Electronic Journal of Mathematical Analysis and Applications
Vol. 7(2) July 2019, pp. 256-266
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

GLOBAL DYNAMICS OF THE SYSTEM OF TWO EXPONENTIAL DIFFERENCE EQUATIONS

MAI NAM PHONG

Abstract. The purpose of this paper is to investigate the boundedness and persistence of the solutions, the global stability of the unique positive equilibrium point and the rate of convergence of solutions of the system of two difference equations which contains exponential terms:

$$x_{n+1} = \frac{a + e^{-(bx_n + cy_n)}}{d + bx_n + cy_n},\ y_{n+1} = \frac{\alpha + e^{-(\beta x_n + \gamma y_n)}}{\delta + \beta x_n + \gamma y_n},$$

where the parameters $a, b, c, d, \alpha, \beta, \gamma, \delta$ and the initial values $x_0,\ y_0$ are positive real numbers. Furthermore, we give some numerical examples to illustrate our theoretical results.

1. Introduction

Difference equations arise in the situations in which the discrete values of the independent variable involve. Many practical phenomena are modeled with the help of difference equations [1, 3, 8]. In engineering, difference equations arise in control engineering, digital signal processing, electrical networks, etc. In social sciences, difference equations arise to study the national income of a country and then its variation with time, Cobweb phenomenon in economics, etc. Recently, there has been a great interest in studying the qualitative properties of difference equations and systems of difference equations of exponential form [4, 6, 11, 12, 13, 14, 15, 19].

In [4], the authors examined the boundedness, the asymptotic behavior, the periodic character of the solutions and the stability character of the positive equilibrium of the difference equation:

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \tag{1}$$

where $a,\ b$ are positive constants and the initial values $x_{-1},\ x_0$ are positive numbers. Furthermore, in (1) the authors used a as the immigration rate and b as the growth rate in the population model. In fact, this was a model suggested by the people from the Harvard School of Public Health; studying the population dynamics of one species x_n.

2010 *Mathematics Subject Classification.* 39A10.

Key words and phrases. Equilibrium points, local stability, global behavior, rate of convergence, positive solutions.

Submitted Oct. 26, 2018.

In [11], the authors studied the boundedness, the asymptotic behavior, the periodicity and the stability of the positive solutions of the difference equation:

$$y_{n+1}=\frac{\alpha+\beta e^{-y_n}}{\gamma+y_{n-1}}, \tag{2}$$

where $\alpha,\ \beta,\ \gamma$ are positive constants and the initial values $y_{-1},\ y_0$ are positive numbers.

In [7], the authors explored the boundedness, the asymptotic behavior and the rate of convergence of the positive solutions of the system of two difference equations:

$$x_{n+1}=\frac{\alpha+\beta e^{-x_n}}{\gamma+y_n},\ y_{n+1}=\frac{\delta+\zeta e^{-y_n}}{\eta+x_n}, \tag{3}$$

where $\alpha,\beta,\gamma,\delta,\zeta,\eta$ are positive constants and the initial values $x_0,\ y_0$ are positive real values.

Motivated by these above papers, in this paper, we will investigate the boundedness, the persistence and the asymptotic behavior of the positive solutions of the following system of exponential form:

$$x_{n+1}=\frac{a+e^{-(bx_n+cy_n)}}{d+bx_n+cy_n},\ y_{n+1}=\frac{\alpha+e^{-(\beta x_n+\gamma y_n)}}{\delta+\beta x_n+\gamma y_n}, \tag{4}$$

where $a,b,c,d,\alpha,\beta,\gamma,\delta$ are positive constants and the initial values $x_0,\ y_0$ are positive real values. Moreover, we establish the rate of convergence of a solution that converges to the equilibrium $E=(\bar{x},\ \bar{y})$ of (4).

2. Global behavior of solutions of system (4)

The following lemma shows that every positive solution $\{(x_n,y_n)\}_{n=0}^{\infty}$ of (4) is bounded and persists.

Lemma 2.1. *Every positive solution of system* (4) *is bounded and persists.*

Proof. Let (x_n,y_n) be an arbitrary solution of (4). From (4) we can see that

$$x_n\le\frac{a+1}{d},\ y_n\le\frac{\alpha+1}{\delta},\ n=1,2,\dots \tag{5}$$

In addition, from (4) and (5) we get

$$x_n\ge\frac{a+e^{-\frac{b(a+1)}{d}-\frac{c(\alpha+1)}{\delta}}}{d+\frac{b(a+1)}{d}+\frac{c(\alpha+1)}{\delta}},\ y_n\ge\frac{\alpha+e^{-\frac{\beta(a+1)}{d}-\frac{\gamma(\alpha+1)}{\delta}}}{\delta+\frac{\beta(a+1)}{d}+\frac{\gamma(\alpha+1)}{\delta}},\ n=2,3,\dots \tag{6}$$

Therefore, from (5) and (6) the proof of lemma is complete. □

The next lemma establishes an invariant set for the system (4)

Lemma 2.2. *Let* $\{(x_n,y_n)\}_{n=0}^{\infty}$ *be a positive solution of the system* (4)*. Then*

$$\Big[\frac{a+e^{-\frac{b(a+1)}{d}-\frac{c(\alpha+1)}{\delta}}}{d+\frac{b(a+1)}{d}+\frac{c(\alpha+1)}{\delta}},\frac{a+1}{d}\Big]\times\Big[\frac{\alpha+e^{-\frac{\beta(a+1)}{d}-\frac{\gamma(\alpha+1)}{\delta}}}{\delta+\frac{\beta(a+1)}{d}+\frac{\gamma(\alpha+1)}{\delta}},\frac{\alpha+1}{\delta}\Big]$$

is an invariant set for the system (4).

Proof. It follows from induction. □

The following result will be useful in establishing the global attractivity character of the equilibrium of system (4).

Theorem 2.3. [2] *Let* $\mathcal{R} = [a_1,\ b_1] \times [c_1,\ d_1]$ *and*

$$f:\ \mathcal{R} \longrightarrow [a_1,\ b_1],\ g:\ \mathcal{R} \longrightarrow [c_1,\ d_1]$$

be a continuous functions such that:

(a) $f(x,\ y)$ *is decreasing in both variables and* $g(x,\ y)$ *is decreasing in both variables for each* $(x,\ y) \in \mathcal{R}$*;*

(b) If $(m_1,\ M_1,\ m_2,\ M_2) \in \mathcal{R}^2$ *is a solution of*

$$\begin{cases} M_1 & = f(m_1,\ m_2),\ m_1 = f(M_1,\ M_2) \\ M_2 & = g(m_1,\ m_2),\ m_2 = g(M_1,\ M_2) \end{cases} \tag{7}$$

then $m_1 = M_1$ *and* $m_2 = M_2$*. Then the following system of difference equations:*

$$x_{n+1} = f(x_n,\ y_n),\ y_{n+1} = g(x_n,\ y_n) \tag{8}$$

has a unique equilibrium $(\bar{x},\ \bar{y})$ *and every solution* $(x_n,\ y_n)$ *of the system* (8) *with* $(x_0,\ y_0) \in \mathcal{R}$ *converges to the unique equilibrium* $(\bar{x},\ \bar{y})$*. In addition, the equilibrium* $(\bar{x},\ \bar{y})$ *is globally asymptotically stable.*

Now we state the main theorem of this section.

Theorem 2.4. *Consider system* (4)*. Suppose that the following relations hold true:*

$$d > b + c, \delta > \beta + \gamma. \tag{9}$$

Then system (4) *has a unique positive equilibrium* $(\bar{x},\ \bar{y})$ *and every positive solution of system* (4) *tends to the unique positive equilibrium* $(\bar{x},\ \bar{y})$ *as* $n \to \infty$*. In addition, the equilibrium* $(\bar{x},\ \bar{y})$ *is globally asymptotically stable.*

Proof. We consider the functions

$$f(u,\ v) = \frac{a + e^{-(bu+cv)}}{d + bu + cv},\quad g(u,\ v) = \frac{\alpha + e^{-(\beta u+\gamma v)}}{\delta + \beta u + \gamma v} \tag{10}$$

where

$$u,\ v\ \in I \times J = \Big[\frac{a + e^{-\frac{b(a+1)}{d} - \frac{c(\alpha+1)}{\delta}}}{d + \frac{b(a+1)}{d} + \frac{c(\alpha+1)}{\delta}}, \frac{a+1}{d}\Big] \times \Big[\frac{\alpha + e^{-\frac{\beta(a+1)}{d} - \frac{\gamma(\alpha+1)}{\delta}}}{\delta + \frac{\beta(a+1)}{d} + \frac{\gamma(\alpha+1)}{\delta}}, \frac{\alpha+1}{\delta}\Big]. \tag{11}$$

It is easy to see that $f(u,v),\ g(u,v)$ are decreasing in both variables for each $(u,v) \in\ I \times J$. In addition, from (10) and (11) we have $f(u,v) \in\ I,\ g(u,v) \in\ J$ as $(u,v) \in\ I \times J$ and so $f:\ I \times J \longrightarrow I,\ g:\ I \times J \longrightarrow J$.

Now let $m_1,\ M_1,\ m_2,\ M_2$ be positive real numbers such that

$$M_1 = \frac{a + e^{-(bm_1+cm_2)}}{d + bm_1 + cm_2},\ m_1 = \frac{a + e^{-(bM_1+cM_2)}}{d + bM_1 + cM_2}, \tag{12}$$

and

$$M_2 = \frac{\alpha + e^{-(\beta m_1+\gamma m_2)}}{\delta + \beta m_1 + \gamma m_2},\ m_2 = \frac{\alpha + e^{-(\beta M_1+\gamma M_2)}}{\delta + \beta M_1 + \gamma M_2}. \tag{13}$$

Moreover arguing as in the proof of Theorem.2.3, it suffices to assume that

$$m_1 \leq M_1,\ m_2 \leq M_2. \tag{14}$$

From (12) we get

$$\begin{aligned} M_1 d + bm_1 M_1 + cm_2 M_1 &= a + e^{-(bm_1+cm_2)}, \\ m_1 d + bm_1 M_1 + cm_1 M_2 &= a + e^{-(bM_1+cM_2)}, \end{aligned} \tag{15}$$

which implies that

$$d(M_1 - m_1) + cM_1(m_2 - M_2) + cM_2(M_1 - m_1) = e^{-(bm_1+cm_2)} - e^{-(bM_1+cM_2)}. \tag{16}$$

From (16) we have

$$\begin{aligned} &d(M_1 - m_1) + cM_1(m_2 - M_2) + cM_2(M_1 - m_1) \\ &= e^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1}[b(M_1 - m_1) + c(M_2 - m_2)], \end{aligned} \tag{17}$$

where $bm_1 + cm_2 \leq \theta_1 \leq bM_1 + cM_2$.
From (17) we get

$$\begin{aligned} &(M_1 - m_1)[d + cM_2 - be^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1}] \\ &= (M_2 - m_2)[cM_1 + ce^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1}], \end{aligned} \tag{18}$$

from which we have

$$(M_2 - m_2) = \frac{d + cM_2 - be^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1}}{cM_1 + ce^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1}}(M_1 - m_1). \tag{19}$$

From (13) we imply that

$$\begin{aligned} M_2\delta + \beta m_1 M_2 + \gamma m_2 M_2 &= \alpha + e^{-(\beta m_1+\gamma m_2)}, \\ m_2\delta + \beta m_2 M_1 + \gamma m_2 M_2 &= \alpha + e^{-(\beta M_1+\gamma M_2)}. \end{aligned} \tag{20}$$

From (20) we obtain

$$\delta(M_2 - m_2) + \beta M_2(m_1 - M_1) + \beta M_1(M_2 - m_2) = e^{-(\beta m_1+\gamma m_2)} - e^{-(\beta M_1+\gamma M_2)}. \tag{21}$$

From (21) we have

$$\begin{aligned} &\delta(M_2 - m_2) + \beta M_2(m_1 - M_1) + \beta M_1(M_2 - m_2) \\ &= e^{-(\beta m_1+\gamma m_2+\beta M_1+\gamma M_2)+\theta_2}[\beta(M_1 - m_1) + \gamma(M_2 - m_2)], \end{aligned} \tag{22}$$

where $\beta m_1 + \gamma m_2 \leq \theta_2 \leq \beta M_1 + \gamma M_2$.
From (22) we get

$$\begin{aligned} &(M_1 - m_1)[\beta M_2 + \beta e^{-(\beta m_1+\gamma m_2+\beta M_1+\gamma M_2)+\theta_2}] \\ &\qquad - (M_2 - m_2)[\delta + \beta M_1 - \gamma e^{-(\beta m_1+\gamma m_2+\beta M_1+\gamma M_2)+\theta_2}] = 0. \end{aligned} \tag{23}$$

From two relations (19) and (23) we obtain

$$\begin{aligned} &(M_1 - m_1)\times \\ &\Big[(d + cM_2 - be^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1})(\delta + \beta M_1 - \gamma e^{-(\beta m_1+\gamma m_2+\beta M_1+\gamma M_2)+\theta_2}) \\ &- c\beta(M_1 + e^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1})(M_2 + \beta e^{-(\beta m_1+\gamma m_2+\beta M_1+\gamma M_2)+\theta_2})\Big] = 0. \end{aligned} \tag{24}$$

By using inequality (9), we have

$$\begin{aligned} &(d + cM_2 - be^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1})(\delta + \beta M_1 - \gamma e^{-(\beta m_1+\gamma m_2+\beta M_1+\gamma M_2)+\theta_2}) \\ &> (d + cM_2 - b)(\delta + \beta M_1 - \gamma) > c\beta(1 + M_1)(1 + M_2). \end{aligned} \tag{25}$$

Moreover, we have

$$\begin{aligned} &c\beta(M_1 + e^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1})(M_2 + \beta e^{-(\beta m_1+\gamma m_2+\beta M_1+\gamma M_2)+\theta_2}) \\ &< c\beta(M_1 + 1)(M_2 + 1). \end{aligned} \tag{26}$$

Then from (24), (25) and (26) imply $m_1 = M_1$, so from (19) we have $m_2 = M_2$. Hence from Theorem 2.3 system (4) has a unique positive equilibrium $(\bar{x},\ \bar{y})$ and every positive solution of system (4) tends to the unique positive equilibrium $(\bar{x},\ \bar{y})$ as $n \to \infty$. In addition, the equilibrium $(\bar{x},\ \bar{y})$ is globally asymptotically stable. This completes the proof of the theorem. □

3. Rate of convergence

In this section we give the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x},\ \bar{y})$ of the systems (4) for all values of parameters. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [9] and [10].

The following results give the rate of convergence of solutions of a system of difference equations

$$\mathbf{x}_{n+1} = [A + B(n)]\mathbf{x}_n \tag{27}$$

where $\mathbf{x}_n$ is a k-dimensional vector, $A \in \mathbf{C}^{k\times k}$ is a constant matrix, and $B:\ \mathbb{Z}^+ \longrightarrow \mathbf{C}^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to\ \infty, \tag{28}$$

where $\|\dot{\|}$ denotes any matrix norm which is associated with the vector norm; $\|\dot{\|}$ also denotes the Euclidean norm in $\mathbb{R}^2$ given by

$$\|\mathbf{x}\| = \|(x,\ y)\| = \sqrt{x^2 + y^2}. \tag{29}$$

Theorem 3.1. *([18]) Assume that condition (28) holds. If $\mathbf{x}_n$ is a solution of system (27), then either $\mathbf{x}_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|} \tag{30}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 3.2. *([18]) Assume that condition (28) holds. If $\mathbf{x}_n$ is a solution of system (27), then either $\mathbf{x}_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|} \tag{31}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

The equilibrium point of the system (4) satisfies the following system of equations

$$\begin{cases} \bar{x} &= \dfrac{a + e^{-(b\bar{x}+c\bar{y})}}{d + b\bar{x} + c\bar{y}} \\ \bar{y} &= \dfrac{\alpha + e^{-(\beta\bar{x}+\gamma\bar{y})}}{\delta + \beta\bar{x} + \gamma\bar{y}} \end{cases}. \tag{32}$$

The map T associated to the system (4) is

$$T(x,\ y) = \begin{pmatrix} f(x,\ y) \\ g(x,\ y) \end{pmatrix} = \begin{pmatrix} \dfrac{a + e^{-(bx+cy)}}{d + bx + cy} \\ \dfrac{\alpha + e^{-(\beta x+\gamma y)}}{\delta + \beta x + \gamma y} \end{pmatrix}. \tag{33}$$

The Jacobian matrix of T is

$$J_T(x,y) =$$
$$\begin{pmatrix} \dfrac{-b[a+(d+bx+cy+1)e^{-(bx+cy)}]}{(d+bx+cy)^2} & \dfrac{-c[a+(d+bx+cy+1)e^{-(bx+cy)}]}{(d+bx+cy)^2} \\ \dfrac{-\beta[\alpha+(\alpha+\beta x+\gamma y+1)e^{-(\beta x+\gamma y)}]}{(\alpha+\beta x+\gamma y)^2} & \dfrac{-\gamma[\alpha+(\alpha+\beta x+\gamma y+1)e^{-(\beta x+\gamma y)}]}{(\alpha+\beta x+\gamma y)^2} \end{pmatrix}. \tag{34}$$

By using the system (32), value of the Jacobian matrix of T at the equilibrium point $E = (\bar{x},\ \bar{y})$ is

$$J_T(\bar{x},\ \bar{y}) =$$
$$\begin{pmatrix} \dfrac{-b[a+(d+b\bar{x}+c\bar{y}+1)e^{-(b\bar{x}+c\bar{y})}]}{(d+b\bar{x}+c\bar{y})^2} & \dfrac{-c[a+(d+b\bar{x}+c\bar{y}+1)e^{-(b\bar{x}+c\bar{y})}]}{(d+b\bar{x}+c\bar{y})^2} \\ \dfrac{-\beta[\alpha+(\alpha+\beta\bar{x}+\gamma\bar{y}+1)e^{-(\beta\bar{x}+\gamma\bar{y})}]}{(\alpha+\beta\bar{x}+\gamma\bar{y})^2} & \dfrac{-\gamma[\alpha+(\alpha+\beta\bar{x}+\gamma\bar{y}+1)e^{-(\beta\bar{x}+\gamma\bar{y})}]}{(\alpha+\beta\bar{x}+\gamma\bar{y})^2} \end{pmatrix}. \tag{35}$$

Our goal in this section is to determine the rate of convergence of every solution of the system (4) in the regions where the parameters $a, b, c, d, \alpha, \beta, \gamma, \delta \in (0,\ \infty)$, $(d > b+c, \delta > \beta+\gamma)$ and initial conditions x_0 and y_0 are arbitrary, nonnegative numbers.

Theorem 3.3. *The error vector* $\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \end{pmatrix}$ *of every solution* $(x_n, y_n) \neq (\bar{x}, \bar{y})$ *of* (4) *satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \text{ for some } i = 1,\ 2, \tag{36}$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \text{ for some } i = 1,\ 2, \tag{37}$$

where $|\lambda_i(J_T(E))|$ *is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium* $J_T(E)$.

Proof. First, we will find a system satisfied by the error terms. The error terms are given as

$$\begin{aligned} x_{n+1} - \bar{x} =& \frac{a+e^{-(bx_n+cy_n)}}{d+bx_n+cy_n} - \frac{a+e^{-(b\bar{x}+c\bar{y})}}{d+b\bar{x}+c\bar{y}} \\ =& \frac{a(d+b\bar{x}+c\bar{y}) - a(d+bx_n+cy_n)}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})} \\ &+ \frac{(d+b\bar{x}+c\bar{y})e^{-(bx_n+cy_n)} - (d+bx_n+cy_n)e^{-(b\bar{x}+c\bar{y})}}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})} \\ =& \frac{b(\bar{x}-x_n)+c(\bar{y}-y_n)+[b(\bar{x}-x_n)+c(\bar{y}-y_n)]e^{-(bx_n+cy_n)}}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})} \\ &+ \frac{(d+bx_n+cy_n)[e^{-(bx_n+cy_n)} - e^{-(b\bar{x}+c\bar{y})}]}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})} \\ =& \frac{-b(x_n-\bar{x})-c(y_n-\bar{y})+[b(\bar{x}-x_n)+c(\bar{y}-y_n)]e^{-(bx_n+cy_n)}}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(d+bx_n+cy_n)e^{-(b\bar{x}+c\bar{y})}[e^{-(bx_n-b\bar{x}+cy_n-c\bar{y})}-1]}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}\\
&=\frac{-b(x_n-\bar{x})-c(y_n-\bar{y})+[b(\bar{x}-x_n)+c(\bar{y}-y_n)]e^{-(bx_n+cy_n)}}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}\\
&+\frac{(d+bx_n+cy_n)e^{-(b\bar{x}+c\bar{y})}[-b(x_n-\bar{x})-c(y_n-\bar{y})]}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}\\
&+\frac{\mathcal{O}_1\left((x_n-\bar{x})\right)+\mathcal{O}_2\left((y_n-\bar{y})\right)}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}\\
&=\frac{-b\left[a+e^{-(bx_n+cy_n)}+(d+bx_n+cy_n)e^{-(b\bar{x}+c\bar{y})}\right]}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}(x_n-\bar{x})\\
&+\frac{-c\left[a+e^{-(bx_n+cy_n)}+(d+bx_n+cy_n)e^{-(b\bar{x}+c\bar{y})}\right]}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}(y_n-\bar{y})\\
&+\frac{1}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}\mathcal{O}_1\left((x_n-\bar{x})\right)\\
&+\frac{1}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}\mathcal{O}_2\left((y_n-\bar{y})\right)
\end{aligned}
\tag{38}
$$

By calculating similarly, we get

$$
\begin{aligned}
y_{n+1}-\bar{y}=&\ \frac{-\gamma\left[\alpha+e^{-(\beta x_n+\gamma y_n)}+(\delta+\beta x_n+\gamma y_n)e^{-(\beta\bar{x}+\gamma\bar{y})}\right]}{(\delta+\beta x_n+\gamma y_n)(\delta+\beta\bar{x}+\gamma\bar{y})}(x_n-\bar{x})\\
&+\frac{-\beta\left[\alpha+e^{-(\beta x_n+\gamma y_n)}+(\delta+\beta x_n+\gamma y_n)e^{-(\beta\bar{x}+\gamma\bar{y})}\right]}{(\delta+\beta x_n+\gamma y_n)(\delta+\beta\bar{x}+\gamma\bar{y})}(y_n-\bar{y})\\
&+\frac{1}{(\delta+\beta x_n+\gamma y_n)(\delta+\beta\bar{x}+\gamma\bar{y})}\mathcal{O}_3\left((x_n-\bar{x})\right)\\
&+\frac{1}{(\delta+\beta x_n+\gamma y_n)(\delta+\beta\bar{x}+\gamma\bar{y})}\mathcal{O}_4\left((y_n-\bar{y})\right)
\end{aligned}
\tag{39}
$$

From (38) and (39) we have

$$
\begin{aligned}
x_{n+1}-\bar{x}\approx&\ \frac{-b\left[a+e^{-(bx_n+cy_n)}+(d+bx_n+cy_n)e^{-(b\bar{x}+c\bar{y})}\right]}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}(x_n-\bar{x})\\
&+\frac{-c\left[a+e^{-(bx_n+cy_n)}+(d+bx_n+cy_n)e^{-(b\bar{x}+c\bar{y})}\right]}{(d+bx_n+cy_n)(d+b\bar{x}+c\bar{y})}(y_n-\bar{y})\\
y_{n+1}-\bar{y}\approx&\ \frac{-\gamma\left[\alpha+e^{-(\beta x_n+\gamma y_n)}+(\delta+\beta x_n+\gamma y_n)e^{-(\beta\bar{x}+\gamma\bar{y})}\right]}{(\delta+\beta x_n+\gamma y_n)(\delta+\beta\bar{x}+\gamma\bar{y})}(x_n-\bar{x})\\
&+\frac{-\beta\left[\alpha+e^{-(\beta x_n+\gamma y_n)}+(\delta+\beta x_n+\gamma y_n)e^{-(\beta\bar{x}+\gamma\bar{y})}\right]}{(\delta+\beta x_n+\gamma y_n)(\delta+\beta\bar{x}+\gamma\bar{y})}(y_n-\bar{y}).
\end{aligned}
\tag{40}
$$

Set

$$e_n^1=x_n-\bar{x} \text{ and } e_n^2=y_n-\bar{y}.$$

Then system (40) can be represented as:

$$
\begin{aligned}
e_{n+1}^1&\approx a_ne_n^1+b_ne_n^2\\
e_{n+1}^2&\approx c_ne_n^1+d_ne_n^2
\end{aligned}
$$

where

$$\begin{aligned} a_n &= \frac{-b\left[a + e^{-(bx_n+cy_n)} + (d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}\right]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})}, \\ b_n &= \frac{-c\left[a + e^{-(bx_n+cy_n)} + (d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}\right]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})}, \\ c_n &= \frac{-\gamma\left[\alpha + e^{-(\beta x_n+\gamma y_n)} + (\delta + \beta x_n + \gamma y_n)e^{-(\beta\bar{x}+\gamma\bar{y})}\right]}{(\delta + \beta x_n + \gamma y_n)(\delta + \beta\bar{x} + \gamma\bar{y})}, \\ d_n &= \frac{-\beta\left[\alpha + e^{-(\beta x_n+\gamma y_n)} + (\delta + \beta x_n + \gamma y_n)e^{-(\beta\bar{x}+\gamma\bar{y})}\right]}{(\delta + \beta x_n + \gamma y_n)(\delta + \beta\bar{x} + \gamma\bar{y})}. \end{aligned}$$

Taking the limits of $a_n,\ b_n,\ c_n$ and d_n as $n \to \infty$, we obtain

$$\begin{aligned} \lim_{n\to\infty} a_n &= \frac{-b\left[a + (d + b\bar{x} + c\bar{y} + 1)e^{-(b\bar{x}+c\bar{y})}\right]}{(d + b\bar{x} + c\bar{y})^2} := A_1, \\ \lim_{n\to\infty} b_n &= \frac{-c\left[a + (d + b\bar{x} + c\bar{y} + 1)e^{-(b\bar{x}+c\bar{y})}\right]}{(d + b\bar{x} + c\bar{y})^2} := B_1, \\ \lim_{n\to\infty} c_n &= \frac{-\gamma\left[\alpha + (\delta + \beta\bar{x} + \gamma\bar{y} + 1)e^{-(\beta\bar{x}+\gamma\bar{y})}\right]}{(\delta + \beta\bar{x} + \gamma\bar{y})^2} := C_1, \\ \lim_{n\to\infty} d_n &= \frac{-\beta\left[\alpha + (\delta + \beta\bar{x} + \gamma\bar{y} + 1)e^{-(\beta\bar{x}+\gamma\bar{y})}\right]}{(\delta + \beta\bar{x} + \gamma\bar{y})^2} := D_1, \end{aligned}$$

that is

$$\begin{aligned} a_n = A_1 + \alpha_n,\ b_n = B_1 + \beta_n, \\ c_n = C_1 + \gamma_n,\ d_n = D_1 + \delta_n, \end{aligned}$$

where $\alpha_n \to 0,\ \beta_n \to 0,\ \gamma_n \to 0$ and $\delta_n \to 0$ as $n \to \infty$.

Now, we have system of the form (27):

$$\mathbf{e}_{n+1} = (A + B(n))\mathbf{e}_n,$$

where $A = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix},\quad B(n) = \begin{pmatrix} \alpha_n & \beta_n \\ \delta_n & \gamma_n \end{pmatrix}$ and

$$\|B(n)\| \to 0 \text{ as } n \to\ \infty.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \end{pmatrix} = A \begin{pmatrix} e^1_n \\ e^2_n \end{pmatrix}.$$

The system is exactly linearized system of (4) evaluated at the equilibrium $E = (\bar{x},\ \bar{y})$. Then Theorem 3.1 and Theorem 3.2 imply the result. □

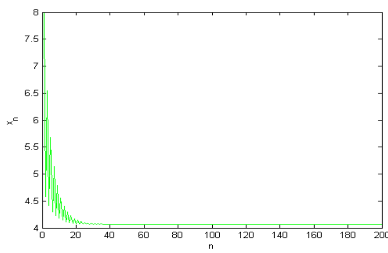
4. Examples

In order to verify our theoretical results and to support our theoretical discussion, we consider several interesting numerical examples. These examples represent different types of qualitative behavior of solutions of the systems (4). All plots in this section are drawn with Matlab.

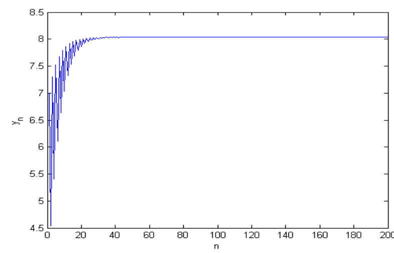
Example 4.1. *Let* $a = 30, b = 0.0007, c = 0.8, d = 0.95, (d > b + c); \alpha = 35, \beta = 0.85, \gamma = 0.0006, \delta = 0.9, (\delta > \beta + \gamma)$. *Then system* (4) *can be written as*

$$x_{n+1} = \frac{30 + e^{-(0.0007x_n + 0.8y_n)}}{0.95 + 0.0007x_n + 0.8y_n}, \; y_{n+1} = \frac{35 + e^{-(0.85x_n + 0.0006y_n)}}{0.9 + 0.85x_n + 0.0006y_n}, \tag{41}$$

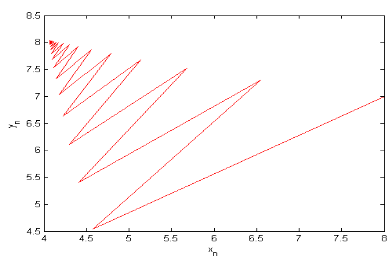
with initial conditions $x_0 = 8$ *and* $y_0 = 7$.



(a) Plot of x_n for the system (41)



(b) Plot of y_n for the system (41)



(c) An attractor of the system (41)

FIGURE 1. Plots for the system (41)

In Figure 1, the plot of x_n is shown in Figure 1 (a), the plot of y_n is shown in Figure 1 (b), and an attractor of the system (41) is shown in Figure 1 (c).

Example 4.2. *Let* $a = 25, b = 0.00009, c = 0.7, d = 0.89, (d > b + c); \alpha = 15, \beta = 0.8, \gamma = 0.00003, \delta = 0.85, (\delta > \beta + \gamma)$. *Then system* (4) *can be written as*

$$x_{n+1} = \frac{25 + e^{-(0.00009x_n + 0.7y_n)}}{0.89 + 0.00009x_n + 0.7y_n}, \; y_{n+1} = \frac{15 + e^{-(0.8x_n + 0.00003y_n)}}{0.95 + 0.8x_n + 0.00003y_n}, \tag{42}$$

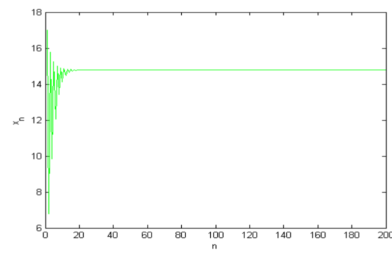
with initial conditions $x_0 = 17$ *and* $y_0 = 4$.

In Figure 2, the plot of x_n is shown in Figure 2 (a), the plot of y_n is shown in Figure 2 (b), and an attractor of the system (42) is shown in Figure 2 (c).

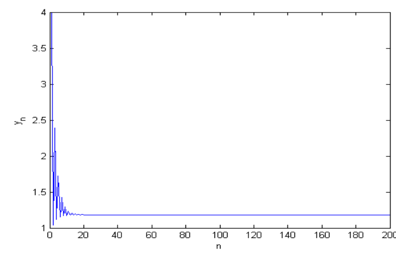
Example 4.3. *Let* $a = 20, b = 0.01, c = 0.6, d = 0.005, (d < b + c); \alpha = 25, \beta = 0.8, \gamma = 0.02, \delta = 0.09, (\delta < \beta + \gamma)$. *Then system* (4) *can be written as*

$$x_{n+1} = \frac{20 + e^{-(0.01x_n + 0.6y_n)}}{0.005 + 0.01x_n + 0.6y_n}, \; y_{n+1} = \frac{25 + e^{-(0.8x_n + 0.02y_n)}}{0.09 + 0.8x_n + 0.02y_n}, \tag{43}$$

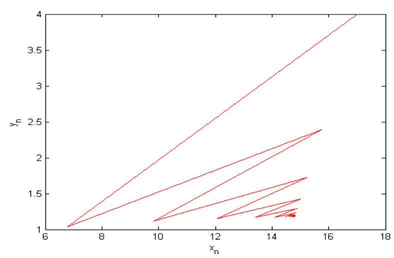
with initial conditions $x_0 = 15$ *and* $y_0 = 2$.



(a) Plot of x_n for the system (42)

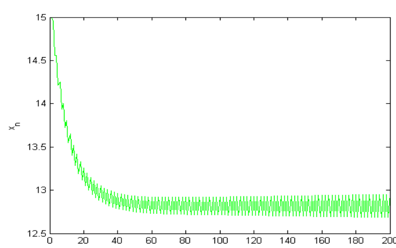


(b) Plot of y_n for the system (42)

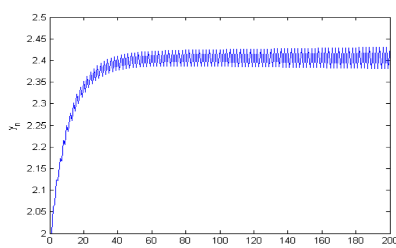


(c) An attractor of the system (42)

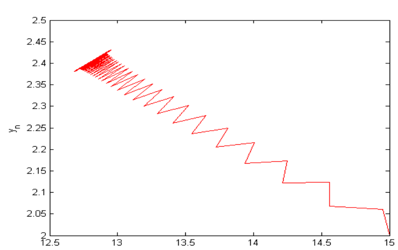
FIGURE 2. Plots for the system (42)



(a) Plot of x_n for the system (43)



(b) Plot of y_n for the system (43)



(c) Phase portrait of the system (43)

FIGURE 3. Plots for the system (43)

In this case, the unique positive equilibrium point of the system (43) is unstable. Moreover, in Figure 3, the plot of x_n is shown in Figure 3 (a), the plot of y_n is shown in Figure 3 (b), and a phase portrait of the system (43) is shown in Figure 3 (c).

References

[1] Agarwal, R.P., *Difference Equations and Inequalities*, Second Ed. Dekker, New York, (2000).

[2] Burgić, DŽ., Nurkanović, Z., *An example of globally asymptotically stable anti-monotonic system of rational difference equations in the plane*, Sarajevo Journal of Mathematics. 5(18) (2009) 235–245.

[3] S. Elaydi, *An Introduction to Difference Equations*, 3rd ed., Springer-Verlag, New York, 2005.

[4] El-Metwally, E., Grove, E.A., Ladas, G., Levins, R., Radin, M., *On the difference equation,* $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$, Nonlinear Anal. 47 (2001) 4623–4634.

[5] Elsayed, E.M., "Dynamics and behavior of a higher order rational difference equation", *J. Nonlinear Sci. Appl.* 9 (2016), pp. 1463474.

[6] Khuong, V. V., Phong, M. N., On the system of two difference equations of exponential form, *Int. J. Difference Equ.* 8(5) (2013), 215 - 223.

[7] Khuong, V. V., Phong, M. N., Global behavior and the rate of convergence of a system of two difference equations of exponential form, *PanAmer. Math. J.* 27(1)(2017), pp. 67-78.

[8] Kocic, V. L., Ladas, G., *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer Academic, Dordrecht, (1993).

[9] Kulenović, M. R. S., Nurkanović, Z.,*The rate of convergence of solution of a three dimensional linear fractional systems of difference equations*, Zbornik radova PMF Tuzla - Svezak Matematika. 2 (2005) 1–6.

[10] Kulenović, M.R.S., Nurkanović, M., *Asymptotic behavior of a competitive system of linear fractional difference equations*, Adv. Difference Equ. Art. ID 19756 (2006) 13pp.

[11] Ozturk, I., Bozkurt, F., Ozen, S., *On the difference equation* $y_{n+1} = (\alpha + \beta e^{-y_n})/(\gamma + y_{n-1})$, Appl. Math. Comput.181 (2006) 1387–1393.

[12] Papaschinopoluos, G., Ellina, G., Papadopoulos, K.B., *Asymptotic behavior of the positive solutions of an exponential type system of difference equations*, Appl. Math. Comput. 245 (2014) 181–190.

[13] Papaschinopoluos, G., Fotiades, N., Schinas, C.J., *On a system of difference equations including negative exponential terms*, J. Difference Equ. Appl. 20(5-6) (2014) 717–732.

[14] Papaschinopoluos, G., Radin, M.A., Schinas, C.J., *On a system of two difference equations of exponential form:* $x_{n+1} = a + bx_{n-1}e^{-y_n}$, $y_{n+1} = c + dy_{n-1}e^{-x_n}$, Math. Comput. Model. 54 (2011) 2969–12977.

[15] Papaschinopoluos, G., Radin, M.A., Schinas, C.J., *Study of the asymptotic behavior of the solutions of three systems of difference equations of exponential form*, Appl. Math. Comput. 218 (2012) 5310–5318.

[16] Phong, M.N.,*A note on a system of two nonlinear difference equations*, Electronic Journal of Mathematical Analysis and Applications, 3(1) (2015) 170 –179.

[17] Phong, M.N., Khuong, V.V., Asymptotic behavior of a system of two difference equations of exponential form, *Int. J. Nonlinear Anal. Appl.* **7**(2)(2016), pp. 319-329.

[18] Pituk, M.,*More on Poincare's and Peron's theorems for difference equations*, J. Difference Equ. Appl. 8 (2002) 201–216.

[19] Stefanidou, G., Papaschinopoluos, G., Schinas, C.J.,*On a system of two exponential type difference equations*, Com. Appl. Nonlinear Anal. 17(2) (2010) 1–13.

Mai Nam Phong

Department of Mathematical Analysis, University of Transport and Communications, Hanoi City, Vietnam

E-mail address: mnphong@utc.edu.vn