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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR *GH*-CONVEX FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Some inequalities of Hermite-Hadamard type for GH-convex functions defined on positive intervals are given. Applications for special means are provided as well.

1. Introduction

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X,\ x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0, 1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [21, p. 2], [22, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x) + f(y)}{2}, \tag{1.1}$$

which can be derived from the classical Hermite-Hadamard inequality for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

For related results, see [1]-[20], [23]-[25], [26]-[35] and [36]-[45].

Let X be a linear space and C a convex subset in X. A function $f : C \to \mathbb{R}\setminus\{0\}$ is called *AH-convex (concave)* on the convex set C if the following inequality holds

$$f((1-\lambda)x + \lambda y) \leq (\geq) \frac{1}{(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)}} = \frac{f(x) f(y)}{(1-\lambda) f(y) + \lambda f(x)} \tag{AH}$$

for any $x, y \in C$ and $\lambda \in [0, 1]$.

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An important case which provides many examples is that one in which the function is assumed to be positive for any $x \in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{f(x)}+\lambda\frac{1}{f(y)} \le (\ge) \frac{1}{f((1-\lambda)x+\lambda y)}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Therefore we can state the following fact:

Criterion 1. *Let X be a linear space and C a convex subset in X. The function $f : C \to (0,\infty)$ is AH-convex (concave) on C if and only if $\frac{1}{f}$ is concave (convex) on C in the usual sense.*

If we apply the Hermite-Hadamard inequality (1.1) for the function $\frac{1}{f}$ then we state the following result:

Proposition 1. *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0,\infty)$ is AH-convex (concave) on C, then*

$$\frac{f(x)+f(y)}{2f(x)f(y)} \le (\ge) \int_0^1 \frac{d\lambda}{f((1-\lambda)x+\lambda y)} \le (\ge) \frac{1}{f\left(\frac{x+y}{2}\right)} \tag{1.2}$$

for any $x, y \in C$.

Following [4], we can introduce the concept of *GH-convex (concave)* function $f : I \subset (0,\infty) \to \mathbb{R}$ on an interval of positive numbers I as satisfying the condition

$$f\left(x^{1-\lambda}y^{\lambda}\right) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{f(x)}+\lambda\frac{1}{f(y)}} = \frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)}. \tag{1.3}$$

Since

$$f\left(x^{1-\lambda}y^{\lambda}\right) = f\circ\exp\left[(1-\lambda)\ln x+\lambda\ln y\right]$$

and

$$\frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)} = \frac{f\circ\exp(\ln x)\, f\circ\exp(\ln y)}{(1-\lambda)f\circ\exp(y)+\lambda f\circ\exp(x)}$$

then $f : I \subset (0,\infty) \to \mathbb{R}$ is GH-convex (concave) on I if and only if $f\circ\exp$ is *AH-convex (concave)* on $\ln I := \{x|\ x = \ln t,\ t \in I\}$.

Motivated by the above results, in this paper we establish some Hermite-Hadamard type inequalities for GH-convex (concave) functions. Some examples for special means are provided as well.

2. Results

As a direct consequence of Hermite-Hadamard inequality we have:

Theorem 1. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be GH-convex (concave) on $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2f(a)f(b)} \le (\ge) \frac{1}{\ln b-\ln a}\int_a^b \frac{1}{tf(t)}dt \le (\ge) \frac{1}{f\left(\sqrt{ab}\right)}. \tag{2.1}$$

From a different perspective we have:

Theorem 2. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be GH-convex (concave) on $[a, b]$. Then*

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \leq (\geq) \frac{G^2(f(a), f(b))}{L(f(a), f(b))}, \tag{2.2}$$

where, for $p, q > 0$, $G(p, q) := \sqrt{pq}$ is the geometric-mean while

$$L(p, q) := \begin{cases} \frac{p-q}{\ln p - \ln q} \text{ if } p \neq q \\ \\ q \text{ if } p = q \end{cases}$$

is the logarithmic-mean.

Using the following well known inequality $G(a, b) \leq L(a, b)$ we have a simpler upper bound

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \leq \frac{G^2(f(a), f(b))}{L(f(a), f(b))} \leq G(f(a), f(b)) \tag{2.3}$$

provided that $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is GH-convex on $[a, b]$.

We have also the complementary result:

Theorem 3. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be GH-convex (concave) on $[a, b]$. Then*

$$f\left(\sqrt{ab}\right) \leq (\geq) \frac{\int_a^b \frac{1}{t} f(t) f\left(\frac{ab}{t}\right) dt}{\int_a^b \frac{1}{t} f(t) dt}. \tag{2.4}$$

We observe that by Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\int_a^b \frac{1}{t} f(t) f\left(\frac{ab}{t}\right) dt \leq \left(\int_a^b \frac{1}{t^2} f^2(t) dt\right)^{1/2} \left(\int_a^b f^2\left(\frac{ab}{t}\right) dt\right)^{1/2}. \tag{2.5}$$

If we change the variable $\frac{ab}{t} = s$, then $dt = -\frac{ab}{s^2} ds$ and we have

$$\int_a^b f^2\left(\frac{ab}{t}\right) dt = ab \int_a^b \frac{1}{s^2} f^2(s) ds.$$

From (2.5) we get

$$\begin{aligned} \int_a^b \frac{1}{t} f(t) f\left(\frac{ab}{t}\right) dt &\leq \left(\int_a^b \frac{1}{t^2} f^2(t) dt\right)^{1/2} \left(ab \int_a^b \frac{1}{s^2} f^2(s) ds\right)^{1/2} \\ &= \sqrt{ab} \int_a^b \frac{1}{t^2} f^2(t) dt. \end{aligned}$$

Now, if $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is GH-convex, then from (2.4) we have

$$f\left(\sqrt{ab}\right) \leq \frac{\int_a^b \frac{1}{t} f(t) f\left(\frac{ab}{t}\right) dt}{\int_a^b \frac{1}{t} f(t) dt} \leq \sqrt{ab} \frac{\int_a^b \frac{1}{t^2} f^2(t) dt}{\int_a^b \frac{1}{t} f(t) dt}. \tag{2.6}$$

If the function $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is monotonic either nonincreasing or nondecreasing, then the functions $f(\cdot)$ and $f\left(\frac{ab}{\cdot}\right)$ have opposite monotonicities. By the Čebyšev weighted integral inequality for asynchronous functions g and h and the positive weight $w \geq 0$, namely

$$\int_a^b w(t) dt \int_a^b w(t) g(t) h(t) dt \leq \int_a^b w(t) g(t) dt \int_a^b w(t) h(t) dt,$$

we have
$$\int_a^b \frac{1}{t}dt \int_a^b \frac{1}{t} f(t) f\left(\frac{ab}{t}\right) dt \le \int_a^b \frac{1}{t} f(t)\, dt \int_a^b \frac{1}{t} f\left(\frac{ab}{t}\right) dt,$$
i.e.,
$$\int_a^b \frac{1}{t} f(t) f\left(\frac{ab}{t}\right) dt \le \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} f(t)\, dt \int_a^b \frac{1}{t} f\left(\frac{ab}{t}\right) dt.$$
So, if $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is GH-convex and monotonic on $[a, b]$, then from (2.4) we have
$$f\left(\sqrt{ab}\right) \le \frac{\int_a^b \frac{1}{t} f(t) f\left(\frac{ab}{t}\right) dt}{\int_a^b \frac{1}{t} f(t)\, dt} \le \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} f\left(\frac{ab}{t}\right) dt \tag{2.7}$$
or, equivalently
$$f\left(\sqrt{ab}\right) \le \frac{\int_a^b \frac{1}{t} f(t) f\left(\frac{ab}{t}\right) dt}{\int_a^b \frac{1}{t} f(t)\, dt} \le \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} f(t)\, dt. \tag{2.8}$$

Theorem 4. *Let $f : I \subset (0, \infty) \to (0, \infty)$ be GH-convex (concave) on I. If $x, y \in \mathring{I}$, the interior of I, then there exists $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$ such that*
$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{\varphi(y)\, y}{f(y)} (\ln y - \ln x). \tag{2.9}$$

In particular, we have:

Corollary 1. *Let $f : I \subset (0, \infty) \to (0, \infty)$ be GH-convex (concave) on I and differentiable on $\mathring{I}$. If $x, y \in \mathring{I}$, then*
$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{f'(y)\, y}{f(y)} (\ln y - \ln x). \tag{2.10}$$

We also have:

Theorem 5. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be GH-convex (concave) on $[a, b]$. Then*
$$\int_a^b \frac{1}{s} f^2(s)\, ds \le (\ge) \left[(\ln b - \ln u) f(b) + (\ln u - \ln a) f(a)\right] f(u), \tag{2.11}$$
for any $u \in [a, b]$.

If we take in (2.11) $u = G(a, b) = \sqrt{ab}$, then we get
$$\frac{1}{\ln b - \ln a} \int_a^b \frac{1}{s} f^2(s)\, ds \le (\ge) A(f(a), f(b))\, f(G(a, b)). \tag{2.12}$$

If we take in (2.11) either $u = a$ or $u = b$, then we have
$$\frac{1}{\ln b - \ln a} \int_a^b \frac{1}{s} f^2(s)\, ds \le (\ge) f(b) f(a). \tag{2.13}$$

Also, by taking in (2.11) $u = I(a, b)$, the *identric mean*, that is defined by
$$I(a, b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } b \ne a \\ a \text{ if } b = a, \end{cases}$$

then we get

$$\int_a^b \frac{1}{s} f^2(s)\,ds \tag{2.14}$$
$$\leq (\geq) [(\ln b - \ln I(a,b)) f(b) + (\ln I(a,b) - \ln a) f(a)] f(I(a,b)).$$

Since simple calculations show that

$$\ln b - \ln I(a,b) = \frac{L(a,b) - a}{L(a,b)}$$

and

$$\ln I(a,b) - \ln a = \frac{b - L(a,b)}{L(a,b)},$$

then the inequality (2.14) is equivalent to

$$\int_a^b \frac{1}{s} f^2(s)\,ds \leq (\geq) f(I(a,b)) \left[\frac{L(a,b) - a}{L(a,b)} f(b) + \frac{b - L(a,b)}{L(a,b)} f(a)\right]. \tag{2.15}$$

3. Proofs

Since $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is *GH*-convex (concave) on $[a,b]$, hence $f \circ \exp$ is *AH*-convex (concave) on $[\ln a, \ln b]$. By the inequality (1.2) for $f \circ \exp$ and $\ln a$, $\ln b$ we have

$$\frac{f \circ \exp(\ln a) + f \circ \exp(\ln b)}{2 f \circ \exp(\ln a)\, f \circ \exp(\ln b)} \leq (\geq) \int_0^1 \frac{d\lambda}{f \circ \exp((1-\lambda)\ln a + \lambda \ln b)} \tag{3.1}$$
$$\leq (\geq) \frac{1}{f \circ \exp\left(\frac{\ln a + \ln b}{2}\right)}$$

that is equivalent to

$$\frac{f(a) + f(b)}{2 f(a) f(b)} \leq (\geq) \int_0^1 \frac{d\lambda}{f(a^{1-\lambda} b^{\lambda})} \leq (\geq) \frac{1}{f\left(\sqrt{ab}\right)}. \tag{3.2}$$

If we change the variable $t = a^{1-\lambda} b^{\lambda}$, then $(1-\lambda)\ln a + \lambda \ln b = \ln t$, which gives $\lambda = \frac{\ln t - \ln a}{\ln b - \ln a}$ and $d\lambda = \frac{1}{(\ln b - \ln a)t} dt$. We have then

$$\int_0^1 \frac{d\lambda}{f(a^{1-\lambda} b^{\lambda})} = \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t f(t)} dt$$

and by (3.2) we obtain the desired result (2.1).

From the definition of *GH*-convex (concave) functions on $[a,b]$ and by integration we get

$$\int_0^1 f\left(a^{1-\lambda} b^{\lambda}\right) d\lambda \leq (\geq) f(a) f(b) \int_0^1 \frac{d\lambda}{(1-\lambda) f(a) + \lambda f(b)}. \tag{3.3}$$

If $f(a) = f(b)$, then the integral

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(a) + \lambda f(b)} \tag{3.4}$$

reduces to $\frac{1}{f(a)}$.

If $f(a) \neq f(b)$, then by changing the variable $u = (1-\lambda) f(a) + \lambda f(b)$ in (3.4) we have

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(a) + \lambda f(b)} = \frac{1}{f(b) - f(a)} \int_{f(a)}^{f(b)} \frac{du}{u} = \frac{1}{L(f(a), f(b))}.$$

Also, as above, if we change the variable $t = a^{1-\lambda} b^{\lambda}$, then

$$\int_0^1 f\left(a^{1-\lambda} b^{\lambda}\right) d\lambda = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt.$$

Replacing these values in (3.3), we get the desired result (2.2).

If we take in the definition of GH-convex functions $\lambda = \frac{1}{2}$, then we get

$$f(\sqrt{xy}) \leq (\geq) \frac{2f(x) f(y)}{f(y) + f(x)}. \tag{3.5}$$

If we replace in (3.5), $x = a^{1-\lambda} b^{\lambda}$ and $y = a^{\lambda} b^{1-\lambda}$, then we get

$$f\left(\sqrt{ab}\right) \left[f\left(a^{1-\lambda} b^{\lambda}\right) + f\left(a^{\lambda} b^{1-\lambda}\right)\right] \leq (\geq) 2f\left(a^{1-\lambda} b^{\lambda}\right) f\left(a^{\lambda} b^{1-\lambda}\right). \tag{3.6}$$

By integrating this inequality over λ on $[0,1]$ we obtain

$$\begin{aligned} & f\left(\sqrt{ab}\right) \left[\int_0^1 f\left(a^{1-\lambda} b^{\lambda}\right) d\lambda + \int_0^1 f\left(a^{\lambda} b^{1-\lambda}\right) d\lambda\right] \\ & \leq (\geq) 2 \int_0^1 f\left(a^{1-\lambda} b^{\lambda}\right) f\left(a^{\lambda} b^{1-\lambda}\right) d\lambda. \end{aligned} \tag{3.7}$$

Observe that

$$\int_0^1 f\left(a^{\lambda} b^{1-\lambda}\right) d\lambda = \int_0^1 f\left(a^{1-\lambda} b^{\lambda}\right) d\lambda = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt$$

and

$$\begin{aligned} \int_0^1 f\left(a^{1-\lambda} b^{\lambda}\right) f\left(a^{\lambda} b^{1-\lambda}\right) d\lambda &= \int_0^1 f\left(a^{1-\lambda} b^{\lambda}\right) f\left(\frac{ab}{a^{1-\lambda} b^{\lambda}}\right) d\lambda = \\ &= \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t) f\left(\frac{ab}{t}\right)}{t} dt. \end{aligned}$$

Making use of (3.7) we deduce the desired result (2.4).

The following lemma is of interest in itself:

Lemma 1. *Let $f : I \subset \mathbb{R} \to (0, \infty)$ be AH-convex (concave) on I. If $x, y \in \mathring{I}$, the interior of I, then there exists $\varphi(y) \in \left[f'_{-}(y), f'_{+}(y)\right]$ such that*

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{\varphi(y)}{f(y)} (y - x) \tag{3.8}$$

holds.

Proof. Let $x, y \in \mathring{I}$. Since the function $\frac{1}{f}$ is concave (convex) then the lateral derivatives $f'_{-}(y), f'_{+}(y)$ exists for $y \in \mathring{I}$ and $\left(\frac{1}{f}\right)'_{-(+)}(y) = -\frac{f'_{-(+)}(y)}{f^2(y)}$.

Since $\frac{1}{f}$ is concave (convex) then we have the *gradient inequality*

$$\frac{1}{f(y)} - \frac{1}{f(x)} \geq (\leq) \lambda(y)(y - x) = -\lambda(y)(x - y)$$

with $\lambda(y) \in \left[-\frac{f'_+(y)}{f^2(y)}, -\frac{f'_-(y)}{f^2(y)}\right]$, which is equivalent to

$$\frac{1}{f(y)} - \frac{1}{f(x)} \geq (\leq) \frac{\varphi(y)}{f^2(y)}(x-y) \tag{3.9}$$

with $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$.

The inequality (3.9) can be also written as

$$1 - \frac{f(y)}{f(x)} \geq (\leq) \frac{\varphi(y)}{f(y)}(x-y)$$

or as

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{\varphi(y)}{f(y)}(y-x)$$

and the inequality (3.8) is proved. □

Now, since $f : I \subset (0,\infty) \to (0,\infty)$ is GH-convex (concave) on I, then the function $f \circ \exp$ is AH-convex (concave) on $\ln I$.

Let $u, v \in \ln \mathring{I}$, then by (3.8) we have

$$\frac{f(e^v)}{f(e^u)} - 1 \leq (\geq) \frac{\varphi(e^v)e^v}{f(e^v)}(v-u) \tag{3.10}$$

with $\varphi(e^v) \in \left[f'_-(e^v), f'_+(e^v)\right]$.

If $x, y \in \mathring{I}$ and we take $u = \ln x$, $v = \ln y$ in (3.10) then we get

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{\varphi(y)\,y}{f(y)}(\ln y - \ln x)$$

with $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$.

This proves Theorem 4.

The following lemma is of interest in itself.

Lemma 2. *Let $g : [c,d] \subset (0,\infty) \to (0,\infty)$ be AH-convex (concave) on $[c,d]$, then we have the inequality*

$$\frac{1}{d-c}\int_c^d g^2(t)\,dt \leq (\geq) \left[\frac{d-s}{d-c}g(d) + \frac{s-c}{d-c}g(c)\right]g(s) \tag{3.11}$$

for any $s \in [c,d]$.

Proof. If the function $g : [c,d] \subset (0,\infty) \to (0,\infty)$ is AH-convex (concave) on $[c,d]$, then the function g is differentiable almost everywhere on $[c,d]$ and we have the inequality

$$\frac{g(t)}{g(s)} - 1 \leq (\geq) \frac{g'(t)}{g(t)}(t-s) \tag{3.12}$$

for every $s \in [c,d]$ and almost every $t \in [c,d]$.

Multiplying (3.12) by $g(t) > 0$ and integrating over $t \in [c,d]$ we have

$$\frac{1}{g(s)}\int_c^d g^2(t)\,dt - \int_c^d g(t)\,dt \leq (\geq) \int_c^d g'(t)(t-s)\,dt. \tag{3.13}$$

Integrating by parts we also have

$$\int_c^d g'(t)(t-s)\,dt = g(d)(d-s) + g(c)(s-c) - \int_c^d g(t)\,dt$$

and by (3.13) we get the desired result (3.11). □

Now, since $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GH-convex (concave) on I, then the function $g=f\circ\exp$ is AH-convex (concave) on $[c,d]=[\ln a,\ln b]$.

From (3.11) we then have for $s=\ln u$, $u\in[a,b]$ that

$$\begin{aligned}&\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}f^2\circ\exp(t)\,dt\\&\leq(\geq)\left[\frac{\ln b-\ln u}{\ln b-\ln a}f\circ\exp(\ln b)+\frac{\ln u-\ln a}{\ln b-\ln a}f\circ\exp(\ln a)\right]f\circ\exp(\ln u).\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}f^2\circ\exp(t)\,dt\\&\leq(\geq)\left[\frac{\ln b-\ln u}{\ln b-\ln a}f(b)+\frac{\ln u-\ln a}{\ln b-\ln a}f(a)\right]f(u),\end{aligned}\tag{3.14}$$

for any $u\in[a,b]$.

If we make the change of variable $s=\exp(t)$, then $t=\ln s$, $dt=\frac{ds}{s}$ and by (3.14) we get the desired inequality (2.11).

4. Applications

Consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=t^p$, $p\in\mathbb{R}\setminus\{0\}$. By the weighted geometric mean-harmonic mean inequality, we have

$$\begin{aligned}f\left(x^{1-\lambda}y^{\lambda}\right)&=\left(x^{1-\lambda}y^{\lambda}\right)^p=(x^p)^{1-\lambda}(y^p)^{\lambda}\\&\geq\frac{1}{\frac{1-\lambda}{x^p}+\frac{\lambda}{y^p}}=\frac{x^py^p}{(1-\lambda)y^p+\lambda x^p}\\&=\frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)},\end{aligned}\tag{4.1}$$

for any $x,y\in[a,b]$ and $\lambda\in[0,1]$, which shows that f is GG-concave on $[a,b]$.

For $p\in\mathbb{R}\setminus\{0,-1\}$, we define the *p-logarithmic mean* as

$$L_p(a,b):=\begin{cases}\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, \text{ if } b\neq a\\[2ex] b \text{ if } b=a.\end{cases}$$

We observe that

$$L_p^p(a,b)=\frac{1}{b-a}\int_a^b t^p dt,\ p\in\mathbb{R}\setminus\{0,-1\}.$$

If we write the inequality (2.1) for the GG-concave function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=t^p$, $p\in\mathbb{R}\setminus\{0\}$, then

$$H^{-1}(a^p,b^p)\geq\frac{1}{\ln b-\ln a}\int_a^b t^{-p-1}dt\geq G^{-1}(a^p,b^p).\tag{4.2}$$

Observe that

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b t^{-p-1}dt&=\frac{b-a}{\ln b-\ln a}\frac{1}{b-a}\int_a^b t^{-p-1}dt\\&=L(a,b)\,L_{-p-1}^{-p-1}(a,b)\end{aligned}$$

for $p\in\mathbb{R}\setminus\{0,-1\}$ and by (4.2) we get

$$H^{-1}\left(a^{p},b^{p}\right)\geq L\left(a,b\right)L_{-p-1}^{-p-1}\left(a,b\right)\geq G^{-1}\left(a^{p},b^{p}\right), \tag{4.3}$$

for $p\in\mathbb{R}\setminus\{0,-1\}$.

Now, if we use (2.2), then we get

$$L\left(a,b\right)L_{p-1}^{p-1}\left(a,b\right)\geq\frac{G^{2p}\left(a,b\right)}{L\left(a^{p},b^{p}\right)}, \tag{4.4}$$

for $p\in\mathbb{R}\setminus\{0,1\}$.

From (2.4) we also have

$$L_{p-1}^{p-1}\left(a,b\right)\geq G^{p}\left(a,b\right)L\left(a,b\right), \tag{4.5}$$

for $p\in\mathbb{R}\setminus\{0,1\}$.

Moreover, if we use the inequality (2.12) we have

$$L\left(a,b\right)L_{2p-1}^{2p-1}\left(a,b\right)\geq A\left(a^{p},b^{p}\right)G^{p}\left(a,b\right), \tag{4.6}$$

for $p\in\mathbb{R}\setminus\left\{0,\frac{1}{2}\right\}$.

From (2.15) we finally have

$$L\left(a,b\right)L_{2p-1}^{2p-1}\left(a,b\right)\geq I^{p}\left(a,b\right)\left[\frac{L\left(a,b\right)-a}{b-a}b^{p}+\frac{b-L\left(a,b\right)}{b-a}a^{p}\right], \tag{4.7}$$

for $p\in\mathbb{R}\setminus\left\{0,\frac{1}{2}\right\}$.

Now, for $q>0$ consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp(-qt)$. Then by the weighted arithmetic mean - geometric mean - harmonic mean inequality, we have for any $x,y\in[a,b]$ and $\lambda\in[0,1]$ that

$$\begin{aligned}
f\left(x^{1-\lambda}y^{\lambda}\right)&=\exp\left(-qx^{1-\lambda}y^{\lambda}\right)\geq\exp\left(-q\left[\left(1-\lambda\right)x+\lambda y\right]\right)\\
&=\left[\exp\left(-qx\right)\right]^{1-\lambda}\left[\exp\left(-qy\right)\right]^{\lambda}\\
&\geq\frac{1}{\left(1-\lambda\right)\frac{1}{\exp(-qx)}+\lambda\frac{1}{\exp(-qy)}}\\
&=\frac{\exp\left(-qx\right)\exp\left(-qy\right)}{\left(1-\lambda\right)\exp\left(-qy\right)+\lambda\exp\left(-qx\right)}\\
&=\frac{f\left(x\right)f\left(y\right)}{\left(1-\lambda\right)f\left(y\right)+\lambda f\left(x\right)},
\end{aligned}$$

which shows that f is GH-concave on $[a,b]$.

We consider the following *α-exponential integral mean*

$$Ei_{\alpha}\left(a,b\right):=\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{\exp\left(\alpha t\right)}{t}dt,$$

where $b>a>0$ and $\alpha\in\mathbb{R}$.

By (2.1) for the GH-convex function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp(-qt)$ where $q>0$, we get that

$$\frac{\exp\left(-qa\right)+\exp\left(-qb\right)}{2\exp\left(-q\left(a+b\right)\right)}\geq Ei_{q}\left(a,b\right)\geq\exp\left(q\sqrt{ab}\right). \tag{4.8}$$

From (2.2) we have for $q>0$ that

$$Ei_{-q}\left(a,b\right)\geq\frac{\exp\left(-q\left(a+b\right)\right)}{L\left(\exp\left(-qa\right),\exp\left(-qb\right)\right)}. \tag{4.9}$$

Observe, however, that

$$\begin{aligned} L\left(\exp\left(-qa\right), \exp\left(-qb\right)\right) &= \frac{\exp\left(-qb\right) - \exp\left(-qa\right)}{q\left(a-b\right)} \\ &= \frac{\exp\left(qa\right) - \exp\left(qb\right)}{q\left(a-b\right)\exp\left(q\left(a+b\right)\right)} \\ &= \frac{E\left(qa, qb\right)}{\exp\left(q\left(a+b\right)\right)}, \end{aligned}$$

where E is defined by

$$E\left(c, d\right) := \frac{\exp d - \exp c}{d-c}, \ c \neq d.$$

Then by (4.9) we get

$$E\left(qa, qb\right) Ei_{-q}\left(a, b\right) \geq 1. \tag{4.10}$$

From (2.12) we also have

$$Ei_{-2q}\left(a, b\right) \geq A\left(\exp\left(-qa\right), \exp\left(-qb\right)\right) \exp\left(-q\sqrt{ab}\right), \tag{4.11}$$

where $q > 0$ and $b > a > 0$.

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S. S. Dragomir
[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia, [2]School of Computational & Applied Mathematics,, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa

E-mail address: `sever.dragomir@vu.edu.au`