}{R_0} < |z| < R_0\}$ where $1 < R_0 \le +\infty$. Let $a_j\,(j = 1, 2, ..., q)$ be q distinct complex numbers in $\overline{C}$ and $k_j\,(j = 1, 2, ..., q)$ be positive integers or ∞, such that:*
$k_1 \ge k_2 \ge ... \ge k_q$, and $\overline{E}_{k_j)}(a_j, f_1) = \overline{E}_{k_j)}(a_j, f_2)$, $(j = 1, 2, ..., q)$.
Then:
(i) if $q = 7$, then $f_1(z) \equiv f_2(z)$.
(ii) if $q = 6$ and $k_3 \ge 2$, then $f_1(z) \equiv f_2(z)$.
(iii) if $q = 5$, $k_3 \ge 3$ and $k_5 \ge 2$, then $f_1(z) \equiv f_2(z)$.
(iv) if $q = 5$ and $k_4 \ge 4$, then $f_1(z) \equiv f_2(z)$.
(v) if $q = 5$, $k_3 \ge 5$ and $k_4 \ge 3$, then $f_1(z) \equiv f_2(z)$.
(vi) if $q = 5$, $k_3 \ge 6$ and $k_4 \ge 2$, then $f_1(z) \equiv f_2(z)$.

From Theorem B, we can get the following theorem immediately.

Theorem C.(See [4], Theorem 3.2). *Let f_1 and f_2 are two transcendental or admissible meromorphic functions on the annulus $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let $a_j\,(j = 1, 2, 3, 4, 5)$ be five distinct complex numbers in $\overline{C}$. If $\overline{E}(a_j, f_1) = \overline{E}(a_j, f_2)$, $(j = 1, 2, 3, 4, 5)$, then $f_1(z) \equiv f_2(z)$.*

Remark 1.1. *In the set $E(a, f) = \{z \in A : f(z) - a = 0\}$*, each zero with multiplicity m is counted m times where as in $\overline{E}(a, f)$, we ignore the multiplicity. Also in $\overline{E}_{k)}(a, f)$ the set of zeros of $f - a$ with multiplicities no greater than k, each zero is counted only once.

Definition 1.1.([9]) For $B \subset A$ and $a \in \overline{C}$, $\overline{N}_0^B\left(r, \frac{1}{f-a}\right)$ denotes the reduced counting function of those zeros of $f - a$ on A , which belong to the set B.
In 2016, Hong-Yan and Hua Wang [9] investigated the following problem on two meromorphic functions partially sharing five or more values.

Theorem D.. *Let f and g be two transcendental or admissible meromorphic functions on the annulus $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let*

$a_1, a_2, ..., a_q$ $(q \geq 5)$ be q distinct complex numbers or ∞. Suppose that $k_1 \geq k_2 \geq ... \geq k_q$, m are positive integers or infinity; $1 \leq m \leq q$ and $\delta_j\,(\geq 0)\,(j = 1, 2, ..., q)$ are such that:
$(1 + \frac{1}{k_m})\sum_{j=m}^{q} \frac{1}{1+k_j} + \sum_{j=1}^{q} \delta_j + 3 < (q - m - 1)(1 + \frac{1}{k_m}) + m.$
Let $B_j = \overline{E}_{k_j)}(a_j, f)\backslash\overline{E}_{k_j)}(a_j, g)$ for $j = 1, 2, ..., q$. If $\overline{N}_0^{B_j}\left(r, \frac{1}{f-a_j}\right) \leq \delta_j T_0(r, f)$ and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{f-a_j}\right)}{\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{g-a_j}\right)} > \frac{k_m}{(1 + k_m)\sum_{j=m}^{q} \frac{k_j}{1+k_j} - 2(1 + k_m) + (m - 2 - \sum_{j=1}^{q} \delta_j)k_m}$$

then $f(z) \equiv g(z)$.

It is a natural question that on what conditions the derivatives of two meromorphic functions partially share five or more values on annuli. In this article we are giving a positive answer to the above question by generalizing and improving the previous results.

Theorem 1.1. *Let f and g be two transcendental or admissible meromorphic functions on the annulus $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, a_2, ..., a_q$ $(q \geq 5)$ be q distinct complex numbers or ∞. Suppose that $k_1 \geq k_2 \geq ... \geq k_q$, m are positive integers or infinity; $1 \leq m \leq q$ and $\delta_j\,(\geq 0)\,(j = 1, 2, ..., q)$ are such that:*

$$\begin{aligned}(1 + \frac{1}{k_m})\sum_{j=m}^{q} \frac{1}{1 + k_j} + \left(1 + \frac{1}{k_m}\right)\sum_{j=1}^{q} \frac{n}{1 + k_j} + (n + 1)\sum_{j=1}^{q} \delta_j + 2n + 3 \\ < (q + qn - m - 2n - 1)\left(1 + \frac{1}{k_m}\right) + m.\end{aligned} \tag{1.1}$$

Let $B_j = \overline{E}_{k_j)}(a_j, f^{(n)})\backslash\overline{E}_{k_j)}(a_j, g^{(n)})$ for $j = 1, 2, ..., q$. If

$$\overline{N}_0^{B_j}\left(r, \frac{1}{f^{(n)} - a_j}\right) \leq \delta_j (n + 1) T_0(r, f) \tag{1.2}$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right)}{\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{g^{(n)}-a_j}\right)}$$

$$> \frac{k_m(n + 1)}{(1 + k_m)\sum_{j=m}^{q} \frac{k_j}{1+k_j} + (qn - 2n - 2)(1 + k_m) + (m - n - 2 - (n + 1)\sum_{j=1}^{q} \delta_j)k_m - \sum_{j=1}^{q} \frac{n}{1+k_j}(1 + k_m)}, \tag{1.3}$$

then $f(z) \equiv g(z)$.

For n=0 we get Theorem D.
For n=1 we get the following result.

Corollary 1. Under the same conditions of Theorem 1.1 and
$(1+\frac{1}{k_m})\sum_{j=m}^{q} \frac{1}{1+k_j}+\left(1 + \frac{1}{k_m}\right)\sum_{j=1}^{q} \frac{1}{1+k_j}+2\sum_{j=1}^{q} \delta_j+5 < (2q-m-3)\left(1 + \frac{1}{k_m}\right)+ m.$

Let $B_j = \overline{E}_{k_j)}(a_j, f')\backslash\overline{E}_{k_j)}(a_j, g')$ for $j = 1, 2, ..., q$. If $\overline{N}_0^{B_j}\left(r, \frac{1}{f'-a_j}\right) \leq 2\delta_j T_0(r, f)$ and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{f'-a_j}\right)}{\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{g'-a_j}\right)}$$

$$> \frac{2k_m}{(1+k_m)\sum_{j=m}^{q} \frac{k_j}{1+k_j} + (q-4)(1+k_m) + (m-3-2\sum_{j=1}^{q}\delta_j)k_m - \sum_{j=1}^{q}\frac{1}{1+k_j}(1+k_m)},$$

then $f(z) \equiv g(z)$.

Corollary 2.. For $n = 0, m = 1, k_j = \infty$ for $j = 1, 2, \ldots, q$ and :

$$\liminf_{r\to R_0} \frac{\sum_{j=1}^{q} \overline{N}_0\left(r, \frac{1}{f-a_j}\right)}{\sum_{j=1}^{q} \overline{N}_0\left(r, \frac{1}{g-a_j}\right)} > \frac{1}{q-3}.$$

If $\overline{N}_0^{B_j)}\left(r, \frac{1}{f-a_j}\right) \leq \delta_j T_0(r, f)$ where $\delta_j(\geq 0)$ satisfy $0 \leq \sum_{j=1}^{q}\delta_j < k - 3 - \frac{1}{\gamma}$, then $f(z) \equiv g(z)$.
If $q = 5$ and $\overline{E}(a_j, f) = \overline{E}(a_j, g)$, then $\gamma = 1$ and $\delta_j = 0$ for $j = 1, 2, \ldots, 5$. We can obtain $f(z) \equiv g(z)$. Hence Corollary 2 is an improvement of theorem C.
For n=0, m=3 and under the conditions of Corollary 2, we get an improvement of Theorem B.

2. Lemmas

Lemma 2.1. [11]*(The lemma on the logarithmic derivative.) Let f be a nonconstant meromorphic function on the annulus $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 \leq +\infty$, and let $\lambda > 0$. Then:*

$$m_0\left(r, \frac{f'}{f}\right) = S_1(r, f),$$

where (i) in the case $R_0 = +\infty$, $S_1(r, *) = O(log(rT_0(r, *)))$ for $r \in (1, +\infty)$, except for the set $\triangle_r$, such that $\int_{\triangle_r} r^{\lambda-1}dr < +\infty$;
(ii) If $R_0 < +\infty$, then $S_1(r, *) = O(log(\frac{T_0(r,*)}{R_0-r}))$ for $r \in (1, R_0)$, except for the set $\triangle'_r$, such that $\int_{\triangle'_r} \frac{dr}{(R_0-r)^{\lambda-1}} < +\infty$.

Lemma 2.2. [11] *(The second fundamental theorem.) Let f be a nonconstant meromorphic function on the annulus $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, a_2, ..., a_q$ be q distinct complex numbers in the extended complex plane $\overline{C}$. Then:*

$$(q-2)T_0(r, f) < \sum_{j=1}^{q} \overline{N_0}\left(r, \frac{1}{f-a_j}\right) + S_1(r, f),$$

where $S_1(r, f)$ is stated as in Lemma 2.1.

Lemma 2.3. [4]. *Let f be a nonconstant meromorphic function on the annulus $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R < R_0 \leq +\infty$. Let a be an arbitrary complex number and k be a positive integer. Then:*

$(i)\overline{N}_0\left(R, \frac{1}{f-a}\right) \leq \frac{k}{k+1}\overline{N_0}^{k)}\left(R, \frac{1}{f-a}\right) + \frac{1}{k+1}N_0\left(R, \frac{1}{f-a}\right)$
$(ii)\overline{N}_0\left(R, \frac{1}{f-a}\right) \leq \frac{k}{k+1}\overline{N_0}^{k)}\left(R, \frac{1}{f-a}\right) + \frac{1}{k+1}T_0\left(R, f\right) + O(1).$

3. Proofs of the Theorem.

In this section we present the proof of the main result.

Proof of Theorem 1.1. Suppose that $f \neq g$. then by Lemma 2.2 and Lemma 2.3 for any integer $m\,(1 \leq m \leq q)$, we have

$$\begin{aligned}
(q-2)T_0(r, f^{(n)}) &\leq \sum_{j=1}^{q}\overline{N}_0\left(r, \frac{1}{f^{(n)}-a_j}\right) + S(r,f), \\
&= \left\{\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right) + \overline{N}_0^{(k_j+1}\left(r, \frac{1}{f^{(n)}-a_j}\right)\right\} + S(r,f), \\
&\leq \sum_{j=1}^{q}\left\{\overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right) + \frac{1}{k_j+1}N_0^{(k_j+1}\left(r, \frac{1}{f^{(n)}-a_j}\right)\right\} + S(r,f), \\
&\leq \sum_{j=1}^{q}\left\{\frac{k_j}{1+k_j}\overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right) + \frac{1}{k_j+1}N_0\left(r, \frac{1}{f^{(n)}-a_j}\right)\right\} + S(r,f), \\
&\leq \sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right) + \left(\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T_0(r, f^{(n)}) + S(r,f).
\end{aligned}$$

Since $T_0(r, f^{(n)}) \leq T_0(r,f) + n\overline{N}_0(r,f) + S(r,f)$, so we have

$$\begin{aligned}
(q-2)T_0(r, f^{(n)}) &\leq \sum_{j=1}^{m-1}\left(\frac{k_j}{1+k_j} - \frac{k_m}{1+k_m}\right)\overline{N}_0^{k_j)}\left(r, \frac{1}{f-a_j}\right) \\
&+ \left(\sum_{j=1}^{q}\frac{1}{k_j+1}\right)[T_0(r,f) + n\overline{N}_0(r,f)] \\
&+ \sum_{j=1}^{q}\frac{k_m}{1+k_m}\overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right) + S(r,f), \\
&\leq \left[\sum_{j=1}^{m-1}\frac{k_j}{1+k_j} - (m-1)\frac{k_m}{1+k_m} + \sum_{j=1}^{m-1}\frac{1}{1+k_j} + \sum_{j=m}^{q}\frac{1}{1+k_j}\right]T_0(r,f) \\
&+ \sum_{j=1}^{q}\frac{n}{1+k_j}T_0(r,f) + \sum_{j=1}^{q}\frac{k_m}{1+k_m}\overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right) + S(r,f).
\end{aligned}$$

$$\begin{aligned}
(q-2)(n+1)T_0(r,f) &\leq \left[(m-1) - (m-1)\frac{k_m}{1+k_m} + \sum_{j=m}^{q}\frac{1}{1+k_j} + \sum_{j=1}^{q}\frac{n}{1+k_j}\right]T_0(r,f) \\
&+ \sum_{j=1}^{q}\frac{k_m}{1+k_m}\overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right) + S(r,f).
\end{aligned}$$

That is,

$$\left[qn-2n-2+(m-1)\frac{k_m}{1+k_m}+\sum_{j=m}^{q}\frac{k_j}{1+k_j}-\sum_{j=1}^{q}\frac{n}{1+k_j}\right]T_0(r,f)$$
$$\leq \sum_{j=1}^{q}\frac{k_m}{1+k_m}\overline{N}_0^{k_j)}\left(r,\frac{1}{f^{(n)}-a_j}\right)+S(r,f).$$

$$\left[qn-2n-2+(m-1)\frac{k_m}{1+k_m}+\sum_{j=m}^{q}\frac{k_j}{1+k_j}-\sum_{j=1}^{q}\frac{n}{1+k_j}\right]T_0(r,g)$$
$$\leq \sum_{j=1}^{q}\frac{k_m}{1+k_m}\overline{N}_0^{k_j)}\left(r,\frac{1}{g^{(n)}-a_j}\right)+S(r,g).$$

Since $B_j=\overline{E}_{k_j)}(a_j,f^{(n)})\backslash\overline{E}_{k_j)}(a_j,g^{(n)})$, let $D_j=\overline{E}_{k_j)}(a_j,f^{(n)})\backslash B_j$ for $j=1,2,...,q$. Since $\overline{E}_{k_j)}(a_j,f^{(n)})=\overline{E}_{k_j)}(a_j,g^{(n)})$ $(j=1,2,...,q)$,

$$\begin{aligned}\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{f^{(n)}-a_j}\right) &= \sum_{j=1}^{q}\overline{N}_0^{B_j}\left(r,\frac{1}{f^{(n)}-a_j}\right)+\sum_{j=1}^{q}\overline{N}_0^{D_j}\left(r,\frac{1}{f^{(n)}-a_j}\right)\\ &+ N_0\left(r,\frac{1}{f^{(n)}-g^{(n)}}\right),\\ &= \sum_{j=1}^{q}\delta_jT_0(r,f^{(n)})+T_0(r,f^{(n)})+T_0(r,g^{(n)})+O(1),\\ &= \sum_{j=1}^{q}\delta_j(n+1)T_0(r,f)+T_0(r,f)+n\overline{N}_0(r,f)+T_0(r,g)\\ &+ n\overline{N}_0(r,g)+O(1),\\ &= (n+1)\left[1+\left(\sum_{j=1}^{q}\delta_j\right)\right]T_0(r,f)+(n+1)T_0(r,g)+O(1).\end{aligned}$$

and since f, g are transcendental or admissible, it follows from (1.2) and (1.3) that

$$\left(\sum_{j=m}^{q}\frac{k_j}{1+k_j}+qn-2n-2+(m-1)\frac{k_m}{1+k_m}-\sum_{j=1}^{q}\frac{n}{1+k_j}+o(1)\right)\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{f^{(n)}-a_j}\right)$$
$$\leq [1+\sum_{j=1}^{q}\delta_j](n+1)\frac{k_m}{1+k_m}\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{f^{(n)}-a_j}\right)$$
$$+(1+o(1))\sum_{j=1}^{q}(n+1)\frac{k_m}{1+k_m}\overline{N}_0^{k_j)}\left(r,\frac{1}{g^{(n)}-a_j}\right),$$

as $r \to R_0$. Since $1 \geq \frac{k_1}{k_1+1} \geq \frac{k_2}{k_2+1} \geq ... \geq \frac{k_q}{k_q+1} \geq \frac{1}{2}$, it follows that:

$$\left(\sum_{j=m}^{q} \frac{k_j}{1+k_j} + qn - 2n - 2 + (m-1)\frac{k_m}{1+k_m} - \sum_{j=1}^{q} \frac{n}{1+k_j} - (n+1)[1+\sum_{j=1}^{q} \delta_j]\frac{k_m}{1+k_m}\right)$$
$$\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right) = (n+1)\frac{k_m}{1+k_m}\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{g^{(n)}-a_j}\right),$$

which implies

$$\liminf_{r\to\infty} \qquad \frac{\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{f^{(n)}-a_j}\right)}{\sum_{j=1}^{q} \overline{N}_0^{k_j)}\left(r, \frac{1}{g^{(n)}-a_j}\right)}$$
$$\leq \quad \frac{(n+1)\frac{k_m}{1+k_m}}{\sum_{j=m}^{q} \frac{k_j}{1+k_j} + (qn-2n-2) + [m-n-2-(n+1)\sum_{j=1}^{q} \delta_j]\frac{k_m}{1+k_m} - \sum_{j=1}^{q} \frac{n}{1+k_j}}.$$

This is a contradiction to equation (1.1). Thus we have $f(z) \equiv g(z)$.
Open Question. Is it possible to replace the derivative by a linear differential polynomial of the form $L(f) = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_k f^{(K)}$.

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