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FIXED POINTS FOR AN ORDERED F-CONTRACTION MAPPING IN ORDERED PARTIAL METRIC SPACES

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Abstract. The concept of ordered F-contraction in an ordered metric space was introduced by Durmaz et. al. [4] and was proven to be very useful in the existing metric fixed point theory. In this paper, the notion of ordered F-contraction in an ordered metric space will be generalized in an ordered partial metric spaces. In particular, the main results generalize a fixed point theorem due to Durmaz et. al. [4]. Also an illustrative example is provided to validate our results.

1. Introduction and Preliminaries

The study of fixed points for partially ordered sets on complete metric spaces was introduced by Ran and Reurings [9] who proved the analougue of Banach Contraction Principle in partially ordered sets. Nieto and Rodriguez [7] further extended the work of Ran and Reurings [9].

Following Ran and Reurings [9], Durmaz et. al. [4] introduced the concept of ordered F-contraction in an ordered metric space and proved the following fixed point theorem.

Let $(X, d, \preceq)$ be an ordered complete metric space and $T : X \to X$ be an ordered F-contraction. Let T be a non-decreasing map and there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. If T is continuous or X is regular then T has a fixed point.

In partial metric spaces, Matthews [6] proved the analogue of Banach Contraction Principle and established a fixed point theorem as a generalization of metric spaces in partial metric spaces. Matthews [6] provided the following definition:

Definition 1 [6] Let X be non-empty set. A partial metric space is a pair (X, p), where p is a function $p : X \times X \to \mathbb{R}^+$, called the partial metric, such that for all $x, y, z \in X$ the following axioms hold:

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(P1) $x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$; and
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Clearly, by (P1)-(P3), if $p(x, y) = 0$, then $x = y$. But, the converse is in general not true.

The most common example of partial metric spaces is a pair $([0, \infty), p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. More examples of partial metric spaces may be found in [2].

Each partial metric p on X generates a T_0 topology τ_p on X whose basis is the collection of all open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$ where
$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$, and ϵ is a positive real number.

The following definitions and facts will be used to establish our main results:

Definition 2 [6] Let (X, p) be a partial metric space. Then:

(i) a sequence $\{x_n\}$ in (X, p) is said to be convergent to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.
(ii) a sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
(iii) a partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to the topology τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

The following lemma was proved by Bukatin et.al. [2]

Lemma 1[2] Let (X, p) be a partial metric space. Then the mapping $p^s : X \times X \to [0, \infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$ defines a metric on X.

Bukatin et. al. [2] also proved the following lemma:

Lemma 2[2] Let (X, p) be a partial metric space. Then:

(i) a sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).
(ii) a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.

Paesano and Vetro [8] provided the following definition:

Definition 3[8] If (X, p) is a partial metric space and $(X, \preceq)$ is partially ordered set, then $(X, p, \preceq)$ is called an ordered partial metric. We say that $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$ holds. Further a self map $T : X \to X$ is called non-decreasing if $Tx \preceq Ty$ whenever $x \preceq y$ for all $x, y \in X$ and an ordered partial metric space $(X, p, \preceq)$ is regular if the following holds:
For every non decreasing sequence $\{x_n\}$ in X convergent to some $x \in X$, we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Paesano and Vetro [8] provided the following definition:

Definition 4[8] Let $(X, \preceq)$ be a partially ordered set. Let A and B be two non-empty subset of X. Two relation between A and B are denoted and defined as follows;
$(r1)$ $A \prec_1 B$ if for each $a \in A$ there exists $b \in B$ such that $a \preceq b$.
$(r2)$ $A \prec_2 B$ if for each $a \in A$ and $b \in B$, we have $a \preceq b$.

In 2012, Wardowski introduced an F-contraction mapping and defined it as follows:

Definition 5[10] Let (M, d) be a metric space, a mapping $T : M \longrightarrow M$ is said to be an F-contraction on M if there exists $\tau > 0$ such that, for all $x, y \in M$,

$$d(Tx, Ty) \geq 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{1}$$

and $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ a mapping satisfying the following conditions:

- F1: F is strictly increasing, that is for all $x, y \in \mathbb{R}_+$ such that $x \leq y \Rightarrow F(x) \leq F(y)$.
- F2: For each sequence $\{\alpha_n\}_{n\geq 1}$ of positive numbers $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$.
- F3: There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote by Δ_F the set of all functions satisfying the conditions $(F1) - (F3)$

Remark 1: From $(F1)$ and contractive condition (1), we observe that every F-contraction is necessarily continuous.

Durmaz et. al. [4] introduced the concept of Ordered F-contraction in an ordered complete metric space:

Definition 6[4] Let $(X, \preceq, d)$ be an ordered metric space and $T : X \to X$ be a mapping. Let $Y = \{(x, y) \in X \times X : x \leq y, d(Tx, Ty) > 0\}$ we say that T is an ordered F-contraction if $F \in \Delta_F$ and there exists $\tau > 0$ such that;

$$\text{for all}(x, y) \in Y \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{2}$$

Durmaz et. al. [4] proved the following theorem:

Theorem 1[4] Let $(X, d, \preceq)$ be an ordered complete metric space and $T : X \to X$ be an ordered F-contraction. Let T be a non -decreasing map and there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. If T is continuous or X is regular then T has a fixed point.

The purpose of this work is to extend Theorem 1 to an ordered partial metric space in order to obtain a fixed point theorem for an ordered F-contraction in partial metric spaces.

2. Main Results

First we will provide the extension of Definition 6 in an ordered partial metric space which is as follows:

Definition 7 Let $(X, \preceq, p)$ be an ordered partial metric space and $T : X \to X$ be a mapping. Also let $Y = \{(x, y) \in X \times X : x \leq y, p(Tx, Ty) > 0\}$, we say that T is an ordered F-contraction if $F \in \Delta_F$ and there exists $\tau > 0$ such that,

$$\text{for all } (x, y) \in Y \Rightarrow \tau + F(p(Tx, Ty)) \leq F(p(x, y)). \tag{3}$$

Now, we will extend Theorem 1 in partial metric space:

Theorem 2 Let $(X, p, \preceq)$ be an ordered complete partial metric space and $T : X \to X$ be an ordered F-contraction. If the following holds:
(i) T is a non-decreasing map,
(ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and
(iii) X is regular,
then T has a fixed point.

Proof: By assumption (ii) let $x_0 \in X$ such that $x_0 \preceq Tx_0$. We can construct a monotone increasing sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n = 0, 1, 2, ...$ or $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$.
If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = x_{n_0+1}$ then x_{n_0} is a fixed point of T and so this ends the proof. Now suppose that for every $n \in \mathbb{N}$, $x_{n+1} \neq x_n$. Since $x_0 \preceq Tx_0$ and T is non-decreasing then,

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq ... \preceq x_n \preceq ...$$

Also since $x_n \preceq x_{n+1}$ and $p(Tx_n, Tx_{n-1}) > 0$ for all $n \in \mathbb{N}$, thus $(x_n, x_{n+1}) \in Y$ of Definition 7.
From the contractive condition (3) of Definition 7, we can now write,

$$F(p(x_{n+1}, x_n)) = F(p(Tx_n, Tx_{n-1})) \leq F(p(x_n, x_{n-1})) - \tau. \tag{4}$$

We denote $c_n = p(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ then, from (4) we now have,

$$F(c_n) \leq F(c_{n-1}) - \tau \leq F(c_{n-2}) - 2\tau \leq ... \leq F(c_0) - n\tau. \tag{5}$$

From (5) we get $\lim_{n\to\infty} F(c_n) = -\infty$ and by $(F2)$ of Definition 5, we obtain that,

$$\lim_{n\to\infty} (c_n) = 0.$$

By $(F3)$ of Definition 5, there exists $k \in (0, 1)$ such that, $\lim_{n\to\infty} {c_n}^k F(c_n) = 0$.
From (5) the following holds for all $n \in \mathbb{N}$,

$${c_n}^k(F(c_n) - F(c_0)) \leq -{c_n}^k n\tau \leq 0. \tag{6}$$

letting $n \to \infty$ in (6) we obtain that,

$$\lim_{n\to\infty} n{c_n}^k = 0. \tag{7}$$

From (7) there exists $n_1 \in \mathbb{N}$ such that $n{c_n}^k \leq 1$ for all $n \geq n_1$. Hence we have,

$$c_n \leq \frac{1}{n^{\frac{1}{k}}}, \text{ for all } n \geq n_1. \tag{8}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using $(P3)$ of Definition 1 and (8) we now have,

$$\begin{aligned} p(x_n, x_m) \leq & p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + ... + p(x_{m-1}, x_m) - \sum_{j=n+1}^{m-1} p(x_j, x_j) \\ \leq & p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + ... + p(x_{m-1}, x_m) \\ = & c_n + c_{n+1} + ... + c_{m-1} \\ = & \sum_{i=n}^{m-1} c_i \leq \sum_{i=n}^{\infty} c_i \\ \leq & \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ implies that $\lim_{n\to\infty} p(x_n, x_m) = 0$.
By Lemma 1 we get that, for any $n.m \in \mathbb{N}, p^s(x_n, x_m) \leq 2p(x_n, x_m) \to 0$ as $n \to \infty$, this implies that, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p^s and hence converges by Lemma 1. Thus there exists $z \in X$ such that, $\lim_{n\to\infty} p^s(x_n, z) = 0$. Moreover by Lemma 1 we have,

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n.x_m). \tag{9}$$

By the continuity of T we now have,

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_n = Tz.$$

Hence z is a fixed point of T.
Also from the assumption that X is regular then, $x_n \preceq z$ for all $n \in \mathbb{N}$. We now consider the following:
(i) If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = z$ then $Tz = Tx_{n_0} = x_{n_0+1} \preceq z$. Also since $x_{n_0} \preceq x_{n_0+1}$ then $z \preceq Tz$, hence $z = Tz$.
(ii) We now suppose that $x_n \neq z$ for every $n \in \mathbb{N}$ and $p(z, Tz) > 0$. Since $\lim_{n\to\infty} x_n = z$ then, there exists $n_1 \in \mathbb{N}$ such that $p(x_{n+1}, Tz) > 0$ and $p(x_n, z) < \frac{1}{2}p(z, Tz)$, for all $n \geq n_1$ where $(x_n, z) \in Y$.
Therefore from $(F1)$ of Definition 1 we obtain that, for all $n \geq n_1$

$$\tau + F(p(Tx_n, Tz)) \leq F(p(x_n, z)) \leq \frac{1}{2}F(p(z, Tz)). \tag{10}$$

From (10) which we obtain,

$$p(x_{n+1}, Tz) \leq \frac{1}{2}p(z, Tz). \tag{11}$$

Letting $n \to \infty$ in (11) we obtain that, $p(z, Tz) \leq \frac{1}{2}p(z, Tz)$ which is a contradiction. Thus we conclude $p(z, Tz) = 0$. Hence $z = Tz$.
Now we show that z is a unique fixed point of T.

We now show that T has a unique fixed point and this can be proved by showing that for every $x \in X$, $\lim_{n\to\infty} T^n x = z$ where z is a fixed point of T such that $z = \lim_{n\to\infty} T^n x_0$.
Let $x \in X$ and $x_0 \preceq Tx_0$ by hypothesis (ii), consider the following cases:

(iii) if $x \preceq x_0$ or $x_0 \preceq x$ then $T^n x \preceq T^n x_0$ or $T^n x_0 \preceq T^n x$ for all $n \in \mathbb{N}$. If $T^{n_0} x = T^{n_0} x_0$ for some $n_0 \in \mathbb{N}$ then $T^n x \to z$.
We now let $T^n x_0 \neq T^n x$ for all $n \in \mathbb{N}$, then $p(T^n x_0, T^n x) > 0$ and so, $(T^n x_0, T^n x) \in Y$ of Definition 7 for all $n \in \mathbb{N}$. Thus from 3 we obtain,

$$\begin{aligned} F(p(T^n x_0, T^n x)) \leq & F(p(T^{n-1} x_0, T^{n-1} x)) - \tau \\ \leq & F(p(T^{n-2} x_0, T^{n-2} x)) \\ & . \\ & . \\ & . \\ \leq & F(p(x_0, x)) - n\tau. \end{aligned}$$

By (F_2) of Definition 5 we have $\lim_{n\to\infty} p(T^n x_0, T^n x) = 0$ and so,

$$\lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} T^n x = z.$$

(iv) If $x \npreceq x_0$ or $x_0 \npreceq x$ then by using the assumption that every pair of elements has a lower bound and upper bound, there exists $x_1, x_2 \in X$ such that $x_2 \preceq x \preceq x_1$ and $x_2 \preceq x_0 \preceq x_1$ therefore as in case (iii) we now show that,

$$\lim_{n\to\infty} T^n x_1 = \lim_{n\to\infty} T^n x_2 = \lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^n x_0 = z.$$

Hence the fixed point of T is unique.

Corollary 1 Let $(X, \preceq, p)$ be an ordered partial metric space and a map $T : X \to X$ is an ordered F-contraction, if every pair of elements $(x, y) \in X \times X$ has lower and upper bounds then, T has a unique fixed point.

Example 2 Let $X = \{\frac{1}{n^4}; n \in \mathbb{N}\} \cup \{0\}$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Define an order relation $\preceq$ on X as $x \leq y \Leftrightarrow [x = y$ or $x, y \in X$ with $x \leq y]$ where $\leq$ is usual order. It is clear that $(X, \leq, p)$ is an ordered complete partial metric space. Let $T : X \to X$ be given by,

$$T(x) = \begin{cases} \frac{1}{(n+1)^2} & , \quad x = \frac{1}{n^4} \\ x & , \quad x \in \{0\} \end{cases}.$$

It is clear that T is non decreasing and for $x_0 = 0$ we have $x_0 \preceq T x_0$. Also taking $F(\alpha) = \ln(\alpha)$ for all $\alpha > 0$, obviously F satisfies the conditions $(F1 - F3)$ and T is an ordered F-contraction with $\tau = \ln(2)$. We now consider that,

$$\begin{aligned} Y =& \{(x, y) \in X \times X : x \leq y, p(Tx, Ty) > 0\} \\ =& \{(x, y) \in X \times X : x, y \in X \text{ and } x < y\}. \end{aligned}$$

Therefore, to observe the contractive condition (3) it is enough to show that,

$$\begin{aligned} \text{for all } (x, y) \in Y \Rightarrow & \ln(2) + F(p(Tx, Ty)) \leq F(p(x, y)) \\ \Leftrightarrow & x, y \in X \text{ and } x < y \Rightarrow \frac{p(Tx, Ty)}{p(x, y)} \leq \frac{1}{2}. \end{aligned}$$

$$\Leftrightarrow x, y \in X \text{ and } x < y \Rightarrow \frac{\max\{Tx, Ty\}}{\max\{x, y\}} \leq \frac{1}{2}. \tag{12}$$

Thus, we see that (12) is true. Also T is continuous (and X is regular). Therefore, all conditions of Theorem 2 are satisfied and so T has a fixed point in X.

Remark 2 As we observe in Example 2, if the assumption that every pair of elements has a lower bound and upper bound is not satisfied then, a fixed point of T may not be unique.

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