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ON TRIPLED FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS VIA GENERAL CONTRACTIVE CONDITIONS IN COMPLEX VALUED METRIC SPACES

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Abstract. We present some common tripled fixed point results for weakly compatible mappings satisfying a general contractive condition under rational expressions in complex valued metric spaces. Our results represent extend and unify some known results in the literature (see [7]). Finally, we give an example for this generalization.

1. Introduction

Fixed point theory is very important theory in various branches as determining the existence and uniqueness of solutions of many mathematical equations in mathematical science, engineering and applications in other fields. Banach's contraction principle plays an important role as the most widely used fixed point theorem in all analysis. In 2011, Azam et al. [1] introduced the concept of complex valued metric spaces which is more general than ordinary metric spaces. Recently, Both of Berinde and Borcut [2,4] introduced the concept of tripled fixed point for nonlinear contractive mappings in partially complete metric spaces.

2. Preliminaries

In this section, we recall some definitions and properties of complex valued metric space which will be needed in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2).$$

Consequently, one can deduce that $z_1 \precsim z_2$ if one of the following conditions satisfied:

(p1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
(p2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(j_2)$,
(p3) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
(p4) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

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In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (p_1), (p_2) and (p_3) is satisfied and we write $z_1 \prec z_2$ if only (p_3) is satisfied.

Remark 2.1: We can easily check the followings notes:
(1) $a, b \in \mathbb{R},\ a \leq b \Rightarrow az \precsim bz \quad \forall\, z \in \mathbb{C}$.
(2) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| \precsim |z_2|$.
(3) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Remark 2.2: Let (X, d) be a complex valued metric space. Then one can say
(1) $|d(x, y)|$ or $|d(u, v)| < |1 + d(x, y) + d(u, v)| \quad \forall\ x, y, u, v \in X$.
(2) $|d(x, y)| > 0, \quad$ if $x \neq y$.

Definition 2.1 [1] Let X be a nonempty set and $\mathbb{C}$ be the set of all complex numbers. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$, the following conditions are satisfied:
(CVM_1) $0 \precsim d(x, y)$ and $d(x, y) = 0$ iff $x = y$,
(CVM_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(CVM_3) $d(x, y) \precsim d(x, z) + d(z, y) \quad \forall\ x, y, z \in X$.
Then (X, d) is called a complex valued metric space.

Example 2.1 [8] Let $X = \mathbb{C}$ be a set of complex numbers. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x_1, x_2) = e^{ik}\, |x_1 - x_2| \quad \forall\ x_1, x_2 \in X,$$

where $k \in [0, \frac{\pi}{2}]$.Then (X, d) is called a complex valued metric space.

Definition 2.2 [6] Let $S : X \times X \longrightarrow CB(X)$ and $T : X \longrightarrow X$ be two given mappings. Then an element $(x, y) \in X \times X$ is called
(1) A coupled coincidence point of a pair (S, T), if $Tx \in S(x, y)$ and $Ty \in S(y, x)$,
(2) A common coupled fixed point of a pair (S, T), if

$$x = Tx \in S(x, y) \ \text{ and } \ y = Ty \in S(y, x).$$

Definition 2.3 [2,4] Let (X, d) be a complex valued metric space. Then the element $(x, y, z) \in X \times X \times X$ is called tripled fixed point of the mapping $S : X \times X \times X \longrightarrow X$ if $x = S(x, y, z), y = S(y, z, x)$ and $z = S(z, x, y)$.

Definition 2.4 [3] Let X be a complex valued metric space and (S, T) be a pair of self-mappings. The pair (S, T) is said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$. i.e., they commute at their coincidence points.

Definition 2.5 [1] Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d) and $x \in X$. Then

(i) x is called the limit of $\{x_n\}$ if for every $\varepsilon > 0$ there exist $x_0 \in \mathbb{N}$ such that $d(x_n, x) \prec \varepsilon$ for all $x > x_0$ and we can write $\lim_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exist $x_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec \varepsilon$ for all $x > x_0$, where $m \in \mathbb{N}$.

(iii) (X, d) is said to be a complete complex valued metric space if every Cauchy sequence is convergent in (X, d).

Lemma 2.1 [1] Let (X,d) be a complex valued metric space. Then a sequence $\{x_n\}$ in X converges to x if and only if $|d(x_n,x)|\to 0$ as $n\to\infty$.

Lemma 2.2 [1] Let (X,d) be a complex valued metric space. Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $|d(x_n,x_{n+m})|\to 0$ as $n\to\infty$, where $m\in\mathbb{N}$.

Lemma 2.3 [5] Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence such that $\lim\limits_{n\to\infty} x_n = x$. Then $\lim\limits_{n\to\infty} d(x_n,a) = d(x,a) \quad \forall\ a\in X$.

3. Main Results

Now, we state and prove the first main results:
Theorem 3.1 Let (X,d) be a complete complex valued metric space. Suppose $P,Q: X\times X\times X\to X$ and $S,T: X\to X$ be self-mappings satisfy:

$$d\big(P(x,y,z),Q(u,v,w)\big) \precsim \beta_1 \frac{[d(Tx,Su)+d(Ty,Sv)+d(Tz,Sw)]}{3} + \beta_2 \frac{[d(Tx,P(x,y,z)).d(Tx,Q(u,v,w)) + d(Su,Q(u,v,w))d(Su,P(x,y,z))]}{1+d(Tx,Q(u,v,w))+d(Su,P(x,y,z))}, \tag{1}$$

for all $x,y,z,u,v,w\in X$, β_1,β_2 are non-negative reals with $0\le\beta_1+\beta_2<1$, where the pairs (P,S) and (Q,T) are weakly compatible. Then P,Q,S and T have a unique common tripled fixed point in $X\times X\times X$.

Proof. Let x_0,y_0 and z_0 be arbitrary points in X. Then we can define three sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ in X such that,

$$\begin{cases} x_{2n+1}=Sx_{2n+1}=P(x_{2n},y_{2n},z_{2n}), & x_{2n+2}=Tx_{2n+2}=Q(x_{2n+1},y_{2n+1},z_{2n+1}) \\ y_{2n+1}=Sy_{2n+1}=P(y_{2n},z_{2n},x_{2n}), & y_{2n+2}=Ty_{2n+2}=Q(y_{2n+1},z_{2n+1},x_{2n+1}) \\ z_{2n+1}=Sz_{2n+1}=P(z_{2n},x_{2n},y_{2n}), & z_{2n+2}=Tz_{2n+2}=Q(z_{2n+1},x_{2n+1},y_{2n+1}) \end{cases}, \tag{2}$$

for all $n=0,1,2,\dots$.

Then, we have

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d\big(P(x_{2n},y_{2n},z_{2n}),Q(x_{2n+1},y_{2n+1},z_{2n+1})\big) \\ &\precsim \beta_1 \frac{[d(Tx_{2n},Sx_{2n+1})+d(Ty_{2n},Sy_{2n+1})+d(Tz_{2n},Sz_{2n+1})]}{3} \\ &\quad + \beta_2 \Bigg[\frac{d(Tx_{2n},P(x_{2n},y_{2n},z_{2n})).d(Tx_{2n},Q(x_{2n+1},y_{2n+1},z_{2n+1}))}{1+d(Tx_{2n},Q(x_{2n+1},y_{2n+1},z_{2n+1}))+d(Sx_{2n+1},P(x_{2n},y_{2n},z_{2n}))} \\ &\qquad + \frac{d(Sx_{2n+1},Q(x_{2n+1},y_{2n+1},z_{2n+1})).d(Sx_{2n+1},P(x_{2n},y_{2n},z_{2n}))}{1+d(Tx_{2n},Q(x_{2n+1},y_{2n+1},z_{2n+1}))+d(Sx_{2n+1},P(x_{2n},y_{2n},z_{2n}))}\Bigg] \\ &\precsim \beta_1 \frac{[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})]}{3} \\ &\quad + \beta_2 \left[\frac{d(x_{2n},x_{2n+1}).d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+2}).d(x_{2n+1},x_{2n+1})}{1+d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}\right] \end{aligned}$$

$$\precsim \beta_1 \frac{[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})]}{3} + \beta_2\, d(x_{2n},x_{2n+1})$$
$$= \big(\frac{\beta_1}{3}+\beta_2\big)\, d(x_{2n},x_{2n+1}) + \frac{\beta_1}{3}\, d(y_{2n},y_{2n+1}) + \frac{\beta_1}{3}\, d(z_{2n},z_{2n+1}).$$

This implies that

$$|d(x_{2n+1},x_{2n+2})| \le \big(\frac{\beta_1}{3}+\beta_2\big)\, |d(x_{2n},x_{2n+1})| + \frac{\beta_1}{3}\, |d(y_{2n},y_{2n+1})| + \frac{\beta_1}{3}\, |d(z_{2n},z_{2n+1})|. \tag{3}$$

Also, we find that

$$d(y_{2n+1},y_{2n+2}) = d\big(P(y_{2n},z_{2n},x_{2n}),Q(y_{2n+1},z_{2n+1},x_{2n+1})\big)$$
$$\precsim \beta_1 \frac{[d(Ty_{2n},Sy_{2n+1})+d(Tz_{2n},Sz_{2n+1})+d(Tx_{2n},Sx_{2n+1})]}{3}$$
$$+\beta_2 \Big[\frac{d(Ty_{2n},P(y_{2n},z_{2n},x_{2n})).d(Ty_{2n},Q(y_{2n+1},z_{2n+1},x_{2n+1}))}{1+d(Ty_{2n},Q(y_{2n+1},z_{2n+1},x_{2n+1}))+d(Sy_{2n+1},P(y_{2n},z_{2n},x_{2n}))}$$
$$+\frac{d(Sy_{2n+1},Q(y_{2n+1},z_{2n+1},x_{2n+1})).d(Sy_{2n+1},P(y_{2n},z_{2n},x_{2n}))}{1+d(Ty_{2n},Q(y_{2n+1},z_{2n+1},x_{2n+1}))+d(Sy_{2n+1},P(y_{2n},z_{2n},x_{2n}))}\Big]$$
$$\precsim \beta_1 \frac{[d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})+d(x_{2n},x_{2n+1})]}{3}$$
$$+\beta_2 \Big[\frac{d(y_{2n},y_{2n+1}).d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+2}).d(y_{2n+1},y_{2n+1})}{1+d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+1})}\Big]$$
$$\precsim \beta_1 \frac{[d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})+d(x_{2n},x_{2n+1})]}{3} + \beta_2\, d(y_{2n},y_{2n+1})$$
$$= \frac{\beta_1}{3}\, d(x_{2n},x_{2n+1}) + \big(\frac{\beta_1}{3}+\beta_2\big)\, d(y_{2n},y_{2n+1}) + \frac{\beta_1}{3}\, d(z_{2n},z_{2n+1}).$$

This implies that

$$|d(y_{2n+1},y_{2n+2})| \le \frac{\beta_1}{3}\, |d(x_{2n},x_{2n+1})| + \big(\frac{\beta_1}{3}+\beta_2\big)\, |d(y_{2n},y_{2n+1})| + \frac{\beta_1}{3}\, |d(z_{2n},z_{2n+1})|. \tag{4}$$

Similarly, we can show that

$$|d(z_{2n+1},z_{2n+2})| \le \frac{\beta_1}{3}\, |d(x_{2n},x_{2n+1})| + \frac{\beta_1}{3}\, |d(y_{2n},y_{2n+1})| + \big(\frac{\beta_1}{3}+\beta_2\big)\, |d(z_{2n},z_{2n+1})|. \tag{5}$$

By adding (3), (4) and (5), we have

$$|d(x_{2n+1},x_{2n+2})| + |d(y_{2n+1},y_{2n+2})| + |d(z_{2n+1},z_{2n+2})|$$
$$\le \big(\beta_1+\beta_2\big)\big[\,|d(x_{2n},x_{2n+1})| + |d(y_{2n},y_{2n+1})| + |d(z_{2n},z_{2n+1})|\,\big]$$
$$\le \rho\,\big[\,|d(x_{2n},x_{2n+1})| + |d(y_{2n},y_{2n+1})| + |d(z_{2n},z_{2n+1})|\,\big],$$

where $0 \le \rho = \beta_1+\beta_2 < 1$.

Consequently, we can write

$$|d(x_n,x_{n+1})| + |d(y_n,y_{n+1})| + |d(z_n,z_{n+1})|$$
$$\le \rho\,\big[\,|d(x_{n-1},x_n)| + |d(y_{n-1},y_n)| + |d(z_{n-1},z_n)|\,\big]$$

$$\leq ... \leq \rho^n \left[|d(x_0,x_1)| + |d(y_0,y_1)| + |d(z_0,z_1)| \right].$$

Now, for any $m > n$, we have

$$\begin{aligned}
&|d(x_n,x_m)| + |d(y_n,y_m)| + |d(z_n,z_m)| \\
&\quad \leq \left[|d(x_n,x_{n+1})| + |d(y_n,y_{n+1})| + |d(z_n,z_{n+1})| \right] \\
&\quad \leq \left[|d(x_{n+1},x_{n+2})| + |d(y_{n+1},y_{n+2})| + |d(z_{n+1},z_{n+2})| \right] \\
&\quad \leq ... \leq \left[|d(x_{m-1},x_m)| + |d(y_{m-1},y_m)| + |d(z_{m-1},z_m)| \right] \\
&\quad \leq \left[\rho^n + \rho^{n+1} + ... + \rho^{m-1}\right] \left[|d(x_0,x_1)| + |d(y_0,y_1)| + |d(z_0,z_1)| \right] \\
&\quad \leq \rho^n \left[1 + \rho + ... + \rho^{m-n-1}\right] \left[|d(x_0,x_1)| + |d(y_0,y_1)| + |d(z_0,z_1)| \right] \\
&\quad \leq \frac{\rho^n}{1-\rho} \left[|d(x_0,x_1)| + |d(y_0,y_1)| + |d(z_0,z_1)| \right] \longrightarrow 0, \ \text{ as } \ m \longrightarrow 0.
\end{aligned}$$

From above, we can deduce that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X. Since (X,d) is complete, then there exists $x,y,z \in X$ such that $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $z_n \longrightarrow z$ as $n \longrightarrow \infty$.

Now, we prove that $x = P(x,y,z), y = P(y,z,x)$ and $z = P(z,x,y)$. Let on contrary that $x \neq P(x,y,z), y \neq P(y,z,x)$ and $z \neq P(z,x,y)$, then

$$\begin{aligned}
&d(x,P(x,y,z)) \precsim d(x,x_{2n+2}) + d(x_{2n+2},P(x,y,z)) \\
&\precsim d(x,x_{2n+2}) + d(Q(x_{2n+1},y_{2n+1},z_{2n+1}),P(x,y,z)) \\
&\precsim d(x,x_{2n+2}) + \beta_1 \frac{[d(Tx,Sx_{2n+1}) + d(Ty,Sy_{2n+1}) + d(Tz,Sz_{2n+1})]}{3} \\
&\qquad + \beta_2 \left[\frac{d(Tx,P(x,y,z)).d(Tx,Q(x_{2n+1},y_{2n+2},z_{2n+1}))}{1 + d(Tx,Q(x_{2n+1},y_{2n+2},z_{2n+1})) + d(Sx_{2n+1},P(x,y,z))} \right. \\
&\qquad\qquad \left. + \frac{d(Sx_{2n+1},Q(x_{2n+1},y_{2n+2},z_{2n+1})).d(Sx_{2n+1},P(x,y,z))}{1 + d(Tx,Q(x_{2n+1},y_{2n+2},z_{2n+1})) + d(Sx_{2n+1},P(x,y,z))} \right] \\
&\precsim d(x,x_{2n+2}) + \beta_1 \frac{[d(x,x_{2n+1}) + d(y,y_{2n+1}) + d(z,z_{2n+1})]}{3} \\
&\qquad + \beta_2 \left[\frac{d(x,P(x,y,z)).d(x,Q(x_{2n+1},y_{2n+2},z_{2n+1}))}{1 + d(x,Q(x_{2n+1},y_{2n+2},z_{2n+1})) + d(x_{2n+1},P(x,y,z))} \right. \\
&\qquad\qquad \left. + \frac{d(x_{2n+1},Q(x_{2n+1},y_{2n+2},z_{2n+1})).d(x_{2n+1},P(x,y,z))}{1 + d(x,Q(x_{2n+1},y_{2n+2},z_{2n+1})) + d(x_{2n+1},P(x,y,z))} \right] \\
&\precsim d(x,x_{2n+2}) + \beta_1 \frac{[d(x,x_{2n+1}) + d(y,y_{2n+1}) + d(z,z_{2n+1})]}{3} \\
&\quad + \beta_2 \left[\frac{d(x,P(x,y,z)).d(x,Q(x_{2n+1},y_{2n+2},z_{2n+1}))}{1 + d(x,Q(x_{2n+1},y_{2n+2},z_{2n+1})) + d(x_{2n+1},P(x,y,z))} \right. \\
&\qquad\qquad \left. + \frac{d(x_{2n+1},Q(x_{2n+1},y_{2n+2},z_{2n+1})).d(x_{2n+1},P(x,y,z))}{1 + d(x,Q(x_{2n+1},y_{2n+2},z_{2n+1})) + d(x_{2n+1},P(x,y,z))} \right] \\
&\precsim d(x,x_{2n+2}) + \beta_1 \frac{[d(x,x_{2n+1}) + d(y,y_{2n+1}) + d(z,z_{2n+1})]}{3}
\end{aligned}$$

$$+\beta_2\left[\frac{d(x,P(x,y,z)).d(x,x_{2n+2})+d(x_{2n+1},x_{2n+2}).d(x_{2n+1},P(x,y,z))}{1+d(x,x_{2n+2})+d(x_{2n+1},P(x,y,z))}\right].$$

i.e., $d(x,P(x,y,z))\precsim 0$, then $|d(x,P(x,y,z))|\longrightarrow 0$ as $n\longrightarrow\infty$. Consequently, $x=P(x,y,z)$. Similarly, we can show that $y=P(y,z,x)$, $z=P(z,x,y)$. In addition to $x=Q(x,y,z)$, $y=P(y,z,x)$ and $z=Q(z,x,y)$. This implies that (x,y,z) is a common tripled fixed point of P and Q.

To show the uniqueness, Suppose that $(x^*,y^*,z^*)\neq(x,y,z)$ be another common tripled fixed point of P and Q in X. Then

$$\begin{aligned}
&d(x,x^*)=d(P(x,y,z),Q(x^*,y^*,z^*))\\
&\precsim\beta_1\frac{[d(Tx,Sx^*)+d(Ty,Sy^*)+d(Tz,Sz^*)]}{3}\\
&+\beta_2\left[\frac{d(Tx,P(x,y,z)).d(Tx,Q(x^*,y^*,z^*))+d(Sx^*,Q(x^*,y^*,z^*)).d(Sx^*,P(x,y,z))}{1+d(Tx,Q(x^*,y^*,z^*))+d(Sx^*,P(x,y,z))}\right]\\
&\precsim\beta_1\frac{[d(x,x^*)+d(y,y^*)+d(z,z^*)]}{3}+\beta_2\left[\frac{d(x,x).d(x,x^*)+d(x^*,x^*).d(x^*,x)}{1+d(x,x^*)+d(x^*,x)}\right].
\end{aligned}$$

This tends that

$$|d(x,x^*)|\leq\frac{\beta_1}{3}\Bigg[|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|\Bigg].\qquad(6)$$

By a similar way, we find

$$|d(y,y^*)|\leq\frac{\beta_1}{3}\Bigg[|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|\Bigg],\qquad(7)$$

also,

$$|d(z,z^*)|\leq\frac{\beta_1}{3}\Bigg[|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|\Bigg].\qquad(8)$$

By adding (6), (7) and (8), we get

$$|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|\leq\beta_1\Bigg[|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|\Bigg],$$

that is

$$(1-\beta_1)\Bigg[|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|\Bigg]\leq 0,$$

which is a contraction, where $1-\beta_1\neq 0$ since $\beta_1<1$, then

$$|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|=0.$$

i.e., $x=x^*, y=y^*, z=z^*$. Consequently, x,y,z are a unique common tripled fixed point of P,Q,S and T.

Example 3.1. Let (X,d) be a complete complex valued metric space, where $X=[0,1]$ and define $d:X\times X\longrightarrow X$ by $d(x,y)=i\,|x-y|$. Define four mappings as follow:

(i) $P,Q:X\times X\times X\to X$ by

$$P(x,y,z)=\frac{x}{9}+\frac{y}{18}+\frac{z}{27}\quad\text{and}\quad Q(u,v,w)=\frac{u}{18}+\frac{v}{36}+\frac{w}{54}.$$

(ii) $S, T : X \to X$ by $\quad Sx = \dfrac{x}{3} \quad$ and $\quad Tx = \dfrac{x^2}{4}$.

Let $x_n = \frac{1}{n}$, $y_n = \frac{1}{n^2}$ and $z_n = \frac{1}{n^3}$ be three sequences in X. Then

$$\lim_{n\to\infty} P(x_n, y_n, z_n) = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Q(x_n, y_n, z_n) = \lim_{n\to\infty} Tx_n = 0,$$
$$\lim_{n\to\infty} P(y_n, z_n, x_n) = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Q(y_n, z_n, x_n) = \lim_{n\to\infty} Ty_n = 0,$$

and $\displaystyle\lim_{n\to\infty} P(z_n, x_n, y_n) = \lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Q(z_n, x_n, y_n) = \lim_{n\to\infty} Tz_n = 0.$

Clearly, the pairs (P, S) and (Q, T) are weakly compatible.

Taking $\beta_1 = \frac{9}{2}$, and $\beta_2 = \frac{-3}{2}$, we can verify that inequality (1) is satisfied for all $x, y, z, u, v, w \in X$. Also, $(0, 0, 0)$ is the unique common tripled fixed point in $X \times X \times X$.

Taking $S = T = I$, where I is the identity mapping in Theorem 3.1, we get the following corollary:

Corollary 3.1 [[7],Theorem 2.1] Let (X, d) be a complete complex valued metric space and $P, Q : X \times X \times X \to X$, satisfy:

$$d\big(P(x,y,z), Q(u,v,w)\big) \precsim \beta_1 \frac{[d(x,u) + d(y,v) + d(z,w)]}{3}$$
$$+ \beta_2 \frac{[d(x, P(x,y,z)).d(x, Q(u,v,w)) + d(u, Q(u,v,w))d(u, P(x,y,z))]}{1 + d(x, Q(u,v,w)) + d(u, P(x,y,z))}, \quad (9)$$

for all $x, y, z, u, v, w \in X$, where β_1, β_2 are non-negative reals with $0 \leq \beta_1 + \beta_2 < 1$. Then P, Q, S and T have a unique common tripled fixed point in $X \times X \times X$.

The following theorem is a new version of Theorem 3.1 with various contractive condition.

Theorem 3.2 Let (X, d) be a complete complex valued metric space. Suppose $P, Q : X \times X \times X \to X$ and $S, T : X \to X$ be self-mappings satisfy:

$$d\big(P(x,y,z), Q(u,v,w)\big) \precsim \beta_1\, d(Tx, Su) + \beta_2\, d(Ty, Sv) + \beta_3\, d(Tz, Sw)$$
$$+ \beta_4 \frac{d(Su, P(x,y,z)).d(Tx, Q(u,v,w))}{1 + d(Tx, Su) + d(Ty, Sv) + d(Tz, Sw)}, \quad (10)$$

for all $x, y, z, u, v, w \in X$, $\beta_1, \beta_2, \beta_3$ and β_4 are non-negative reals with $0 \leq \beta_1 + \beta_2 + \beta_3 + \beta_4 < 1$, where the pairs (P, S) and (Q, T) are weakly compatible. Then P, Q, S and T have a unique common tripled fixed point as in $X \times X \times X$.

Proof. Let x_0, y_0 and z_0 be arbitrary points in X. Then we can define three sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X, as (2). Then, we get

$$d(x_{2n+1}, x_{2n+2}) = d\big(P(x_{2n}, y_{2n}, z_{2n}), Q(x_{2n+1}, y_{2n+1}, z_{2n+1})\big)$$
$$\precsim \beta_1\, d(Tx_{2n}, Sx_{2n+1}) + \beta_2\, d(Ty_{2n}, Sy_{2n+1}) + \beta_3\, d(Tz_{2n}, Sz_{2n+1})$$

$$+\beta_4 \frac{d(Sx_{2n+1}, P(x_{2n}, y_{2n}, z_{2n})).d(Tx_{2n}, Q(x_{2n+1}, y_{2n+1}, z_{2n+1}))}{1 + d(Tx_{2n}, Sx_{2n+1}) + d(Ty_{2n}, Sy_{2n+1}) + d(Tz_{2n}, Sz_{2n+1})}$$

$$\precsim \beta_1\, d(x_{2n}, x_{2n+1}) + \beta_2\, d(y_{2n}, y_{2n+1}) + \beta_3\, d(z_{2n}, z_{2n+1}) + \beta_4 \frac{d(x_{2n+1}, x_{2n+1}).d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}) + d(z_{2n}, z_{2n+1})}.$$

Consequently, one can say

$$|d(x_{2n+1}, x_{2n+2})| \precsim \beta_1\, |d(x_{2n}, x_{2n+1})| + \beta_2\, |d(y_{2n}, y_{2n+1})| + \beta_3\, |d(z_{2n}, z_{2n+1})|. \tag{11}$$

Similarly,

$$|d(y_{2n+1}, y_{2n+2})| \precsim \beta_1\, |d(y_{2n}, y_{2n+1})| + \beta_2\, |d(z_{2n}, z_{2n+1})| + \beta_3\, |d(x_{2n}, x_{2n+1})|, \tag{12}$$

and

$$|d(z_{2n+1}, z_{2n+2})| \precsim \beta_1\, |d(z_{2n}, z_{2n+1})| + \beta_2\, |d(x_{2n}, x_{2n+1})| + \beta_3\, |d(y_{2n}, y_{2n+1})|. \tag{13}$$

By adding (11), (12) and (13), we get

$$\begin{aligned} &|d(x_{2n+1}, x_{2n+2})| + |d(y_{2n+1}, y_{2n+2})| + |d(z_{2n+1}, z_{2n+2})| \\ &\quad \leq (\beta_1 + \beta_2 + \beta_3)\Big[|d(z_{2n}, z_{2n+1})| + |d(x_{2n}, x_{2n+1})| + |d(y_{2n}, y_{2n+1})| \Big] \\ &\quad \leq \rho \Big[|d(z_{2n}, z_{2n+1})| + |d(x_{2n}, x_{2n+1})| + |d(y_{2n}, y_{2n+1})| \Big], \end{aligned}$$

where $0 \leq \rho = \beta_1 + \beta_2 + \beta_3 < 1$.

Consequently, one can write

$$\begin{aligned} &|d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| + |d(z_n, z_{n+1})| \\ &\quad \leq \rho \Big[|d(x_{n-1}, x_n)| + |d(y_{n-1}, y_n)| + |d(z_{n-1}, z_n)| \Big], \\ &\quad \leq ... \leq \rho^n \Big[|d(x_0, x_1)| + |d(y_0, y_1)| + |d(z_0, z_1)| \Big], \end{aligned}$$

Now, for any $m > n$, we have

$$\begin{aligned} &|d(x_n, x_m)| + |d(y_n, y_m)| + |d(z_n, z_m)| \\ &\quad \leq \Big[|d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| + |d(z_n, z_{n+1})| \Big] \\ &\quad \leq \Big[|d(x_{n+1}, x_{n+2})| + |d(y_{n+1}, y_{n+2})| + |d(z_{n+1}, z_{n+2})| \Big] \\ &\quad \leq ... \leq \Big[|d(x_{m-1}, x_m)| + |d(y_{m-1}, y_m)| + |d(z_{m-1}, z_m)| \Big], \\ &\quad \leq \big[\rho^n + \rho^{n+1} + ... + \rho^{m-1}\big] \big[|d(x_0, x_1)| + |d(y_0, y_1)| + |d(z_0, z_1)| \big] \\ &\quad \leq \rho^n \big[1 + \rho + ... + \rho^{m-n-1}\big] \big[|d(x_0, x_1)| + |d(y_0, y_1)| + |d(z_0, z_1)| \big] \end{aligned}$$

$$\leq \frac{\rho^n}{1-\rho}\left[\,|d(x_0,x_1)|+|d(y_0,y_1)|+|d(z_0,z_1)|\,\right] \longrightarrow 0, \text{ as } m \longrightarrow 0.$$

From above, we show that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequence in X. Since (X,d) is complete, then there exists $x, y, z \in X$ such that $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $z_n \longrightarrow z$ as $n \longrightarrow \infty$.

Now, we show that $x = P(x,y,z), y = P(y,z,x)$ and $z = P(z,x,y)$. Let on contrary that $x \neq P(x,y,z), y \neq P(y,z,x)$ and $z \neq P(z,x,y)$, then

$$\begin{aligned}
&d(x,P(x,y,z)) \precsim d(x,x_{2n+2}) + d(x_{2n+2},P(x,y,z))\\
&\precsim d(x,x_{2n+2}) + d(Q(x_{2n+1},y_{2n+1},z_{2n+1}),P(x,y,z))\\
&\precsim d(x,x_{2n+2}) + \beta_1\, d(Tx,Sx_{2n+1}) + \beta_2\, d(Ty,Sy_{2n+1}) + \beta_3\, d(Tz,Sz_{2n+1})]\\
&\qquad\qquad + \beta_4\ \frac{d(Sx_{2n+1},P(x,y,z)).d(Tx,Q(x_{2n+1},y_{2n+2},z_{2n+1}))}{1+d(Tx,Sx_{2n+1})+d(Ty,Sy_{2n+1})+d(Tz,Sz_{2n+1})}\\
&\precsim d(x,x_{2n+2}) + \beta_1\, d(x,x_{2n+1}) + \beta_2\, d(y,y_{2n+1}) + \beta_3\, d(z,z_{2n+1})]\\
&\qquad\qquad + \beta_4\ \frac{d(x_{2n+1},P(x,y,z)).d(x,x_{2n+2})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(z,z_{2n+1})}.
\end{aligned}$$

Therefore, $d(x,P(x,y,z)) \precsim 0$, then $|d(x,P(x,y,z))| \longrightarrow 0$ as $n \longrightarrow \infty$. Consequently, $x = P(x,y,z)$. Similarly, we can show that $y = P(y,z,x)$, $z = P(z,x,y)$. In addition to $x = Q(x,y,z)$, $y = P(y,z,x)$ and $z = Q(z,x,y)$. This implies that (x,y,z) is a common tripled fixed point of P and Q.

To prove the uniqueness, Suppose that $(x^*,y^*,z^*) \neq (x,y,z)$ be another common tripled fixed point of P and Q in X. Then

$$\begin{aligned}
&d(x,x^*) = d(P(x,y,z),Q(x^*,y^*,z^*))\\
&\precsim \beta_1\, d(Tx,Sx^*) + \beta_2\, d(Ty,Sy^*) + \beta_3\, d(Tz,Sz^*)\\
&\qquad\qquad + \beta_4\ \frac{d(Sx^*,P(x,y,z)).d(Tx,Q(x^*,y^*,z^*))}{1+d(Tx,Sx^*)+d(Ty,Sy^*)+d(Tz,Sz^*)}\\
&\precsim \beta_1\, d(x,x^*) + \beta_2\, d(y,y^*) + \beta_3\, d(z,z^*)\\
&\qquad\qquad + \beta_4\ \frac{d(x^*,x).d(x,x^*)}{1+d(x,x^*)+d(y,y^*)+d(z,z^*)}.
\end{aligned}$$

Consequently, one can write

$$|d(x,x^*)| \leq \beta_1\ |d(x,x^*)| + \beta_2\ |d(y,y^*)| + \beta_3\ |d(z,z^*)| + \beta_4\ |d(x,x^*)|,$$

that is,

$$|d(x,x^*)| \leq (\beta_1+\beta_4)\ |d(x,x^*)| + \beta_2\ |d(y,y^*)| + \beta_3\ |d(z,z^*)|. \tag{14}$$

By a similar way, we can say that

$$|d(y,y^*)| \leq (\beta_1+\beta_4)\ |d(y,y^*)| + \beta_2\ |d(z,z^*)| + \beta_3\ |d(x,x^*)|, \tag{15}$$

and

$$|d(z,z^*)| \leq (\beta_1+\beta_4)\ |d(z,z^*)| + \beta_2\ |d(x,x^*)| + \beta_3\ |d(y,y^*)|. \tag{16}$$

By adding (14), (15) and (16), we get

$$|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)| \leq (\beta_1+\beta_2+\beta_3+\beta_4)\Big[\,|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|\,\Big],$$

that is

$$\big[1-(\beta_1+\beta_2+\beta_3+\beta_4)\big]\Big[\,|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|\,\Big] \leq 0.$$

This tends that

$$|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)| = 0,$$

where $\big[1-(\beta_1+\beta_2+\beta_3+\beta_4)\big] \neq 0$, since $0 \leq \beta_1+\beta_2+\beta_3+\beta_4 < 1$.
i.e., $x = x^*$, $y = y^*$, $z = z^*$. Consequently, x, y, z are a unique common tripled fixed point of P, Q, S and T.

Taking $P = Q = F$ and $S = T = I$ where I is the identity mapping in Theorem 3.1, we get the following corollary:

Corollary 3.2 [[7],Theorem 2.2] Let (X, d) be a complete complex valued metric space and $F : X \times X \times X \to X$, satisfy:

$$d\big(F(x,y,z),F(u,v,w)\big) \precsim \beta_1\, d(x,u)+\beta_2\, d(y,v)+\beta_3\, d(z,w) + \beta_4\, \frac{d(F(x,y,z),u).d(F(u,v,w),x)}{1+d(x,u)+d(y,v)+d(z,w)}, \tag{17}$$

for all $x, y, z, u, v, w \in X$, where $\beta_1, \beta_2, \beta_3$ and β_4 are non-negative reals with $0 \leq \beta_1+\beta_2+\beta_3+\beta_4 < 1$. Then F has a unique tripled fixed point.

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