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MINIMUM DEGREE ENERGY OF GRAPHS

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Dedicated to the Platinum Jubilee year of Dr. V. R. Kulli

ABSTRACT. Let G be a graph of order n. Then an $n \times n$ symmetric matrix is called the minimum degree matrix $MD(G)$ of a graph G, if its $(i,j)^{th}$ entry is $min\{d_i, d_j\}$ whenever $i \neq j$, and zero otherwise, where d_i and d_j are the degrees of i^{th} and j^{th} vertices of G, respectively. In the present work, we obtain the characteristic polynomial of the minimum degree matrix of graphs obtained by some graph operations. In addition, bounds for the largest minimum degree eigenvalue and minimum degree energy of graphs are obtained.

1. INTRODUCTION

Throughout this paper by a graph $G = (V, E)$ we mean a finite undirected graph without loops and multiple edges of *order* n and *size* m. Let $V = V(G)$ and $E = E(G)$ be the vertex set and edge set of G, respectively. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to it in G. The graph G is *r-regular* if and only if the degree of each vertex in G is r. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of G and let $d_i = d_G(v_i)$. Basic notations and terminologies can be found in [8, 12, 14]. In literature, there are several graph polynomials defined on different graph matrices such as adjacency matrix [8, 12, 14], Laplacian matrix [15], signless Laplacian matrix [9, 18], seidel matrix [5], degree sum matrix [13, 19], distance matrix [1] etc. The purpose of this paper is to study the characteristic polynomial of the *minimum degree matrix* and to obtain bounds for the largest *minimum degree eigenvalue* and *minimum degree energy*.

Note that there are several matrices associated with graphs. In general, we can say $M(G)$ is a matrix defined on a graph G, whose elements are given by

$$M_{ij} = \mathcal{X}(d_i, d_j) \quad for \quad \mathcal{R}(i,j),$$

where $\mathcal{X}(d_i, d_j)$ is a function on degree of vertices v_i and v_j while $\mathcal{R}(i,j)$ is the relation between the vertices v_i and v_j. Suppose $x_1, x_2, ..., x_n$ are the eigenvalues

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of the matrix M, then the corresponding *energy* can be defined as

$$E_M(G) = \sum_{i=1}^{n} |x_i|.$$

The most extensively studied such matrix is the *Adjacency matrix* [12] $A(G)$, where

$$A_{ij} = \begin{cases} 1 & if \;\; v_iv_j \in E(G), \\ 0 & otherwise. \end{cases}$$

In which,

$$\mathcal{X}(d_i, d_j) = \begin{cases} 1 & for\ \mathcal{R}(i,j) = v_iv_j \in E(G), \\ 0 & for\ \mathcal{R}(i,j) = v_iv_j \notin E(G) \;\; or \;\; v_i = v_j. \end{cases}$$

Recently, the analogous concepts of degree sum matrix [19], degree exponent matrix [20] etc., were put forward. The energy $E_A(G)$ with respect to adjacency marix is smaller than the energy with respect to any other matrix. The degree exponent energy $E_{DE}(G)$ is larger among all other energies defined so far. The *minimum degree energy* $E_{MD}(G)$ is much closer to the energy $E_A(G)$ with respect to adjacency matrix, $E_{MD}(G)$ lies between $E_A(G)$ and $E_{DE}(G)$. For a complete graph K_n of order n, $E_{MD}(K_n) = (n-1)E_A(K_n)$ and $E_{DE}(K_n) = (n-1)^{n-1}E_A(K_n)$. These observations motivated us to study the *minimum degree energy.* The *minimum degree matrix* of a graph G of order n is an $n \times n$ symmetric matrix $MD(G) = [md_{ij}]$, whose elements are defined as

$$md_{ij} = \begin{cases} min\{d_i, d_j\} & if \;\; i \neq j, \\ 0 & otherwise. \end{cases}$$

Let I be the identity matrix and J be the matrix whose all entries are equal to 1. The *minimum degree polynomial* of a graph G is defined as

$$P_{MD}(G;\xi) = det(\xi I - MD(G)).$$

The *eigenvalues* of the matrix $MD(G)$, denoted by $\xi_1, \xi_2, ..., \xi_n$ are called the *minimum degree eigenvalues* of G and their collection is called the *minimum degree spectra* of G. It is easy to see that, if G is an *r-regular* graph, then $MD(G) = rJ - rI$. Therefore, for an *r-regular* graph G of order n, we have

$$P_{MD}(G;\xi) = [\xi - r(n-1)][\xi + r]^{n-1}. \tag{1.1}$$

Let G be a graph of order n with *minimum degree eigenvalues* $\xi_1, \xi_2, ..., \xi_n$. Then the minimum degree energy $E_{MD}(G)$ of a graph G is defined as

$$E_{MD}(G) = \sum_{i=1}^{n} |\xi_i|.$$

Example 1. *Let $G = K_2 \cdot K_3$ be a graph (see Figure 1). Then we have the minimum degree matrix:*

$$MD(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix},$$

minimum degree polynomial: $P_{MD}(G;\xi) = \xi^4 - 15\xi^2 - 28\xi - 12$,
minimum degree eigenvalues: $2+\sqrt{7}, -2, -2, 2-\sqrt{7}$ *and*
the minimum degree energy: $E_{MD}(G) = 8$.

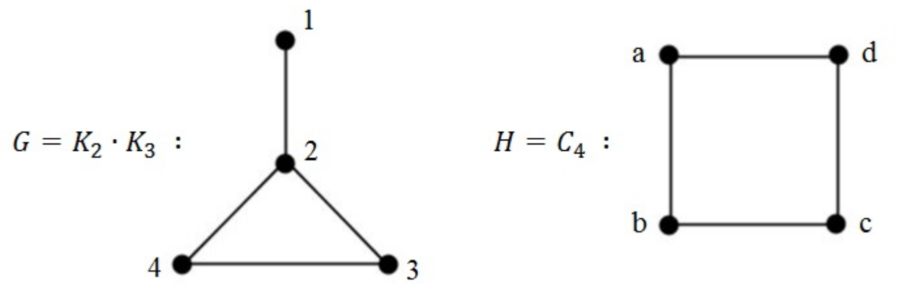

FIGURE 1.

Example 2. *Let $H = C_4$ (see Figure 1) be a 2-regular graph. Then we have the minimum degree matrix:*

$$MD(H) = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix},$$

minimum degree polynomial: $P_{MD}(H;\xi) = \xi^4 - 24\xi^2 - 64\xi - 48$,
minimum degree eigenvalues: $6, -2, -2, -2$ *and*
minimum degree energy: $E_{MD}(H) = 2r(n-1) = 12$, *where r is the degree of the vertices in H.*

2. Minimum degree polynomial of graphs obtained by graph operations

In this section, we obtain the *minimum degree polynomial* of graphs obtained by some graph operations.

The *line graph* [12] $L(G)$ of a graph G is a graph whose vertex set is one-to-one correspondence with the edge set of the graph G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G.

The k^{th} *iterated line graph* [6, 7, 12] of G is defined as $L^k(G) = L(L^{k-1}(G))$, $k = 1, 2, ...$, where $L^0(G) \cong G$ and $L^1(G) \cong L(G)$.

Theorem 2.1. *Let G be an r-regular graph of order n and n_k be the order of $L^k(G)$. Then the minimum degree polynomial of $L^k(G)$, $k = 1, 2, ...$ is*

$$P_{MD}(L^k(G);\xi) = [\xi + 2^k r - 2^{k+1} + 2]^{n_k - 1}[\xi - (2^k r - 2^{k+1} + 2)(n_k - 1)].$$

Proof. The line graph of a regular graph is a regular graph. In particular, the line graph of a regular graph G of order n and of degree r is a regular graph of order $n_1 = \frac{1}{2}nr$ and degree $r_1 = 2r - 2$ [6, 7]. Thus, the order and degree of $L^k(G)$ are

$$n_k = \frac{n}{2^k}\prod_{i=0}^{k-1}(2^i r - 2^{i+1} + 2) \quad \textit{and} \quad r_k = 2^k r - 2^{k+1} + 2.$$

Hence the result follows from (1.1). □

We use the following lemma in order to prove the following theorems.

Lemma 2.2. [20] *If a, b, c and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\xi+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\xi+b)I_{n_2} - bJ_{n_2} \end{vmatrix} \tag{2.1}$$

of order $n_1 + n_2$ can be expressed in the simplified form as

$$(\xi+a)^{n_1-1}(\xi+b)^{n_2-1}\{[\xi-(n_1-1)a][\xi-(n_2-1)b] - n_1n_2cd\}.$$

The *subdivision graph* [12] $S(G)$ of a graph G is the graph obtained by inserting a new vertex on each edge of G.

Theorem 2.3. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD}(S(G);\xi) &= (\xi+2)^{m-1}(\xi+r)^{n-1}\{\xi^2 - [(n-1)r + 2(m-1)]\xi \\ &\quad +2(n-1)(m-1)r - [min\{2,r\}]^2mn\}. \end{aligned}$$

Proof. Let G be an r-regular graph of order n. Then the subdivision graph of the graph G has two types of vertices. The n vertices are of degree r and the remaining m vertices are of degree 2. Hence

$$MD(S(G)) = \begin{bmatrix} r(J_n - I_n) & min\{2,r\}J_{n\times m} \\ min\{2,r\}J_{m\times n} & 2(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD}(S(G);\xi) &= |\xi I - MD(S(G))| \\ &= \begin{vmatrix} (\xi+r)I_n - rJ_n & -min\{2,r\}J_{n\times m} \\ -min\{2,r\}J_{m\times n} & (\xi+2)I_m - 2J_m) \end{vmatrix}. \end{aligned}$$

Using Lemma 2.2, we get the required result. □

The *semitotal point graph* [21] $T_2(G)$ is a graph which is obtained from the graph G by inserting a vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it.

Theorem 2.4. *Let G be an r-regular graph of order n and size m. Then*

$$P_{MD}(T_2(G);\xi) = (\xi+2r)^{n-1}(\xi+2)^{m-1}\{[\xi-2(n-1)r][\xi-2(m-1)] - [min\{2,2r\}]^2mn\}.$$

Proof. Let G be an r-regular graph of order n. Then the semitotal point graph of the graph G has two types of vertices. The n vertices are of degree $2r$ and the remaining m vertices are of degree 2. Hence

$$MD(T_2(G)) = \begin{bmatrix} 2r(J_n - I_n) & min\{2,2r\}J_{n\times m} \\ min\{2,2r\}J_{m\times n} & 2(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD}(T_2(G);\xi) &= |\xi I - MD(T_2(G))| \\ &= \begin{vmatrix} (\xi+2r)I_n - 2rJ_n & -min\{2,2r\}J_{n\times m} \\ -min\{2,2r\}J_{m\times n} & (\xi+2)I_m - 2J_m) \end{vmatrix}. \end{aligned}$$

Using Lemma 2.2, we get the required result. □

The *semitotal line graph* [11] $T_1(G)$ is a graph which is obtained from the graph G by inserting a new vertex into every edge of G, and joining by edges those pairs of new vertices which lie on adjacent edges.

Theorem 2.5. *Let G be an r-regular graph of order n and size m. Then*

$$P_{MD}(T_1(G);\xi) = (\xi + r)^{n-1}(\xi + 2r)^{m-1}\{[\xi - (n-1)r][\xi - 2(m-1)r] - r^2mn\}.$$

Proof. Let G be an r-regular graph of order n. Then the semitotal line graph of the graph G has two types of vertices. The n vertices are of degree r and the remaining m vertices are of degree $2r$. Hence

$$MD(T_1(G)) = \begin{bmatrix} r(J_n - I_n) & rJ_{n\times m} \\ rJ_{m\times n} & 2r(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD}(T_1(G);\xi) &= |\xi I - MD(T_1(G))| \\ &= \begin{vmatrix} (\xi + r)I_n - rJ_n & -rJ_{n\times m} \\ -rJ_{m\times n} & (\xi + 2r)I_m - 2rJ_m) \end{vmatrix}. \end{aligned}$$

Using Lemma 2.2, we get the required result. □

The *total graph* [12] $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident.

Theorem 2.6. *Let G be an r-regular graph of order n and size m. Then*

$$P_{MD}(T(G);\xi) = (\xi + 2r)^{n+m-1}[\xi - 2(n + m - 1)r].$$

Proof. The total graph of a regular graph of degree r is a regular graph of degree $2r$ with $n + m$ vertices. Hence the result follows from (1.1). □

The graph G^{+k} is a graph obtained from the graph G by attaching k ($k \geq 1$) pendant edges to each vertex of G. If G is a graph of order n and size m, then G^{+k} is graph of order $n + nk$ and size $m + nk$.

Theorem 2.7. *Let G be an r-regular graph of order n and size m. Then*

$$P_{MD}(G^{+k};\xi) = (\xi + r + k)^{n-1}(\xi + 1)^{nk-1}\{[\xi - (n-1)(r + k)][\xi - (nk - 1)] - kn^2\}.$$

Proof. The graph G^{+k} of a regular graph G of degree r has two types of vertices. The n vertices are of degree $r + k$ and the remaining nk vertices are of degree 1. Hence

$$MD(G^{+k}) = \begin{bmatrix} (r + k)(J_n - I_n) & J_{n\times nk} \\ J_{nk\times n} & (J_{nk} - I_{nk}) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD}(G^{+k};\xi) &= |\xi I - MD(G^{+k})| \\ &= \begin{vmatrix} (\xi + r + k)I_n - (r + k)J_n & -J_{n\times nk} \\ -J_{nk\times n} & (\xi + 1)I_{nk} - J_{nk}) \end{vmatrix}. \end{aligned}$$

Using Lemma 2.2, we get the required result. □

The *union* [12] of the graphs G_1 and G_2 is a graph $G_1 \cup G_2$ whose vertex set is $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and the edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Theorem 2.8. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*

$$P_{MD}(G\cup H;\xi) = P_{MD}(G;\xi)P_{MD}(H;\xi) - (\xi+r_1)^{n_1-1}(\xi+r_2)^{n_2-1}n_1n_2[min\{r_1, r_2\}]^2.$$

Proof. The minimum degree matrix of $G \cup H$ will be of the form

$$\begin{aligned} MD(G \cup H) &= \begin{bmatrix} MD(G) & min\{r_1, r_2\}J_{n_1 \times n_2} \\ min\{r_1, r_2\}J_{n_2 \times n_1} & MD(H) \end{bmatrix} \\ &= \begin{bmatrix} r_1(J_{n_1} - I_{n_1}) & min\{r_1, r_2\}J_{n_1 \times n_2} \\ min\{r_1, r_2\}J_{n_2 \times n_1} & r_2(J_{n_2} - I_{n_2}) \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} P_{MD}(G \cup H; \xi) &= |\xi I - MD(G \cup H)| \\ &= \begin{vmatrix} (\xi + r_1)I_{n_1} - r_1 J_{n_1} & -min\{r_1, r_2\}J_{n_1 \times n_2} \\ -min\{r_1, r_2\}J_{n_2 \times n_1} & (\xi + r_2)I_{n_2} - r_2 J_{n_2}) \end{vmatrix}. \end{aligned}$$

Using Lemma 2.2, we get

$$P_{MD}(G \cup H; \xi) = (\xi + r_1)^{n_1 - 1}(\xi + r_2)^{n_2 - 1}\{[\xi - (n_1 - 1)r_1][\xi - (n_2 - 1)r_2] - n_1 n_2 [min\{r_1, r_2\}]^2\}. \tag{2.2}$$

Since G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2. Then by (1.1), we have

$$P_{MD}(G; \xi) = (\xi + r_1)^{n_1 - 1}[\xi - (n_1 - 1)r_1] \tag{2.3}$$

and

$$P_{MD}(H; \xi) = (\xi + r_2)^{n_2 - 1}[\xi - (n_2 - 1)r_2]. \tag{2.4}$$

The result follows by substituting (2.3) and (2.4) in (2.2). □

The *join* [8, 12] $G_1 \nabla G_2$ of two graphs G_1 and G_2 is the graph obtained from G_1 and G_2 by joining every vertex of G_1 to all vertices of G_2.

Theorem 2.9. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*

$$\begin{aligned} P_{MD}(G \nabla H; \xi) &= (\xi + R_1)^{n_1 - 1}(\xi + R_2)^{n_2 - 1}\{(\xi - (n_1 - 1)R_1)(\xi - (n_2 - 1)R_2) \\ &\quad - min\{R_1, R_2\}n_1 n_2\}, \end{aligned}$$

where $R_1 = r_1 + n_2$ and $R_2 = r_2 + n_1$.

Proof. If G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2, then $G \nabla H$ has two types of vertices, the n_1 vertices with degree $R_1 = r_1 + n_2$ and the remaining n_2 vertices are of degree $R_2 = r_2 + n_1$. Hence

$$MD(G \nabla H) = \begin{bmatrix} R_1(J_{n_1} - I_{n_1}) & min\{R_1, R_2\}J_{n_1 \times n_2} \\ min\{R_1, R_2\}J_{n_2 \times n_1} & R_2(J_{n_2} - I_{n_2}) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD}(G \nabla H; \xi) &= |\xi I - MD(G \nabla H)| \\ &= \begin{vmatrix} (\xi + R_1)I_{n_1} - R_1 J_{n_1} & -min\{R_1, R_2\}J_{n_1 \times n_2} \\ -min\{R_1, R_2\}J_{n_2 \times n_1} & (\xi + R_2)I_{n_2} - R_2 J_{n_2}) \end{vmatrix}. \end{aligned}$$

Using Lemma 2.2, we get the required result. □

Let K_n denotes the complete graph of order n, and kG is the *union* of k $(k \geq 1)$ copies of G. A *windmill graph* is the *join* of K_1 and kK_2. It has $2k + 1$ vertices. If $r_1 = 0, r_2 = 1, n_1 = 1$ and $n_2 = 2k$, then by Theorem 2.9, we have the following corollary.

Corollary 2.10. *Let $G = K_1 \triangledown kK_2$ be a windmill graph. Then*
$$P_{MD}(K_1 \triangledown kK_2; \xi) = (\xi + 2)^{2k-1}[\xi^2 - 2(2k-1)\xi - 8k^2].$$

The *product* [12] $G \times H$ of graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if and only if $[a = b$ and $xy \in E(H)]$ or $[x = y$ and $ab \in E(G)]$.

Theorem 2.11. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*
$$P_{MD}(G \times H; \xi) = (\xi + (r_1 + r_2))^{n_1n_2-1}[\xi - (r_1 + r_2)(n_1n_2 - 1)].$$

Proof. Since the graphs G and H are regular graphs of degree r_1 and r_2, respectively. Therefore, the graph obtained by the cartesian product of G and H is a regular graph of degree $r_1 + r_2$ with n_1n_2 vertices. Hence the result follows from (1.1). □

The *composition* [12] $G[H]$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G[H]$ if and only if [a is adjacent to b] or [$a = b$ and x is adjacent to y].

Theorem 2.12. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*
$$P_{MD}(G[H]; \xi) = (\xi + (n_2r_1 + r_2))^{n_1n_2-1}[\xi - (n_2r_1 + r_2)(n_1n_2 - 1)].$$

Proof. Since the graphs G and H are regular graphs of degree r_1 and r_2, respectively. Therefore, the graph obtained by the composition of two graphs G and H is a regular graph of degree $n_2r_1 + r_2$ with n_1n_2 vertices. Hence the result follows from (1.1). □

The *corona* [12] $G \circ H$ of graphs G and H is a graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and then joining by an edge each vertex of the i^{th} copy of H is named (H, i) with the i^{th} vertex of G.

Theorem 2.13. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*
$$\begin{aligned} P_{MD}(G \circ H; \xi) &= (\xi + R_1)^{n_1-1}(\xi + R_2)^{n_1n_2-1}\{(\xi - (n_1 - 1)R_1)(\xi - (n_1n_2 - 1)R_2) \\ &\quad -min\{r_1 + n_2, r_2 + 1\}n_1^2n_2\}, \end{aligned}$$
where $R_1 = r_1 + n_2$ and $R_2 = r_2 + 1$.

Proof. If G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2, then $G \circ H$ has two types of vertices, the n_1 vertices are of degree $R_1 = r_1 + n_2$ and the remaining n_1n_2 vertices are of degree $R_2 = r_2 + 1$. Hence
$$MD(G \circ H) = \begin{bmatrix} R_1(J_{n_1} - I_{n_1}) & min\{R_1, R_2\}J_{n_1 \times n_1n_2} \\ min\{R_1, R_2\}J_{n_1n_2 \times n_1} & R_2(J_{n_1n_2} - I_{n_1n_2}) \end{bmatrix}.$$
Therefore,
$$\begin{aligned} P_{MD}(G \circ H; \xi) &= |\xi I - MD(G \circ H)| \\ &= \begin{vmatrix} (\xi + R_1)I_{n_1} - R_1J_{n_1} & -min\{R_1, R_2\}J_{n_1 \times n_1n_2} \\ -min\{R_1, R_2\}J_{n_1n_2 \times n_1} & (\xi + R_2)I_{n_1n_2} - R_2J_{n_1n_2}) \end{vmatrix}. \end{aligned}$$
Using Lemma 2.2, we get the required result. □

The **Cauchy-Schwarz inequality** [2] states that, if $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are real n-vectors, then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{2.5}$$

3. Bounds for the largest Minimum degree eigenvalue

Since $trace(MD(G)) = 0$, the eigenvalues of $MD(G)$ satisfies the following relations:

$$\sum_{i=1}^{n} \xi_i = 0, \tag{3.1}$$

further,

$$\begin{aligned}\sum_{i=1}^{n} \xi_i^2 &= trace([MD(G)]^2) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} d_{ji} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 \\ &= 2\sum_{i<j} (min\{d_i, d_j\})^2.\end{aligned}$$

Thus,

$$\sum_{i=1}^{n} \xi_i^2 = 2\mathcal{M}, \ where\ \mathcal{M} = \sum_{i<j} (min\{d_i, d_j\})^2. \tag{3.2}$$

The following results are useful throughout the paper. Let $M_1 = max_{1\leq i\leq n}(a_i)$; $M_2 = max_{1\leq i\leq n}(b_i)$; $m_1 = min_{1\leq i\leq n}(a_i)$; and $m_2 = min_{1\leq i\leq n}(b_i)$.

Theorem 3.1. [17] *Let a_i and b_i are nonnegative real numbers. Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2. \tag{3.3}$$

Theorem 3.2. [16] *Let a_i and b_i are nonnegative real numbers. Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{4}\left(M_1M_2 - m_1m_2\right)^2. \tag{3.4}$$

Theorem 3.3. [3] *Let a_i and b_i are nonnegative real numbers. Then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b), \tag{3.5}$$

where a, b, A and B are real constants such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for each $i, 1 \leq i \leq n$. Further, $\alpha(n) = n\lfloor \frac{n}{2} \rfloor(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Theorem 3.4. [10] *Let a_i and b_i are nonnegative real numbers. Then*

$$\sum_{i=1}^{n} b_i^2 + pP\sum_{i=1}^{n} a_i^2 \leq (p+P)\left(\sum_{i=1}^{n} a_i b_i\right), \tag{3.6}$$

where p and P are real constants such that $pa_i \leq b_i \leq Pa_i$ for each $i, 1 \leq i \leq n$.

Theorem 3.5. *Let G be an r-regular graph of order n. Then G has only one positive minimum degree eigenvalue $\xi = r(n-1)$.*

Proof. Let G be a connected r-regular graph of order n and $\{v_1, v_2, ..., v_n\}$ are the vertices of G. Let $d_i = r$ be the degree of $v_i, i = 1, 2, ..., n$. Then

$$d_{ij} = \begin{cases} min\{d_i, d_j\} = r & if \;\; i \neq j, \\ 0 & otherwise. \end{cases}$$

Therefore, the characteristic polynomial of $MD(G)$ is,

$$\begin{aligned} P_{MD}(G;\xi) &= det(\xi I - MD(G)) \\ \Longrightarrow det(\xi I - MD(G)) &= det(\xi I - rA(K_n)) \\ &= r^n \left|\frac{\xi}{r} I - A(K_n)\right| \\ &= r^n \left(\frac{\xi}{r} - n + 1\right)\left(\frac{\xi}{r} + 1\right)^{n-1} \\ &= (\xi - r(n-1))(\xi + r)^{n-1}. \end{aligned}$$

Thus, $(\xi - r(n-1))(\xi + r)^{n-1} = 0$, will give

$$\xi = \begin{cases} r(n-1) & 1 \; time, \\ -r & (n-1) \; times. \end{cases}$$

□

Theorem 3.6. *Let G be any graph of order n. Then*

$$\xi_1 \leq \sqrt{\frac{2\mathcal{M}(n-1)}{n}}. \tag{3.7}$$

Proof. Let $\xi_1, \xi_2, ..., \xi_n$ are the minimum degree eigenvalues of the graph G. Then by substituting $a_i = 1$ and $b_i = \xi_i$ for $i = 2, 3, ..., n$ in (2.5), we get

$$\left(\sum_{i=2}^{n} \xi_i\right)^2 \leq (n-1)\left(\sum_{i=2}^{n} \xi_i^2\right). \tag{3.8}$$

Again from (3.1) and (3.2), we have

$$\sum_{i=2}^{n} \xi_i = -\xi_1 \quad and \quad \sum_{i=2}^{n} \xi_i^2 = 2\mathcal{M} - \xi_1^2.$$

Therefore, (3.8) becomes

$$(-\xi_1)^2 \leq (n-1)(2\mathcal{M} - \xi_1^2).$$

Hence,

$$\xi_1 \leq \sqrt{\frac{2\mathcal{M}(n-1)}{n}}.$$

Equality in (3.7) holds for regular graphs. □

4. Bounds for the Minimum degree energy of graphs

Theorem 4.1. *Let G be an r-regular graph of order n. Then $-r$ and $r(n-1)$ are minimum degree eigenvalues of G with multiplicity $(n-1)$ and 1 respectively and $E_{MD}(G) = 2r(n-1)$.*

Proof. To find the minimum degree polynomial, we have the following determinant.

$$\begin{aligned} |\xi I - MD(G)| &= \begin{vmatrix} \xi & -r & -r & ... & -r \\ -r & \xi & -r & ... & -r \\ -r & -r & \xi & ... & -r \\ ... & ... & ... & ... & ... \\ -r & -r & -r & ... & \xi \end{vmatrix} \\ &= (\xi + r)^{n-1} \begin{vmatrix} \xi & -r & -r & ... & -r \\ -1 & 1 & 0 & ... & 0 \\ -1 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ -1 & 0 & 0 & ... & 1 \end{vmatrix} \\ &= (\xi - r(n-1))(\xi + r)^{n-1}. \end{aligned}$$

Thus,

$$E_{MD}(G) = 2r(n-1).$$

□

Theorem 4.2. *Let G be a graph of order n and size m. Then*

$$E_{MD}(G) \geq \sqrt{2n\mathcal{M} - \frac{n^2}{4}(|\xi_1| - |\xi_n|)^2}, \tag{4.1}$$

where $|\xi_1|$ and $|\xi_2|$ are maximum and minimum of the absolute value of ξ_i's.

Proof. Suppose $\xi_1, \xi_2, ..., \xi_n$ are the eigenvalues of $MD(G)$. Then by substituting $a_i = 1$ and $b_i = |\xi_i|$ in 3.4, we get

$$\begin{aligned} \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\xi_i|^2 - \left(\sum_{i=1}^{n} |\xi_i| \right)^2 &\leq \frac{n^2}{4}(|\xi_1| - |\xi_n|)^2 \\ 2\mathcal{M}n - (E_{MD}(G))^2 &\leq \frac{n^2}{4}(|\xi_1| - |\xi_n|)^2 \\ E_{MD}(G) &\geq \sqrt{2n\mathcal{M} - \frac{n^2}{4}(|\xi_1| - |\xi_n|)^2}. \end{aligned}$$

□

Corollary 4.3. *If G is an r-regular graph of order n, then*

$$E_{MD}(G) \geq nr\sqrt{(n-1) - \frac{(n-2)^2}{4}}.$$

Theorem 4.4. *Let G be a graph of order n. Then*

$$\sqrt{2\mathcal{M}} \leq E_{MD}(G) \leq \sqrt{2n\mathcal{M}}.$$

Proof. For upper bound: Let $\xi_1, \xi_2, ..., \xi_n$ be the minimum degree eigenvalues of G. Then by substituting $a_i = 1$ and $b_i = |\xi_i|$ in (2.5), we get

$$\begin{aligned} \left(\sum_{i=1}^{n} |\xi_i|\right)^2 &\leq \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\xi_i|^2 \\ (E_{MD}(G))^2 &\leq 2\mathcal{M}n \\ E_{MD}(G) &\leq \sqrt{2n\mathcal{M}}. \qquad (4.2) \end{aligned}$$

For lower bound: We have,

$$(E_{MD}(G))^2 = \left(\sum_{i=1}^{n} |\xi_i|\right)^2 \geq \sum_{i=1}^{n} |\xi_i|^2 = 2\mathcal{M}.$$

Which implies,

$$E_{MD}(G) \geq \sqrt{2\mathcal{M}}. \qquad (4.3)$$

Combining (4.2) and (4.3), we get the desired result. □

Theorem 4.5. *Let G be a graph of order n and let Δ be the absolute value of the determinant of $MD(G)$. Then*

$$\sqrt{2\mathcal{M} + n(n-1)\Delta^{2/n}} \leq E_{MD}(G) \leq \sqrt{2n\mathcal{M}}.$$

Proof. For lower bound: By definition of *minimum degree energy*, we have

$$\begin{aligned} (E_{MD}(G))^2 &= \left(\sum_{i=1}^{n} |\xi_i|\right)^2 \\ &= \sum_{i=1}^{n} \xi_i^2 + 2\sum_{i<j} |\xi_i||\xi_j| \\ &= 2\mathcal{M} + 2\sum_{i<j} |\xi_i||\xi_j| \\ &= 2\mathcal{M} + \sum_{i \neq j} |\xi_i||\xi_j|. \qquad (4.4) \end{aligned}$$

Since the arithmetic mean is not smaller than geometric mean for nonnegative numbers, then we get

$$\begin{aligned} \frac{1}{n(n-1)} \sum_{i \neq j} |\xi_i||\xi_j| &\geq \left(\prod_{i \neq j} |\xi_i||\xi_j|\right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{i=1}^{n} |\xi_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \prod_{i=1} |\xi_i|^{(2/n)} \\ &= \Delta^{2/n}. \end{aligned}$$

Therefore,

$$\sum_{i \neq j} |\xi_i||\xi_j| \geq n(n-1)\Delta^{2/n}. \qquad (4.5)$$

Combining (4.4) and (4.5), we get

$$E_{MD}(G) \geq \sqrt{2\mathcal{M} + n(n-1)\Delta^{2/n}}. \tag{4.6}$$

For upper bound: Consider a nonnegative quantity

$$Y = \sum_{i=1}^{n}\sum_{j=1}^{n}(|\xi_i| - |\xi_j|)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(|\xi_i|^2 - |\xi_j|^2 - 2|\xi_i||\xi_j|).$$

By direct expansion, we get

$$Y = n\sum_{i=1}^{n}|\xi_i|^2 + n\sum_{j=1}^{n}|\xi_j|^2 - 2\left(\sum_{i=1}^{n}|\xi_i|\right)\left(\sum_{j=1}^{n}|\xi_j|\right).$$

Now, by definition of *minimum degree energy* and (3.2), we have

$$Y = 4n\mathcal{M} - 2(E_{MD}(G))^2.$$

Since $Y \geq 0$,

$$4n\mathcal{M} - 2(E_{MD}(G))^2 \geq 0$$

$$E_{MD}(G) \leq \sqrt{2n\mathcal{M}}. \tag{4.7}$$

Combining (4.6) and (4.7), we get the desired result. □

Corollary 4.6. *If G is an r-regular graph of order n, then*

$$E_{MD}(G) \leq nr\sqrt{(n-1)}.$$

Theorem 4.7. *Let G be a graph of order n and size m. Let $\xi_1 \geq \xi_2 \geq ... \geq \xi_n$ be a non-increasing arrangement of minimum degree eigenvalues. Then*

$$E_{MD}(G) \geq \sqrt{2n\mathcal{M} - \alpha(n)(|\xi_1| - |\xi_n|)^2}, \tag{4.8}$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Proof. Suppose $\xi_1, \xi_2, ..., \xi_n$ are the *minimum degree eigenvalues* of G. Then by substituting $a_i = |\xi_i| = b_i,\ a = |\xi_n| = b$ and $A = |\xi_1| = B$ in (3.5), we get

$$\left| n\sum_{i=1}^{n}|\xi_i|^2 - \left(\sum_{i=1}^{n}|\xi_i|\right)^2 \right| \leq \alpha(n)(|\xi_1| - |\xi_n|)^2. \tag{4.9}$$

Since $E_{MD}(G) = \sum_{i=1}^{n}|\xi_i|,\ \sum_{i=1}^{n}|\xi_i|^2 = 2\mathcal{M}$. Therefore, (4.9) yields the required result. □

Remark 4.1. *Since $\alpha(n) \leq \frac{n^2}{4}$, from (4.1) and (4.8) one can easily observe that, the inequality in (4.8) is sharper than the inequality in (4.1).*

Theorem 4.8. *Let G be a graph of order n and size m. Let $\xi_1 \geq \xi_2 \geq ... \geq \xi_n$ be a non-increasing arrangement of minimum degree eigenvalues. Then*

$$E_{MD}(G) \geq \frac{|\xi_1||\xi_n|n + 2\mathcal{M}}{|\xi_1| + |\xi_n|}, \tag{4.10}$$

where $|\xi_1|$ and $|\xi_2|$ are maximum and minimum of the absolute value of ξ_i's.

Proof. Suppose $\xi_1, \xi_2, ..., \xi_n$ are the *minimum degree eigenvalues* of G. Then by substituting $b_i = |\xi_i|, a_i = 1, p = |\xi_n|$ and $P = |\xi_1|$ in (3.6), we get

$$\sum_{i=1}^{n} |\xi_i|^2 + |\xi_1||\xi_n| \sum_{i=1}^{n} 1^2 \leq (|\xi_1| + |\xi_n|) \left(\sum_{i=1}^{n} |\xi_i| \right). \tag{4.11}$$

Since $E_{MD}(G) = \sum_{i=1}^{n} |\xi_i|,\ \sum_{i=1}^{n} |\xi_i|^2 = 2\mathcal{M}$. Therefore, (4.11) yields the required result. □

Remark 4.2. *(i) For a graph G, $E(G) \leq E_{MD}(G)$ and equality holds for K_2.*
(ii) For a complete graph K_n, $E_{MD}(K_n) = (n-1)E(K_n)$.

5. Conclusion

The characteristic polynomial of the minimum degree matrix of graphs obtained from the regular graphs by some graph operations is obtained. In this paper, we also computed the minimum degree energy of regular graphs. In addition, the bounds for largest minimum degree eigenvalue and minimum degree energy of graphs are obtained. The (4.8) gives the sharp lower bound for minimum degree energy.

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