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BILATERAL MOCK THETA FUNCTIONS AND FURTHER PROPERTIES

MOHAMMAD AHMAD*, SIRAZUL HAQ AND ABDUL HAKIM KHAN

ABSTRACT. Bilateral mock theta functions were obtained using bilateral basic hypergeometric series. It has been shown that they are related to the basic hypergeometric series ${}_9\phi_8$ and satisfy the characteristic property of the mock theta functions defined by Ramanujan. They have also been expressed in terms of Lerch's functions $f(x, \xi; q, p)$.

1. INTRODUCTION

The mock theta functions were first introduced by Ramanujan [3] in his last letter to G H Hardy in January 1920 . He provided a list of seventeen mock theta functions and labeled them as of third, fifth and seventh order without mentioning the reason for his labeling. Watson [17] added to this set three more third order mock theta functions.
His general definition of a mock theta functions is a functions $f(q)$ defined by a q series convergent when $|q| < 1$ which satisfies the following two conditions.

(a) For every root ξ of unity, there exists a theta function $\theta_\xi(q)$ such that the difference between $f(q)$ and $\theta_\xi(q)$ is bounded as $q \rightarrow \xi$ radially.

(b) there is no single theta function which works for all ξ i.e for every theta function $\theta_\xi(q)$ there is some root of unity ξ for which $f(q)$ minus the theta function $\theta_\xi(q)$ is unbounded as $q \rightarrow \xi$ radially.
(When Ramanujan refers to theta functions, he means sum, products and quotients of series of the form $\sum_{n\in\ z} \xi^n q^{an^2+bn}$ with $a, b \in\ Q$and $\xi = -1, 1$.In bilateral form summation is taken from $-\infty$ to ∞).
Andrews and Hickerson [14] announced the existence of eleven more identities given in the 'lost' note book of Ramanujan involving seven new functions which they labeled as mock theta functions of order six. Y.S. Choi [1] has discovered four functions which he called the mock theta function of order ten. B.Gordon and R.J.McItosh [27] have announced the existence of eight mock theta functions of order eight and R.J.McIntosh [5] has announced the existence of three mock theta

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functions of order two.

Hikami [12, 13] has introduced one mock theta function of order two, one of order four and two of order eight. Very recently Andrews [15] while studying q-orthogonal polynomials found four new mock theta functions and Bringmann et al [11] have also found two more new mock theta functions but they did not mention about the order of their mock theta functions.

Watson and others have only proved the first assertion $1(a)$ and no one has proved the second assertion $1(b)$, Watson attempted to prove $1(b)$ too for the third order mock theta functions but could not do it in all its generality. Watson [17, 18], Dragonett [10] and Andrew and Hickerson [14] have shown that all the mock theta functions defined by Ramanujan, at least satisfy the boundedness condition $1(a)$.

Watson [18] has defined four bilateral series, which he called the 'Complete'or 'Bilateral' forms for four of the ten mock theta functions of order five and expressed them in terms of the transcendental functions $f(x, \xi; q, p)$ studied by M. Lerch [7].S.D. Prasad [2] in 1970 has defined the 'Complete'or 'Bilateral' forms of the five generalized third order mock theta functions.The 'Complete' sixth order mock theta functions were studied by A. Gupta [27] Bhaskar Srivastava [23, 24, 25, 26] have studied bilateral mock theta functions of order five, eight, two and new mock theta functions by Andrew [15] and Bringmann et al [11].

N.J. Fine [6] has reduced the third order mock theta function as a limiting case of ${}_2\phi_1$ and A Gupta [28] has reduced the mock theta functions of order five and seven as the limiting cases of ${}_3\phi_2$ and ${}_4\phi_3$ respectively. Shukla and Ahmad [19, 20, 21, 22], M Ahmad [8] and M Ahmad and Shahab Faruqi [9] have obtained bilateral mock theta functions of order "seven","nine ","eleven","thirteen" and fifteen and reduced them as the limiting cases of a basic hypergoeometric series ${}_4\phi_3,{}_5\phi_4,{}_6\phi_5,{}_7\phi_6$ and ${}_8\phi_7$ on a single base and proved that they satisfy characteristic property $1(a)$ of the mock theta functions defined by Ramanujan.

In section 2 we list few important definitions. In section 3 we define following eight bilateral mock theta functions, namely

$$f_{0,c_8}(q) = \sum_{-\infty}^{\infty} \frac{q^{4n^2-3n}}{(-q;q)_n} \tag{1}$$

$$f_{1,c_8}(q) = \sum_{-\infty}^{\infty} \frac{q^{4n^2-2n}}{(-q;q)_n} \tag{2}$$

$$F_{0,c_8}(q^2) = \sum_{-\infty}^{\infty} \frac{q^{8n^2-6n}}{(q;q^2)_n} \tag{3}$$

$$F_{1,c_8}(q^4) = \sum_{-\infty}^{\infty} \frac{q^{16n^2-8n}}{(q^6;q^4)_n} \tag{4}$$

$$\psi_{0,c_8}(q) = \sum_{-\infty}^{\infty} q^{\frac{7(n^2+3n)}{2}} (-q;q)_n \tag{5}$$

$$\phi_{1,c_8}(q^2) = \sum_{-\infty}^{\infty} q^{7n^2+14n}(-q;q^2)_n \tag{6}$$

$$\phi_{0,c_8}(q^2) = \sum_{-\infty}^{\infty} \frac{q^{8n^2}}{(-q;q^2)_n} \tag{7}$$

$$\psi_{1,c_8}(q) = \sum_{-\infty}^{\infty} \frac{q^{4n^2+4n}}{2(-q;q)_n} \tag{8}$$

In section 4 we show that these bilateral mock theta functions are the limiting cases of basic hypergeometric series ${}_9\phi_8$.
In section 5 we show that these functions satisfies the characteristic property 1(a) of the mock theta functions defined by Ramanujan.
In section 6 we express them in terms of the transcendental function $f(x,\xi;q,p)$ studied by M. Lerch [7]

2. Notation and Definitions

We use the following q notation. Suppose q and z are complex numbers and n is an integar if $n \geq\ 0$ we define $(z)_n = (z,q)_n = \prod_{i=0}^{n-1}(1-q^i z)$

$$(z)_{-n} = (z;q)_{-n} = \frac{(-z)^{-n} q^{\frac{n(n+1)}{2}}}{(q/z;q)_n}$$

and more generally

$$(z_1, z_2, \ldots\ldots, z_r; q)_n = (z_1)_n (z_2)_n, \ldots\ldots\ldots\ldots (z_r)_n$$

For $|q^k| < 1$ let us define

$$(z;q^k)_n = (1-z)(1-z\ q^k), \ldots\ldots\ldots (1-z\ q^{k(n-1)})\ n \geq\ 1\ (z;q^k)_0 = 1$$

$$(z;q^k)_\infty = \lim_{n\to\infty}(z;q^k)_n = \prod_{i\geq\ 0}(1-q^{ki}z)$$

and more generally

$$(z_1, z_2, \ldots\ldots, z_r; q^k)_\infty = (z_1;q^k)_\infty, \ldots\ldots\ldots\ldots (z_r;q^k)_\infty$$

A basic hypergeometric series ${}_{r+1}\phi_r$ on a base q^k is defined as

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \ldots\ldots a_r \\ \\ b_1, b_2, \ldots\ldots\ldots b_r \end{matrix} ; q^k, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots\ldots\ldots a_r; q^k)_n z^n}{(q^k;q^k)_n (b_1, b_2, \ldots\ldots\ldots b_r; q^k)_n},\ (|z|<1)$$

and a bilateral basic hypergeometric series ${}_r\psi_r$ is defined as

$${}_r\psi_r \left[\begin{matrix} a_1, a_2, \ldots\ldots a_r \\ \\ b_1, b_2, \ldots\ldots\ldots b_r \end{matrix} ; q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots\ldots\ldots a_r; q)_n z^n}{(b_1, b_2, \ldots\ldots\ldots b_r; q)_n},\ \left(\left| \frac{b_1, \ldots\ldots b_r}{a_{1,\ldots\ldots a_r}} \right| < |z| < 1 \right)$$

Lerch transcendent is defined as

$$f(x,\xi;p,q) = \sum_{-\infty}^{\infty} \frac{(p\ q)^{n^2} (x\xi)^{-2n}}{(-p\xi^{-2};p^2)_n}$$

This is also equivalent to

$$f(x,\xi;p,q)=\sum_{-\infty}^{\infty}(-p\xi^2;p^2)_n q^{n^2}x^{2n}$$

3. Eight bilateral mock theta functions

: In order to define the functions $f_0c_8(q), f_1c_8(q), F_0c_8(q^2), F_1c_8(q^4), \psi_0c_8(q), \phi_1c_8(q^2), \phi_0c_8(q^2), \psi_1c_8(q)$ following transformation of Slater given on page 142 in[16] between ${}_8\psi_8$ i.e

$$\frac{\left(b_1,\dots,b_8,\frac{q}{a_1},\dots,\frac{q}{a_8},dz,\frac{q}{dz};q\right)_\infty}{\left(c_1,\dots,c_8,\frac{q}{c_1},\dots,\frac{q}{c_8};q\right)_\infty}\,{}_8\Psi_8[\begin{matrix}a_1, & \dots & ,a_8\\ b_1, & \dots & ,b_8\end{matrix};\ q;z]$$
$$=\frac{q}{c_1}\frac{\left(\frac{c_1}{a_1},\dots,\frac{c_1}{a_8},\frac{qb_1}{c_1},\dots,\frac{qb_8}{c_1},\frac{dc_1z}{q},\frac{q^2}{dc_1z};q\right)_\infty}{\left(c_1,\frac{q}{c_1},\frac{c_1}{c_2},\dots,\frac{c_1}{c_8},\frac{qc_2}{c_1},\dots,\frac{qc_8}{c_1};q\right)_\infty}\,{}_8\Psi_8\left[\begin{matrix}\frac{qa_1}{c_1}, & \dots & ,\frac{qa_8}{c_1}\\ \frac{qb_1}{c_1}, & \dots & ,\frac{qb_8}{c_1}\end{matrix};q;z\right]$$
$$+\ \mathrm{idem}(c_1,\dots,c_8) \quad (9)$$

where $d=\frac{a_1\cdots a_8}{c_1\cdots c_8}$ and $|\frac{b_1\cdots b_8}{a_1\cdots a_8}|<|z|<1$ and idem $(c_1,\dots,c_8)$ means that the preceding expression is repeated with $c_1,\dots c_8$ interchanged.

now making $a_1,a_2,.......a_8\to\infty$,$b_1=-q$ $b_2,.........b_8=0$ and $z=\frac{q}{a_1a_2......a_8}$ in (9) we have

$$\frac{(-q,\frac{q}{c_1c_2.........c_8},c_1c_2.........c_8;q)_\infty}{(c_1,c_2,.......c_8,q/c_1,q/c_2,........q/c_8;q)_\infty}\sum_{-\infty}^{\infty}\frac{q^{4n^2-3n}}{(-q;q)_n}$$
$$=\frac{q}{c_1}\frac{(-q^2/c_1,\frac{1}{c_2c_3.........c_8},qc_2c_3,.........c_8;q)_\infty}{(c_1,q/c_1,c_1/c_2,c_1/c_3.......c_1/c_8,qc_2/c_1,qc_3/c_1,........qc_8/c_1;q)_\infty}$$
$$\times\sum_{-\infty}^{\infty}\frac{q^{4n^2+5n}/c_1^{8n}}{(-q^2/c_1;q)_n}+idem(c_1,c_2,......c_8) \quad (10)$$

making $a_1,a_2,.......a_8\to\infty$,$b_1=-q$ $b_2,.........b_8=0$ and $z=\frac{q^2}{a_1a_2......a_8}$ in (9)) we have

$$\frac{(-q,\frac{q^2}{c_1c_2.........c_8},\frac{c_1c_2.........c_8}{q};q)_\infty}{(c_1,c_2,.......c_8,q/c_1,q/c_2,........q/c_8;q)_\infty}\sum_{-\infty}^{\infty}\frac{q^{4n^2-2n}}{(-q;q)_n}$$
$$=\frac{q}{c_1}\frac{(-q^2/c_1,\frac{q}{c_2c_3.........c_8},c_2c_3.........c_8;q)_\infty}{(c_1,q/c_1,c_1/c_2,c_1/c_3.......c_1/c_8,qc_2/c_1,qc_3/c_1,........qc_8/c_1;q)_\infty}$$
$$\times\sum_{-\infty}^{\infty}\frac{q^{4n^2+6n}/c_1^{8n}}{(-q^2/c_1;q)_n}+idem(c_1,c_2,......c_8) \quad (11)$$

making $a_1, a_2, ...a_8 \to \infty$,$b_1 = q$ $b_2, ...b_8 = 0$ and $z = \frac{q^2}{a_1 a_2......a_8}$ in (9) and base changed to q^2 we have

$$\frac{(q, \frac{q^2}{c_1 c_2.........c_8}, c_1 c_2.........c_8; q^2)_\infty}{(c_1, c_2, ...c_8, q^2/c_1, q^2/c_2, ...q^2/c_8; q)_\infty} \sum_{-\infty}^{\infty} \frac{q^{8n^2-6n}}{(q;q^2)_n}$$
$$= \frac{q^2}{c_1} \frac{(q^3/c_1, \frac{1}{c_2 c_3.........c_8}, c_2 c_3.........c_8, q^2; q)_\infty}{(c_1, q^2/c_1, c_2 q^2/c_1...c_8 q^2/c_1, c_1/c_2, ...c_1/c_8; q^2)_\infty}$$
$$\times \sum_{-\infty}^{\infty} \frac{q^{8n^2+10n}/c_1^{8n}}{(q^3/c_1;q^2)_n} + idem(c_1, c_2, ...c_8) \tag{12}$$

making $a_1, a_2, ...a_8 \to \infty$,$b_1 = q^6$ $b_2, ...b_8 = 0$ and $z = \frac{q^8}{a_1 a_2......a_8}$ in (9) and base changed to q^4 we have

$$\frac{(q^6, \frac{q^8}{c_1 c_2.........c_8}, \frac{c_1 c_2.........c_8}{q^4}; q^4)_\infty}{(c_1, c_2, ...c_8, q^4/c_1, q^4/c_2, ...q^4/c_8; q^4)_\infty} \sum_{-\infty}^{\infty} \frac{q^{16n^2-8n}}{(q^6;q^4)_n}$$
$$= \frac{q^4}{c_1} \frac{(q^{10}/c_1, \frac{q^4}{c_2 c_3.........c_8}, c_2 c_3.........c_8; q^4)_\infty}{(c_1, q^4/c_1, q^4 c_2/c_1, c_3 q^4/c_1, ...c_8 q^4/c_1, c_1/c_2, ...c_1/c_8; q^4)_\infty}$$
$$\times \sum_{-\infty}^{\infty} \frac{q^{16n^2+24n}/c_1^{8n}}{(q^{10}/c_1;q^4)_n} + idem(c_1, c_2, ...c_8) \tag{13}$$

making $a_1, a_2, ...a_7 \to \infty$,$a_8 = -q$ $b_1, b_2.........b_8 = 0$ and $z = \frac{-q^{14}}{a_1, a_2, ...a_7}$ in (9) we have

$$\frac{(-1, \frac{q^{15}}{c_1 c_2.........c_8}, \frac{c_1 c_2.........c_8}{q^{14}}; q)_\infty}{(c_1, c_2, ...c_8, q/c_1, q/c_2, ...q/c_8; q)_\infty} \sum_{-\infty}^{\infty} q^{7(n^2+3n)/2} \; (-q;q)_n$$
$$= \frac{q}{c_1} \frac{(-c_1/q, \frac{q^{14}}{c_2 c_3.........c_8}, \frac{c_2 c_3.........c_8}{q^{13}}; q)_\infty}{(c_1, q/c_1, c_1/c_2, c_1/c_3, ...c_1/c_8, qc_2/c_1, qc_3/c_1, ..., qc_8/c_1; q)_\infty}$$
$$\times \sum_{-\infty}^{\infty} q^{7(n^2+5n)/2}/c_1^{7n} \; (-q^2/c_1;q^2)_n + idem(c_1, c_2, ...c_8) \tag{14}$$

making $a_1, a_2, ...a_7 \to \infty$,$a_8 = -q$ $b_1, b_2.........b_8 = 0$ and $z = \frac{-q^{21}}{a_1 a_2......a_7}$ in (9) and base changed to q^2 we have

$$\frac{(-q, \frac{q^{22}}{c_1 c_2.........c_8}, \frac{c_1 c_2.........c_8}{q^{20}}; q^2)_\infty}{(c_1, c_2, ...c_8, q^2/c_1, q^2/c_2, ...q^2/c_8; q^2)_\infty} \sum_{-\infty}^{\infty} (q^{7n^2+14n}) \; (-q;q^2)_n$$
$$= \frac{q^2}{c_1} \frac{(-c_1/q, \frac{q^{20}}{c_2 c_3.........c_8}, \frac{c_2 c_3.........c_8}{q^{18}}; q^2)_\infty}{(c_1, q^2/c_1, c_1/c_2, c_1/c_3, ...c_1/c_8, q^2 c_2/c_1, q^2 c_3/c_1, ..., q^2 c_8/c_1; q^2)_\infty}$$
$$\times \sum_{-\infty}^{\infty} q^{7n^2+28n}/c_1^{7n}(-q^3/c_1;q^2)_n + idem(c_1, c_2, ...c_8) \tag{15}$$

making $a_1, a_2, ...a_8 \to \infty$,$b_1 = -q$ $b_2, b_3.........b_8 = 0$ and $z = \frac{q^8}{a_1 a_2 ......a_8}$ in (9) and base changed to q^2 we have

$$\begin{aligned}
&\frac{(-q, \frac{q^8}{c_1 c_2.........c_8}, \frac{c_1 c_2.........c_8}{q^6}; q)_\infty}{(c_1, c_2, ...c_8, q^2/c_1, q^2/c_2, ...q^2/c_8; q^2)_\infty} \sum_{-\infty}^{\infty} \frac{q^{8n^2}}{(-q; q^2)_n} \\
&= \frac{q^2}{c_1} \frac{(-q^3/c_1, \frac{q^6}{c_2 c_3.........c_8}, \frac{c_2 c_3.........c_8}{q^4}; q^2)_\infty}{(c_1, q^2/c_1, c_1/c_2, c_1/c_3, ...c_1/c_8, q^2 c_2/c_1, q^2 c_3/c_1, ..., q^2 c_8/c_1; q^2)_\infty} \\
&\times \sum_{-\infty}^{\infty} \frac{q^{8n(n+2)}/c_1^{8n}}{(-q^3/c_1; q^2)_n} + idem(c_1, c_2, ...c_8) \qquad (16)
\end{aligned}$$

making $a_1, a_2, ...a_8 \to \infty$,$b_1 = -1$, $b_2, b_3.........b_8 = 0$ and $z = \frac{1}{a_1 a_2 ......a_8}$ in (9) we have

$$\begin{aligned}
&\frac{(-1, \frac{1}{c_1 c_2.........c_8}, q c_1 c_2.........c_8; q)_\infty}{(c_1, c_2, ...c_8, q/c_1, q/c_2, ...q/c_8; q)_\infty} \sum_{-\infty}^{\infty} \frac{q^{4n^2+4n}}{2(-q; q)_n} \\
&= \frac{q}{c_1} \frac{(-q/c_1, \frac{1}{c_2 c_3.........c_8 q}, c_2 c_3.........c_8 q^2; q)_\infty}{(c_1, q/c_1, c_1/c_2, c_1/c_3, ...c_1/c_8, q c_2/c_1, q c_3/c_1, ..., q c_8/c_1; q)_\infty} \\
&\times \sum_{-\infty}^{\infty} \frac{q^{4n^2+4n}/c_1^{8n}}{(-q/c_1; q)_n} + idem(c_1, c_2, ...c_8) \qquad (17)
\end{aligned}$$

4. Bilateral mock theta functions as a limiting cases of a ${}_9\phi_8$:

Bilateral mock theta functions can be reduced as the limiting cases of basic hypergeometric series ${}_9\phi_8$ as follows

$$\begin{aligned}
&f_{0,c_8}(q) = \sum_{-\infty}^{\infty} \frac{q^{4n^2-3n}}{(-q; q)_n} \\
&= \lim_{t \to 0} {}_9\phi_8 \left[\begin{matrix} -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, q \\ -q, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^8 q \right] \\
&+ 2q^7 \lim_{t \to 0} {}_9\phi_8 \left[\begin{matrix} -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q, q; \\ 0, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^7 q^7 \right] \qquad (18)
\end{aligned}$$

$$\begin{aligned}
&f_{1,c_8}(q) = \sum_{-\infty}^{\infty} \frac{q^{4n^2-2n}}{(-q; q)_n} \\
&= \lim_{t \to 0} {}_9\phi_8 \left[\begin{matrix} -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, q \\ -q, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^8 q^2 \right] \\
&+ 2q^6 \lim_{t \to 0} {}_9\phi_8 \left[\begin{matrix} -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q, q \\ 0, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^7 q^6 \right] \qquad (19)
\end{aligned}$$

$$
\begin{aligned}
F_{0,c_8}(q^2) &= \sum_{-\infty}^{\infty} \frac{q^{8n^2-6n}}{(q;q^2)_n} \\
&= \lim_{t\to 0} {}_9\phi_8(q^2) \left[\begin{matrix} -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, q^2 \\ q, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^8 q \right] \\
&+ (q^{14} - q^{13}) \lim_{t\to 0} {}_9\phi_8(q^2) \left[\begin{matrix} -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, q^3, q^2 \\ 0, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; -t^7 q^{13} \right]
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
F_{1,c_8}(q^4) &= \sum_{-\infty}^{\infty} \frac{q^{16n^2-8n}}{(q^6;q^4)_n} \\
&= \lim_{t\to 0} {}_9\phi_8(q^4) \left[\begin{matrix} -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, -1/t, q^4 \\ q^6, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^8 q^8 \right] \\
&+ (q^{24} - q^{26}) \lim_{t\to 0} {}_9\phi_8(q^4) \left[\begin{matrix} -q^4/t, -q^4/t, -q^4/t, -q^4/t, -q^4/t, -q^4/t, -q^4/t, q^2, q^4 \\ 0, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; -t^7 q^{26} \right]
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\psi_{0,c_8}(q) &= \sum_{-\infty}^{\infty} q^{7(n^2+3n)/2} (-q;q)_n \\
&= \lim_{t\to 0} {}_9\phi_8 \left[\begin{matrix} -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q, q \\ 0, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^7 q^7 \right] \\
&+ \frac{1}{2q^7} \lim_{t\to 0} {}_9\phi_8 \left[\begin{matrix} -q/t, -q/t - q/t, -q/t, -q/t, -q/t, -q/t, -q/t, q; \\ -q, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^8/q^7 \right]
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\phi_{1,c_8}(q^2) &= \sum_{-\infty}^{\infty} q^{7n^2+14n} (-q;q^2)_n \\
&= \lim_{t\to 0} {}_9\phi_8(q^2) \left[\begin{matrix} -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q, q^2; \\ 0, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^7 q^7 \right] \\
&+ \frac{1}{q^6 + q^7} \lim_{t\to 0} {}_9\phi_8(q^4) \left[\begin{matrix} -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, q^2; \\ -q^3, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^8/q^6 \right]
\end{aligned} \tag{23}
$$

$$\phi_{0,c_8}(q^2) = \sum_{-\infty}^{\infty} \frac{q^{8n^2}}{(-q;q^2)_n}$$
$$= \lim_{t\to 0} {}_9\phi_8(q^2)\left[\begin{matrix} -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, q^2 \\ -q, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^8/q^8\right]$$
$$+ \quad (q^7+q^8) \lim_{t\to 0} {}_9\phi_8(q^2)\left[\begin{matrix} -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^3, q^2 \\ 0, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^7q^7\right] \tag{24}$$

$$\psi_{1,c_8}(q) = \sum_{-\infty}^{\infty} \frac{q^{4n^2+4n}}{2(-q;q)_n}$$
$$= 1/2 \lim_{t\to 0} {}_9\phi_8\left[\begin{matrix} -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, q \\ -q, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^8\right]$$
$$+ \quad 1/2 \lim_{t\to 0} {}_9\phi_8\left[\begin{matrix} -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q/t, -q, q \\ 0, 0, 0, 0, 0, 0, 0, 0 \end{matrix} ; t^7\right] \tag{25}$$

5. The behavior of the bilateral mock theta functions in the neighborhood of the unit circle

The property of a mock theta function which Ramanujan regarded as characteristic property is that , corresponding to each "rational point" $q = \exp(i\pi \frac{h}{k})$ (with h and k integers) of the unit circle $|q| = 1$, there exist a theta function of q whose difference from the mock theta function is bounded when q approaches this rational point along a radius of the circle.
A rational point $\exp(i\pi \frac{h}{k})$ on the unit cicle is called a point of the first category if h is even and k is odd, a point of the second category if h and k both are odd and a point of the third category if h is odd and k is even.
To see the behavior of the bilateral mock theta function given by equations (1) to (8) in the neighborhood of the unit circle $|q| = 1$ we state and prove the following theorems:
Theorem(5.1) For approach to $|q| = 1$ along a radius of first category $\phi_0 c_8(q^2) = O(1)$
Theorem(5.2) For approach to $|q| = 1$ along a radius of second category $\phi_0 c_8(-q^2) = O(1)$

Proof. of Theorem (5.1) we have

$$\phi_{0,c_8}(q^2) = \sum_{-\infty}^{\infty} \frac{q^{8n^2}}{(-q;q^2)_n}$$

$$= \sum_{0}^{\infty} \frac{q^{8n^2}}{(-q;q^2)_n} + \sum_{1}^{\infty} \frac{q^{8n^2}}{(-q;q^2)_{-n}}$$

$$= \sum_{-\infty}^{\infty} \frac{q^{8n^2}}{(-q;q^2)_n}$$

$$+(q^7+q^8) \lim_{t\to 0} {}_9\phi_8(q^2)\left[\begin{matrix} -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^2/t, -q^3, q^2 \\ 0,0,0,0,0,0,0,0 \end{matrix} ; t^7q^7\right] \quad (26)$$

Now let

$$T_{0,c_8}(q^2) = \sum_{0}^{\infty} \frac{q^{8n^2}}{(-q;q^2)_n} = \sum_{0}^{\infty} \frac{q^{8n^2}}{\prod_{r=1}^{n}(1+q^{2r-1})}$$

Putting $q = \rho \exp(i\pi\ h/k)$ and taking $\rho \to 1-$

$$T_{0,c_8}(q^2) = \sum_{0}^{\infty} \frac{\rho^{8n^2} \exp(i\pi \frac{h}{k} 8n^2)}{\prod_{r=1}^{n}(1+\rho^{2r-1}\exp(i\pi\frac{h}{k}(2r-1))} \quad (27)$$

Again putting $n = uk + v$ in (27), we get

$$T_{0,c_8}(q^2) = \sum_{v=0}^{k-1}\sum_{u=0}^{\infty} \frac{\rho^{8(uk+v)^2} \exp(i\pi \frac{h}{k} 8(uk+v)^2)}{\prod_{r=1}^{uk+v}(1+\rho^{2r-1}\exp(i\pi\frac{h}{k}(2r-1))}$$

$$= \sum_{v=0}^{k-1}\sum_{u=0}^{\infty} a_{v,u} \quad (28)$$

so,

$$\left|\frac{a_{v,u+1}}{a_{v,u}}\right| = \frac{\rho^{8k(2uk+2v+k)}}{\prod_{r=uk+v+1}^{uk+v+k}|(1+\rho^{2r-1}\exp(i\pi\frac{h}{k}(2r-1))|} \quad (29)$$

Further we calculate the denominator of (29) using the inequality, Andrews and Hickerson[14] for $0 < R' \le R \le 1$ and $|z| = 1$

$$|1+Rz| \le \sqrt{R/R'}|1+R'z|$$

so,

$$\prod_{r=uk+v+1}^{uk+v+k} |(1+\rho^{2r-1}\exp(i\pi\frac{h}{k}(2r-1))|$$

$$= \prod_{r=1}^{k} |(1+\rho^{2r+2uk+2v-1}\exp(i\pi\frac{h}{k}(2r+2uk+2v-1))|$$

$$= \prod_{r=1}^{k} |(1+\rho^{2r+2uk+2v-1}\exp(i\pi\frac{h}{k}(2r+2v-1))|$$

$$\ge \prod_{r=1}^{k} \rho^{r+uk-1}|(1+\rho^{2v+1}\exp(i\pi\frac{h}{k}(2r+2v-1))|$$

$(R' = \rho^{2r+2uk+2v-1}$ and $R = \rho^{2v+1})$

$$= \rho^{k/2(2uk+k-1)} \prod_{r=1}^{k} |(1+\rho^{2v+1}\exp(i\pi\frac{h}{k}(2r+2v-1))|$$

$$= \rho^{k/2(2uk+k-1)}|(1+\rho^{k(2v+1)}|$$

since $1+\rho^{2v+1}\exp(i\pi\frac{h}{k}(2r+2v-1))$ runs through the roots of $(x-1)^k-\rho^{k(2v+1)}$

$$\geq \rho^{k/2(2uk+k-1)} \tag{30}$$

Hence from equation (29) and (30), we have

$$|\frac{a_{v,u+1}}{a_{v,u}}| \leq \frac{\rho^{8k(2uk+2v+k)}}{\rho^{k/2(2uk+k-1)}} \leq \rho^{k(15uk+16v+15/2\ k+1/2)} < 1 \leq \epsilon \tag{31}$$

where $0<\epsilon<1$

Hence $\sum_u a_{v,u}$ is uniformly convergent.

Now

$$|T_0c_8(q^2) \leq \sum_{v=0}^{k-1}\sum_{u=0}^{\infty}\epsilon^u|a_{v,0}| = \frac{1}{1-\epsilon}\sum_{v=0}^{k-1}|a_{v,0}|$$

$$= \frac{\frac{1}{1-\epsilon}\sum_{v=0}^{k-1}|\rho^{8v^2}\exp^{i\pi(h/k)8v^2}|}{\prod_{r=1}^{v}|1+\rho^{2r-1}\exp^{i\pi(h/k)2r-1}} \leq \frac{\frac{1}{1-\epsilon}\sum_{v=0}^{k-1}\rho^{8v^2}}{\prod_{r=1}^{v}|1+\rho^{2r-1}\exp^{i\pi(h/k)2r-1}} \tag{32}$$

for fixed k as $\rho\to 1-$

Now the second function on the right of the $\phi_0c_8(q^2)$ in (26) is a bounded function of q when $|q|<1$. Hence $\phi_0c_8(q^2)$ is uniformly convergent and bounded when q lies on the radius of first category i.e $\phi_0c_8(q^2)=O(1)$ □

Proof. of Theorem (5.2): It is clear that when q lies on the radius of the second category $-q$ lies on the radius first category hence from the proof of theorem (5.1) we conclude that $\phi_0c_8(-q^2)=O(1)$.

Similarly it can also be proved that

(a) for approach to $|q|=1$ along a radius of first category

$f_0c_8(q)=O(1), f_1c_8(q)=O(1), F_0c_8(q^2)=O(1), F_1c_8(q^4)=O(1)$ and $\psi_1c_8(q)=O(1)$ also

(b) for approach to $|q|=1$ along a radius of second category

$f_0c_8(-q)=O(1), f_1c_8(-q)=O(1), F_0c_8(-q^2)=O(1), F_1c_8(-q^4)=O(1)$ and $\psi_1c_8(-q)=O(1)$ □

Theorem(5.3) for approach to $|q|=1$ along a radius of third category $\phi_1c_8(q^2)=O(1)$ and $\psi_0c_8(q)=O(1)$

Proof. we give different treatment for the radii of third category .Since on the unit circle if $q=\rho\exp^{i\pi(h/k)}$ with h odd and k even and $0\leq\rho\leq 1$, q approaches the circle along a radius of third category when $\rho\to 1-$.

We now consider the functions $\phi_1c_8(q^2)$ and $\psi_0c_8(q)$.

We have $\phi_1c_8(q^2)=\sum_{-\infty}^{\infty}q^{7n^2+14n}(-q;q^2)_n$

$$= \sum_{0}^{\infty}q^{7n^2+14n}(-q;q^2)_n + \sum_{1}^{\infty}q^{7n^2+14n}(-q;q^2)_{-n}$$

$$= \sum_{0}^{\infty}q^{7n^2+14n}(-q;q^2)_n$$

$$+\frac{1}{q^6+q^7}\lim_{t\to 0}\ {}_9\phi_8(q^2)\left[\begin{matrix}-q^2/t,-q^2/t,-q^2/t,-q^2/t,-q^2/t,-q^2/t,-q^2/t,-q^2/t,q^2;\\ -q^3,0,0,0,0,0,0,0\end{matrix};t^8/q^6\right] \tag{33}$$

Now let

$$k_1 c_8(q^2) = \sum_0^\infty q^{7n^2+14n}(-q;q^2)_n$$

$$= \sum_0^\infty q^{7n^2+14n} \prod_{r=1}^{n}(1+q^{2r-1})$$

Now putting $q = \rho \exp^{i\pi(h/k)}$ and $\rho \to 1-$ (here h is odd and k is even)

$$k_1 c_8(q^2) = \sum_0^\infty \rho^{7n^2+14n} \exp^{i\pi(h/k)(7n^2+14n)} \times \prod_{r=1}^{n}(1+\rho^{2r-1}\exp^{i\pi(h/k)(2r-1)})$$

Again putting $n = 2uk+v$, we get

$$k_1 c_8(q^2) = \sum_{v=0}^{2k-1}\sum_{u=0}^{\infty}(-1)^{2uk+v}\rho^{7(2uk+v)^2+14(2uk+v)} \times \exp^{i\pi(h/k)[7(2uk+v)^2+14(2uk+v)]}$$

$$\times \prod_{r=1}^{2uk+v}(1+\rho^{2r-1})\exp^{i\pi(h/k)(2r-1)}$$

$$= \sum_{v=0}^{2k-1}\sum_{u=0}^{\infty} a_{v,u} \tag{34}$$

Thus

$$\left|\frac{a_{v,u+1}}{a_{v,u}}\right| = \rho^{28k[2uk+v+k+1]} \prod_{r=2uk+v+1}^{2uk+v+2k}\left|1+\rho^{2r-1}\exp^{i\pi(h/k)(2r-1)}\right| \tag{35}$$

Now we calculate $\prod_{r=2uk+v+1}^{2uk+v+2k}\left|1+\rho^{2r-1}\exp^{i\pi(h/k)(2r-1)}\right|$ from (35)as

$$\prod_{r=2uk+v+1}^{2uk+v+2k}\left|1+\rho^{2r-1}\exp^{i\pi(h/k)(2r-1)}\right|$$

$$= \prod_{r=1}^{2k}\left|1+\rho^{2r+4uk+2v-1}\exp^{i\pi(h/k)(2r+4uk+2v-1)}\right|$$

$$= \prod_{r=1}^{2k}\left|1+\rho^{2r+4uk+2v-1}\exp^{i\pi(h/k)(2r+2v-1)}\right|$$

$$= \prod_{r=1}^{2k}\left(1+2\rho^{2r+4uk+2v-1}\cos\frac{(2r+2v-1)h\pi}{k}+\rho^{4r+8uk+4v-2}\right)^{1/2}$$

Since $\beta \le \alpha \le\ 1$ we have

$$\frac{1+2\alpha\cos\theta+\alpha^2}{\alpha} \le \frac{1+2\beta\cos\theta+\beta^2}{\beta}$$

We find immediately that

$$\prod_{r=1}^{2k}\left|1+\rho^{2r+4uk+2v-1}\exp^{i\pi(h/k)(2r+2v-1)}\right|$$

$$\leq \prod_{r=1}^{2k}[\rho^{2r-4k}(1+2\rho^{4uk+2v-1+4k}\cos\frac{(2r+2v-1)h\pi}{k}+\rho^{8uk+4v-2+8k})]^{1/2}$$

$$=\rho^{-k(2k-1)}\prod_{r=1}^{2k}\left|1+\rho^{4uk+2v-1+4k}\exp^{i\pi(h/k)(2r+2v-1)}\right|$$

Now as r runs through the values $1,2,...2k$ the points $\exp^{i\pi(h/k)(2r+2v-1)}$ assumes the positions$1,\exp^{i\pi/k},\exp^{2i\pi/k},.....\exp^{(2k-1)i\pi/k}$ respectively.
Hence

$$\prod_{r=1}^{2k}\left|1+\rho^{4uk+2v-1+4k}\exp^{i\pi(h/k)(2r+2v-1)}\right|$$

$$=\prod_{r=0}^{2k-1}\left|1+\rho^{4uk+2v-1+4k}\exp^{ir\pi(h/k)}\right|$$

$$=1-\rho^{2k(4uk+2v-1+4k)}$$

Thus

$$|\frac{a_{v,u+1}}{a_{v,u}}\leq \rho^{28k(2uk+v+k+1)}\times\rho^{-k(2k-1)}\times(1-\rho^{2k(4uk+2v-1+4k)})$$

$$\leq \rho^{28k(2uk+v+k+1)}\times\rho^{-k(2k-1)}\leq\rho^{56uk^2+28vk+26k^2+29k}<1\leq\epsilon \tag{36}$$

where $0<\epsilon<1$
Hence $\sum_u a_{v,u}$ is uniformly convergent.
Now

$$|k_1c_8(q^2)|\leq\sum_{v=0}^{2k-1}\sum_{u=0}^{\infty}\epsilon^u|a_{v,0}|=\frac{1}{1-\epsilon}\sum_{v=0}^{2k-1}|a_{v,0}|$$

$$=\frac{1}{1-\epsilon}\sum_{v=0}^{2k-1}\left|\rho^{7v^2+14v}\exp^{i\pi(h/k)(7v^2+14v)}\right|\times\prod_{r=1}^{v}|1+\rho^{2r-1}\exp^{i\pi(h/k)(2r-1)}\Big|$$

$$\leq\frac{1}{1-\epsilon}\sum_{v=0}^{2k-1}\rho^{7v^2+14v}\times\prod_{r=1}^{v}\left|1+\rho^{2r-1}\exp^{i\pi(h/k)(2r-1)}\right|=O(1) \tag{37}$$

for fixed k as $\rho\to 1-$
Consequently $k_1c_8(q^2)$ is bounded when q lies on the radius of third category and second function on the right of the definition of $\phi_1c_8(q^2)$ in(33) is a bounded function of q for $|q|<1$.
Hence $\phi_1c_8(q^2)$ is uniformly convergent and bounded when q lies on the radius of third category.
i.e $\phi_1c_8(q^2)=O(1)$.
Similarly it can be proved that $\psi_0c_8(q)=O(1)$ for approach to $|q|=1$ along a radius of third category.

Thus theorem (5.1),(5.2),(5.3) confirms that the bilateral mock theta functions defined in section 3 satisfies the characteristic property of the mock theta functions defined by Ramanujan. $\square$

6. Representations of Bilateral Mock theta functions as Lerch Transcendent:

Lemma (6.1): For $\epsilon = \pm 1$,

$$\sum_{n=-\infty}^{\infty} \frac{q^{\alpha\ n^2} q^{\beta\ n}}{(\epsilon\ q^{\gamma}; q^{\delta})_n} = f\left((-\epsilon)^{-1/2} q^{\frac{2\gamma-2\beta-\delta}{4}}, (-\epsilon)^{1/2} q^{\frac{\delta-2\gamma}{4}}; q^{\frac{2\alpha-\delta}{2}}, q^{\delta/2} \right)$$

$$\sum_{n=-\infty}^{\infty} (-q;q^{\gamma})_n q^{\alpha\ n^2} q^{\beta\ n} = f(q^{\beta/2} q^{\frac{2-\gamma}{4}}; q^{\alpha}, q^{\gamma/2}).$$

Proof. The proof follows from direct substitution and use of basic hypergeometric transformations. As an example we note that $f_0 c_8(q) = \sum_{-\infty}^{\infty} \frac{q^{4n^2-3n}}{(-q;q)_n} = f(q^{7/4}, q^{-1/4}, q^{7/2}, q^{1/2})$ by taking $\alpha = 4, \beta = -3, \epsilon = -1, \gamma = \delta = 1$ and $\psi_0 c_8(q) = \sum_{-\infty}^{\infty} q^{\frac{7(n^2+3n)}{2}} (-q;q)_n = f(q^{21/4}, q^{1/4}, q^{7/2}, q^{1/2})$ by taking $\alpha = 7/2, \beta = 21/2, \gamma = 1$ in the above lemma. In this way all other bilateral mock theta functions defined in section 3 may be expressed in terms of the Lerch transcedent. $\square$

7. conclusion:

With the above analysis and as per the definition of order of a mock theta function suggested by Aggarwal[4] A mock theta function defined in terms of ${}_{r+1}\phi_r$ series be labelled as of order $2r+1$. There may be an additive term with ${}_{r+1}\phi_r$ series consisting of θ- products, since they do not affected the order, it will be rational to label these functions as bilateral mock theta functions of order "seventeen". Alternative expressions of these functions in terms of Hecke type series may give exciting results.

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Mohammad Ahmad*(Corresponding Author)
Department of Mathematics, National Defence Academy, Khadakwasla Pune, India
E-mail address: mahmad_786@rediffmail.com

Sirazul Haq
Department of applied Mathematics,Faculty of Engineering and Technology, Aligarh Muslim University,Aligarh, India
E-mail address: sirajulhaq007@gmail.com

Abdul Hakim Khan
Department of applied Mathematics,Faculty of Engineering and Technology, Aligarh Muslim University,Aligarh, India
E-mail address: ahkhan.amu@gmail.com