Electronic Journal of Mathematical Analysis and Applications
Vol. 7(2) July 2019, pp. 132-137.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

ON THE SPACE OF VECTOR VALUED ENTIRE DIRICHLET FUNCTIONS OF TWO COMPLEX VARIABLES

ARCHNA SHARMA AND VIRENDER SINGH

Abstract. The space of vector valued entire Dirichlet series is considered in this paper and the space is endowed with a norm. The characterization of linear continuous transformations on the space and total sets is obtained.

1. Introduction

Let

$$f(s_1,s_2)=\sum_{m,n=1}^{\infty}a_{m,n}\exp(\lambda_m s_1+\mu_n s_2),\ (s_j=\sigma_j+it_j\ ,\ j=1,2) \tag{1}$$

be a Dirichlet series of two complex variables s_1, s_2, where $a_{m,n}{}'s$ belong to a commutative Banach algebra $(E, ||.||)$ and

$$0<\lambda_1<...<\lambda_m\to\infty \text{ as m}\to\infty\,;\ 0<\mu_1<...<\mu_n\to\infty \text{ as } n\to\infty. \tag{2}$$

Further let

$$\lim_{m+n\to\infty}\sup\frac{\log(m+n)}{\lambda_m+\mu_n}=D<+\infty \tag{3}$$

and

$$\lim_{m+n\to\infty}\sup\frac{\log(||a_{m,n}||)}{\lambda_m+\mu_n}=-\infty. \tag{4}$$

Then $f(s_1, s_2)$ represents an entire function (see [3]). Let X be the class of entire functions defined by vector valued Dirichlet series (1) which satisfy the condition

$$\sum_{m,n=1}^{\infty}||a_{m,n}||\exp(m\lambda_m+n\mu_n)<\infty \tag{5}$$

It is evident that X defines a linear space over C with usual definitions of addition and scalar multiplication. For $f\in X$ with representation as in (1), we indicate by $f\sim\{a_{m,n}\}$. For a scalar α, $\alpha f\sim\{\alpha\, a_{m,n}\}$ and for $f\sim\{a_{m,n}\}$ and $g\sim\{b_{m,n}\}\in X$, define

---

2010 *Mathematics Subject Classification.* 32A15, 30H05.
Key words and phrases. Vector Valued Dirichlet Series, Entire Functions, Linear functional.
Submitted July 9, 2018. Revised Oct. 18, 2018.

$f+g \sim \{a_{m,n}+b_{m,n}\}$ and $f*g = \{a_{m,n}\, b_{m,n}\} = \sum_{m,n=1}^{\infty} a_{m,n}\, b_{m,n}\, \exp\{s_1\lambda_m + s_2\mu_n\}$.

For each $f \in X$, we define the following function

$$||f|| = \sum_{m,n=1}^{\infty} ||a_{m,n}|| \exp(m\lambda_m + n\mu_n)\,. \tag{6}$$

We observe that $||f||$ is well defined and it is easy to see that $||f||$ defines a norm on X.

In an earlier paper [3], Srivastava and Sharma have obtained some growth properties of the entire functions de?ned above. B.L. Srivastava [4] de?ned a Banach algebra of a class of vector valued Dirichlet series. Daoud in her papers [1, 5] obtained the properties of the space of entire functions represented Dirichlet series. It is evident that these results are not directly valid if the coefficients am,nare elements of a Banach space or Banach algebra. In this paper, we construct the space of entire functions using the norm de?ned above and obtain the properties of linear functions on the space.

2. Main Results

We now prove

Theorem 1. *The space X is a Banach algebra.*

Proof. First we show that under the norm defined above, X is complete. For this we assume that $\{f_\alpha\}$ is a Cauchy sequence in X, where $(s_1, s_2) \in C^2$,

$$f_\alpha(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n}^{(\alpha)} \exp(\lambda_m s_1 + \mu_n s_2).$$

For a given $\varepsilon > 0$, there exists a constant $l \geq 1$ such that

$$||f_\alpha - f_\beta|| < \varepsilon \qquad \text{for}\, \alpha, \beta \geq l$$

i.e.

$$\sum_{m,n=1}^{\infty} ||a_{m,n}^{(\alpha)} - a_{m,n}^{(\beta)}|| \exp(m\lambda_m + n\mu_n) < \varepsilon \;, \quad \alpha, \beta \geq l. \tag{7}$$

This implies that $\{a_{m,n}^{(\alpha)}\}$ forms a Cauchy sequence in Banach space E for fixed values of $m\,\text{and}\,n$. Hence there exists a sequence $\{a_{m,n}\} \subseteq E$ such that

$$\lim_{\alpha\to\infty} a_{m,n}^{(\alpha)} = a_{m,n} \;, \quad m, n \geq l.$$

On letting $\beta \to \infty$ in (7), we get

$$\sum_{m,n=1}^{\infty} ||a_{m,n}^{(\alpha)} - a_{m,n}|| \exp(m\lambda_m + n\mu_n) < \varepsilon \;, \qquad \alpha \geq l.$$

Therefore

$$f_\alpha \to f \sim \{a_{m,n}\}\, \text{as}\, \alpha \to \infty.$$

Also, $f \in X$. To see this we have

$$\sum_{m,n=1}^{\infty} ||a_{m,n}|| \exp(m\lambda_m + n\mu_n) \;\leq\; \sum_{m,n=1}^{\infty} ||a^{(\alpha)}_{m,n} - a_{m,n}|| \exp(m\lambda_m + n\mu_n)$$

$$+ \sum_{m,n=1}^{\infty} ||a^{(\alpha)}_{m,n}|| \exp(m\lambda_m + n\mu_n) \;<\; \infty$$

in view of (7) and the fact that $f_\alpha \in X$. Hence X is complete.
Now for $f, g \in X$, we consider the product

$$f * g = \sum_{m,n=1}^{\infty} a_{m,n} b_{m,n} \exp(s_1\lambda_m + s_2\mu_n).$$

as defined earlier. Then

$$||f * g|| \;=\; \sum_{m,n=1}^{\infty} ||a_{m,n} b_{m,n}|| \exp(m\lambda_m + n\mu_n)$$

$$\leq \; \sum_{m,n=1}^{\infty} ||a_{m,n}|| \; ||b_{m,n}|| \exp(m\lambda_m + n\mu_n)$$

$$\leq \left(\sum_{m,n=1}^{\infty} ||a_{m,n}|| \exp(m\lambda_m + n\mu_n) \right) \left(\sum_{m,n=1}^{\infty} ||b_{m,n}|| \exp(m\lambda_m + n\mu_n) \right)$$

$$\leq \; ||f|| \; ||g||$$

Therefore X is a Banach algebra and hence the Theorem (2.1) is proved. □

We observe that linear functionals as defined by other workers on the spaces represented by entire Dirichlet series can not be defined on the space X in the usual sense. To overcome this difficulty we consider the linear transformations from the space X to the space E. The continuity of these transformations $\psi : X \to E$ is defined relative to the norms defined on X and E. We now characterize the linear transformations on the space X.

Next we prove

Theorem 2. *Every continuous linear transformation $\psi : X \to E$ is of the form*

$$\psi(f) = \sum_{m,n=1}^{\infty} a_{m,n} t_{m,n} \exp(m\lambda_m + n\mu_n) \tag{8}$$

where $f \sim \{a_{m,n}\}$ and $\{t_{m,n}\}$ is a bounded sequence in the space E.

Proof. First we assume that $\psi : X \to E$ is a continuous linear transformation. We define the sequence $\{f_{m,n}\} \subseteq X$ by

$$f_{m,n} = \exp\{(s_1 - m)\lambda_m + (s_2 - n)\mu_n\}$$

and let $f^{(N)} = \sum_{m,n=1}^{N} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$. Obviously $f^{(N)} \to f$ as $N \to \infty$ and $||f_{m,n}|| = 1$.
Let

$$\psi(f_{m,n}) = t_{m,n}.$$

Then by the continuity of ψ,

$$\psi(f) = \psi(\lim_{N\to\infty} f^{(N)}) = \psi\left(\lim_{N\to\infty}\sum_{m,n=1}^{N} a_{m,n}\exp(\lambda_m s_1 + \mu_n s_2)\right)$$

$$= \psi\left(\lim_{N\to\infty}\sum_{m,n=1}^{N} a_{m,n} f_{m,n}\exp(m\lambda_m + n\mu_n)\right)$$

$$= \lim_{N\to\infty}\sum_{m,n=1}^{N}\psi\left(a_{m,n}\ \exp(m\lambda_m + n\mu_n) f_{m,n}\right)$$

$$= \sum_{m,n=1}^{\infty} a_{m,n} t_{m,n}\exp(m\lambda_m + n\mu_n).$$

Now,

$$||t_{m,n}|| = ||\psi(f_{m,n})|| \le M||f_{m,n}|| = M$$

for some $M > 0$. Hence $\{t_{m,n}\}$ is a bounded sequence.

Conversely, let $\{t_{m,n}\}$ be a bounded sequence satisfying (8). Then the transformation ψ given by (8) is well defined and linear. Further we note that for any $f \in X$,

$$||\psi(f)|| \le \sum_{m,n=1}^{\infty} ||a_{m,n} t_{m,n}||\exp(m\lambda_m + n\mu_n)$$

$$\le \sum_{m,n=1}^{\infty} ||a_{m,n}||\ ||t_{m,n}||\exp(m\lambda_m + n\mu_n)$$

$$\le K||f||. \tag{9}$$

This proves the continuity of ψ and the theorem follows. □

Following the defnition of total set [6] we give

Definition 1. *A set $A \subseteq X$ is called total if $\psi(f) = 0,\ \forall\ f \in A \Rightarrow \psi = 0$ where $\psi : X \to E$ is a continuous linear transformation.*

We now prove.

Theorem 3. *Let* $f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n}\exp(\lambda_m s_1 + \mu_n s_2)$, where $a_{m,n} \neq 0,\ \forall m, n \geq 1$. Let $D \subset C^2$ be a region having at least one finite limit point. Define

$$f_{u,v}(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n}\exp\{(s_1 + u - m)\lambda_m + (s_2 + v - n)\mu_n\}$$

Then the set

$$A_f = \{f_{u,v} : u, v \in D\}$$

is a total set with respect to the family of continuous linear transformations $\psi : X \to E$.

Proof. We have

$$f_{u,v}(s_1,s_2)=\sum_{m,n=1}^{\infty}a_{m,n}\exp\{(s_1+u-m)\lambda_m+(s_2+v-n)\mu_n\}$$

$$=\sum_{m,n=1}^{\infty}[a_{m,n}\exp\{(u-m)\lambda_m+(v-n)\mu_n\}]\ \exp\{s_1\lambda_m+s_2\mu_n\}$$

and

$$\sum_{m,n=1}^{\infty}\exp(m\lambda_m+n\mu_n)||a_{m,n}\exp(u-m)\lambda_m+(v-n)\mu_n||$$

$$=\sum_{m,n=1}^{\infty}||a_{m,n}||\exp(\gamma_1\lambda_m+\gamma_2\mu_n),$$

where $\gamma_1=Reu$, $\gamma_2=Rev$. The series on the right hand side must converge for every $u,v\in D$, because $f(s_1,s_2)\in X$. Let ψ^* be a linear continuous transformation such that $\psi^*(A_f)=0$ i.e.

$$\psi^*(f_{u,v})=0\quad\forall\ u,v\in D.$$

Using the representation of ψ^* from the previous theorem, we have

$$\sum_{m,n=1}^{\infty}a_{m,n}t_{m,n}\exp(m\lambda_m+n\mu_n)\exp\{(u-m)\lambda_m+(v-n)\mu_n\}=0,\quad\forall\ u,v\in D$$

i.e.

$$\sum_{m,n=1}^{\infty}a_{m,n}t_{m,n}\exp(u\lambda_m+v\mu_n)=0,\qquad\forall\ u,v\in D. \tag{10}$$

Now we define $g\sim a_{m,n}\,t_{m,n}$, since $\{t_{m,n}\}$ is a bounded sequence and $f\sim a_{m,n}\in X$,
$g\sim a_{m,n}\,t_{m,n}\in X$. But by (10),

$$g(u,v)=0\,,\qquad\forall\ u,v\in D.$$

Since D has a finite limit point, therefore $g=0$, implies that

$$a_{m,n}\,t_{m,n}=0\quad\ \forall m,n\geq 1$$

and as $a_{m,n}\neq 0$ for every m,n, we get the result. Hence the Theorem is proved. □

Definition 2. *see* [2, p.307] *An element f of the Banach algebra X is called a topological divisor of zero if there exists a sequence f_n in X such that $||fn||=1$ and $f.f_n\to 0$ as $n\to\infty$.*

We now prove.

Theorem 4. *Every element in X is a topological divisor of zero in X.*

Proof. Let us consider the sequence $\{d_{m,n}\}$ where

$$d_{m,n} = \exp\{-(m\lambda_m + n\mu_n)\}.\exp(s_1\lambda_m + s_2\mu_n) \qquad m, n \geq 1.$$

Obviously $d_{m,n} \in X$ and $||d_{m,n}|| = 1$, $\forall\, m, n \geq 1$. Also by definition of the product in X, we have

$$f.\,d_{m,n} = d_{m,n}\,.f = a_{m,n}\exp(s_1\lambda_m + s_2\mu_n)$$

and

$$||f.\,d_{m,n}|| = ||d_{m,n}\,.f|| = ||a_{m,n}||\exp(m\lambda_m + n\mu_n) \to 0 \quad \text{as } m, n \to \infty.$$

Hence the Theorem is proved. □

3. Conclusion

In this paper some properties of the Banach algebra are studied for vector valued Dirichlet series. Its invertible and singular elements are characterized as well as its topological zero divisors. Also we cosider the linear transformation and the continuity of this transformation $\psi : X \to E$ is de?ned relative to the norms.

Acknowledgements.

The authors are very grateful to the referees for their valuable suggestions and opinions.

References

[1] S Daoud, On the class of entire functions defined by Dirichlet series of several complex variables, Journal of Mathematical Analysis and Applications, Vol. 162, 294-299, 1991.
[2] G.F.Simmons, Introduction to Topology and Modern Analysis, McGraw Hill, New York.
[3] G.S.Srivastava and Archna Sharma, Some growth properties of entire functions represented by vector valued Dirichlet series in two complex variables, Gen. Math. Notes, Vol. 2, No. 1, 134-142, 2011.
[4] B.L. Srivastava, On a Banach algebra of a class of vector-valued Dirichlet series, Ganita, Vol. 54, no.1, 3345-3352, 2003.
[5] S Daoud, On the space of integral dirichlet functions of several complex variables, portugaliae mathematica, Vol. 44, 333-340, 1987.
[6] E. Angus Taylor, Introduction to Functional analysis, John Wiley and Sons, New York 1958.

Archna Sharma
Department of Applied Mathematicas, School of Vocational Studies And Applied Sciences, Gautam Buddha University, Greater Noida, PIN-201312, U.P., India.
E-mail address: archnasharmaiitr@gmail.com

Virender Singh
Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh–202002, India
E-mail address: virenderamu2015@gmail.com