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INEQUALITIES FOR FIBONACCI HYPERBOLIC FUNCTIONS AND THEIR APPLICATIONS

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Abstract. In 2010, Zhu introduced inequalities for hyperbolic functions. In this paper we study inequalities for Fibonacci hyperbolic function. A basic theorem is established and found to be a source of inequalities Fibonacci hyperbolic function, such as the ones of Cusa, Huygens, Wilker, Sandor-Bencze, Carlson, Shafer-Fink type inequality, and the one in the form of Oppenheim's problem. Furthermore, these inequalities described above will be extended by this basic theorem.

1. Introduction

In the study [27], a basic theorem is established and found to be a source of inequalities for circular functions, and these inequalities are extended by this basic theorem. In what follows we are going to present the counterpart of these results for the Fibonacci hyperbolic functions.

In this paper, we first establish the following Cusa-type inequalities in exponential type for Fibonacci hyperbolic functions described as Theorem 9. The using the results of Theorem 9, we obtain Huygens, Wilker, Sandor-Bencze, Carlson, and Shafer-Fink-type inequalities respectively.

One of the simplest and most celebrated integer sequences is the Fibonacci sequence. The Fibonacci sequence is $F_n = \{0, 1, 1, 2, 3, 5, ...\}$ where in each term is the sum of the two proceeding terms, beginning with the values $F_0 = 0$ and $F_1 = 1$. It is interesting to emphasize the fact that the ratio of two consecutive Fibonacci numbers converges to the Golden Mean, or Golden Section, $\phi = \frac{1+\sqrt{5}}{2}$, which appears in modern research in many fields from architecture to physics of the high energy particles or theoretical physics other than mathematical areas by Cigler [3] and Falcón and Plaza [4, 5] et al.

On the other hand, recently, the Fibonacci hyperbolic functions have been defined as $\operatorname{sinFh} x$ and $\operatorname{cosFh} x$ by Stakhov [11]; Stakhov and Rozin [12, 13, 14, 15].

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2. Fibonacci Hyperbolic Functions

There are a number of fundamental results in the Fibonacci numbers theory. One of them was found in 19*th* century by the famous French mathematician Binet. Studying the Golden Section, Fibonacci numbers, he discovered remarkable formulas, Binet's formulas, connecting Fibonacci numbers with the Golden Section $\phi = \frac{1+\sqrt{5}}{2}$. Where ϕ is the root of the characteristic equation $\left(\phi^2 = \phi + 1\right)$ associated to the Fibonacci Numbers sequence. Binet's formulas give connections between the "extended" Fibonacci Numbers and the Golden Section may be written in the following from:

$$F_n = \begin{cases} \frac{\phi^{2k+1}+\phi^{-(2k+1)}}{\sqrt{5}}, & n = 2k+1 \\ \frac{\phi^{2k}+\phi^{-2k}}{\sqrt{5}}, & n = 2k \end{cases} \tag{2.1}$$

where the discrete variable k takes its values from the set $0, \pm 1, \pm 2, \pm 3, \ldots$. The formula (2.1) was replaced with the continuous variable x taken its values from the set of the real numbers and them the following continuous functions called the Fibonacci hyperbolic functions were introduced by Stakhov and Tkachenko [10].

Definition 1. *([12]) Let ϕ be Golden ratio. Hence, the Fibonacci hyperbolic functions are defined as:*

$$\operatorname{sinFh} x = \frac{\phi^{2x} - \phi^{-2x}}{\sqrt{5}} \tag{2.2}$$

$$\operatorname{cosFh} x = \frac{\phi^{2x+1} + \phi^{-(2x+1)}}{\sqrt{5}} \tag{2.3}$$

where $\operatorname{sinFh} x$ *and* $\operatorname{cosFh} x$ *are called, respectively.*

By doing a change of variable $2x + 1 = t$ in functions $\operatorname{sinFh} x$ and $\operatorname{cosFh} x$ and also using $\phi + \phi^{-1} = \sqrt{5}$. Another representations of Fibonacci hyperbolic sine and cosine functions are, respectively, given as follows by Stakhov and Tkachenko [10].

Definition 2. *([12]) Let ϕ be Golden ratio.*

$$\operatorname{sinFh} x = \frac{\phi^{x} - \phi^{-x}}{\phi + \phi^{-1}} \tag{2.4}$$

$$\operatorname{cosFh} x = \frac{\phi^{x} + \phi^{-x}}{\phi + \phi^{-1}}. \tag{2.5}$$

The introduced above Fibonacci hyperbolic functions is connected with the hyperbolic functions by the following simple correlations [12]:

$$\operatorname{sinFh} x = \frac{2}{\sqrt{5}} \sinh\left((\ln \phi)\, x\right) \tag{2.6}$$

$$\operatorname{cosFh} x = \frac{2}{\sqrt{5}} \cosh\left((\ln \phi)\, x\right) \tag{2.7}$$

Now, we shall introduce here inequalities for Hyperbolic functions and their applications related with the inequalities for Fibonacci hyperbolic functions studied before.

Theorem 1. (*Cusa-type inequalities*) *([28]) Let $x > 0$. Then the following are considered.*

(i) If $p \geq \frac{4}{5}$, the double inequality

$$(1-\lambda) + \lambda\,(\cosh x)^p < \left(\frac{\sinh x}{x}\right)^p < (1-\eta) + \eta\,(\cosh x)^p \tag{2.8}$$

holds if and only if $\eta \geq \frac{1}{3}$ and $\lambda \leq 0$.

(*ii*) *If* $p < 0$, *the inequality*

$$\left(\frac{\sinh x}{x}\right)^p < (1-\eta) + \eta\,(\cosh x)^p \tag{2.9}$$

holds if and only if $\eta \le \frac{1}{3}$.

Theorem 2. (*Huygens-type inequalities*) *([28]) Let* $x > 0$. *Then one has the following.*

(*i*) *When* $p \ge \frac{4}{5}$, *the double inequality*

$$(1-\lambda)\left(\frac{x}{\sinh x}\right)^p + \lambda\left(\frac{x}{\tanh x}\right)^p < 1 < (1-\eta)\left(\frac{x}{\sinh x}\right)^p + \eta\left(\frac{x}{\tanh x}\right)^p \tag{2.10}$$

holds if and only if $\eta \ge \frac{1}{3}$ *and* $\lambda \le 0$.

(*ii*) *When* $p < 0$, *the inequality*

$$\left(\frac{\sinh x}{x}\right)^p < (1-\eta) + \eta\,(\cosh x)^p \tag{2.11}$$

holds if and only if $\eta \le \frac{1}{3}$.

Theorem 3. (*Wilker-type inequalities*) *([28]) Let* $x > 0$. *Then the following are considered.*

(*i*) *If* $\alpha > 0$, *the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} + \left(\frac{\tanh x}{x}\right)^{\alpha} > \left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha} \tag{2.12}$$

holds.

(*ii*) *If* $\alpha \ge \frac{4}{5}$, *then the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} + \left(\frac{\tanh x}{x}\right)^{\alpha} > \left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha} > 2 \tag{2.13}$$

holds.

Corollary 1. (*First Wilker-type inequalities*) *([28]) One has that*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 \tag{2.14}$$

holds for all $x \in (0, +\infty)$.

Corollary 2. (*Second Wilker-type inequalities*) *([28]) One has that*

$$\left(\frac{x}{\sinh x}\right)^2 + \left(\frac{x}{\tanh x}\right) > 2 \tag{2.15}$$

holds for all $x \in (0, +\infty)$.

Theorem 4. (*Sandor–Bencze-type inequalities*) *([28]) Let* $x > 0$. *Then the following are considered.*

(*i*) *When* $\alpha \ge \frac{4}{5}$, *One has*

$$\left(\frac{\sinh x}{x}\right)^{\alpha} < \frac{2}{3} + \frac{1}{3}\,(\cosh x)^{\alpha} < \frac{(\cosh x)^{\alpha} + \sqrt{(\cosh x)^{2\alpha} + 8}}{4}. \tag{2.16}$$

(*ii*) *When* $\alpha > 0$, *One has*

$$\left(\frac{x}{\sinh x}\right)^{\alpha} < \frac{2}{3} + \frac{1}{3}\left(\frac{1}{\cosh x}\right)^{\alpha} < \frac{1 + \sqrt{8(\cosh x)^{2\alpha} + 1}}{4(\cosh x)^{\alpha}} < \left(\frac{1}{\cosh x}\right)^{\alpha} + 1. \tag{2.17}$$

Theorem 5. *(Carlson-type inequalities) ([28]) Let* $x > 1$. *Then the following are considered.*

(*i*) *When* $p \ge \frac{4}{5}$, *the double inequality*

$$\frac{3\left(2\sqrt{x-1}\right)^p}{\left(2\sqrt{2}\right)^p + \left(\sqrt{1+x}\right)^p} < \left(\cosh^{-1} x\right)^p < \frac{\left(4^{1/3}\sqrt{x-1}\right)^p}{(1+x)^{p/6}} \tag{2.18}$$

holds.

(ii) When $p<0$, the left inequality of (2.18) holds too.

Theorem 6. *(Shafer-Fink-type inequalities and an extension of the problem of Oppenheim) ([28]) Let $t>0$, $p\geq\frac{4}{5}$ or $p<0$. Then the inequality*

$$\frac{3t^p}{2+\left(\sqrt{1+t^2}\right)^p}<\left(\sinh^{-1}t\right)^p \tag{2.19}$$

holds.

Theorem 7. *(Shafer-Fink-type inequalities and an extension of the problem of Oppenheim) ([28]) Let $t>0$, $p\geq\frac{4}{5}$ or $p<0$. Then the inequality*

$$\frac{6\left(\sqrt{2}\right)^p\left(\sqrt{1+t^2}-1\right)^{p/2}}{4+\left(\sqrt{2}\right)^{2-p}\left(\sqrt{1+t^2}+1\right)^{p/2}}<\left(\sinh^{-1}t\right)^p \tag{2.20}$$

holds.

Theorem 8. *(Shafer-Fink-type inequalities and an extension of the problem of Oppenheim) ([28]) Let $x>0$, $p\geq\frac{4}{5}$ or $p<0$. Then the inequality*

$$\frac{3}{2}\frac{\sinh^p x}{1+\frac{1}{2}\cosh^p x}<x^p \tag{2.21}$$

holds.

We establish the following Cusa like inequalities in type for Fibonacci hyperbolic functions.

Theorem 9. *(Fibonacci Cusa-type inequalities) Let $x>\frac{285}{40612}$. Then the following are considered.*

(i) If $p\geq 115411608$, the double inequality

$$\begin{aligned}\left((0,43041)^p-\lambda\,(0,89443)^p\right)+\lambda\,(\operatorname{cosFh}x)^p&<\left(\tfrac{\operatorname{sinFh}x}{x}\right)^p\\&<\left((0,43041)^p-\eta\,(0,89443)^p\right)+\eta\,(\operatorname{cosFh}x)^p\end{aligned} \tag{3.1}$$

holds if and only if $\eta\geq(0,4812)^{p-1}\,(0,00036506)$ and $\lambda\leq 0$.

(ii) If $p<0$, the inequality

$$\left(\tfrac{\operatorname{sinFh}x}{x}\right)^p<\left((0,43041)^p-\eta\,(0,89443)^p\right)+\eta\,(\operatorname{cosFh}x)^p \tag{3.2}$$

holds if and only if $\eta\leq(0,4812)^{p-1}\,(0,00036506)$.

That is, let $\alpha>0$, the inequality

$$\left(\tfrac{x}{\operatorname{sinFh}x}\right)^\alpha<\left(\left(\tfrac{1}{0,43041}\right)^\alpha-\eta\left(\tfrac{1}{0,89443}\right)^\alpha\right)+\eta\left(\tfrac{1}{\operatorname{cosFh}x}\right)^\alpha \tag{3.3}$$

holds if and only if $\eta\leq(0,4812)^{p-1}\,(0,00036506)$.

Lemma 1. *(see [1, 26]). Let $f,g:[a,b]\to\mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further, let $g'\neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the functions $\left(\frac{f(x)-f\left(b^-\right)}{g(x)-g\left(b^-\right)}\right)$ and $\left(\frac{f(x)-f\left(a^+\right)}{g(x)-g\left(a^+\right)}\right)$ are also increasing (or decreasing) on (a,b).*

Lemma 2. *Let $x\in[1,\infty)$. Then the inequalities*

$$D_1(x)\triangleq-\tfrac{8}{5}\left(\ln\phi\right)^2x^2\operatorname{cosFh}x+\tfrac{4}{5}\left(\ln\phi\right)x\operatorname{sinFh}x+\operatorname{sinFh}^2x\operatorname{cosFh}x>0, \tag{3.4}$$

$$D_2(x)\triangleq\tfrac{4}{5}\left(\ln\phi\right)^2x^2\operatorname{cosFh}x-\tfrac{4}{5}\left(\ln\phi\right)x\operatorname{sinFh}x-\left(\ln\phi\right)x\operatorname{sinFh}x\operatorname{cosFh}^2x+ \tag{3.6}$$

$$+\operatorname{sinFh}^2x\operatorname{cosFh}x<0,$$

$$D_3(x) \triangleq \left(\tfrac{461646424}{5}\right)(\ln\phi)^2 x^2 \operatorname{cosFh} x - (115411608)(\ln\phi)\, x \operatorname{sinFh} x \operatorname{cosFh}^2 x + \quad (3.5)$$
$$+ (115411609) \operatorname{sinFh}^2 x \operatorname{cosFh} x - \left(\tfrac{461646428}{5}\right)(\ln\phi)\, x \operatorname{sinFh} x < 0,$$

hold.

Proof. Using the infinite series of $\sinh x$ and $\cosh x$, we have

$$\begin{aligned}
D_1(x) &= -\tfrac{8}{5}(\ln\phi)^2 x^2 \operatorname{cosFh} x + \tfrac{4}{5}(\ln\phi)\, x \operatorname{sinFh} x + \tfrac{1}{5}(\operatorname{cosFh} 3x - \operatorname{cosFh} x)\\
&= -\tfrac{16}{5\sqrt{5}}(\ln\phi)^2 x^2 \cosh((\ln\phi)\, x) + \tfrac{8}{5\sqrt{5}}(\ln\phi)\, x \sinh((\ln\phi)\, x)\\
&+ \tfrac{2}{5\sqrt{5}}(\cosh((\ln\phi)\, 3x) - \cosh((\ln\phi)\, x))\\
&= \tfrac{1}{5\sqrt{5}}\left[\sum_{n=0}^{\infty} \tfrac{(-16)(\ln\phi)^{2n+2}}{(2n)!} x^{2n+2} + \sum_{n=0}^{\infty} \tfrac{8(\ln\phi)^{2n+2}}{(2n+1)!} x^{2n+2}\right]\\
&+ \tfrac{1}{5\sqrt{5}}\left[\sum_{n=1}^{\infty} \tfrac{2\left(3^{2n}-1\right)(\ln\phi)^{2n}}{(2n)!} x^{2n}\right]\\
&= \tfrac{1}{5\sqrt{5}}\left[\sum_{n=0}^{\infty}\left(\tfrac{(-16)\left(4n^2+6n+2\right)+8(2n+2)+2\left(3^{2n+2}-1\right)}{(2n+2)!}\right)(\ln\phi)^{2n+2} x^{2n+2}\right]\\
&= \tfrac{1}{5\sqrt{5}}\left[\sum_{n=2}^{\infty}\left(\tfrac{(-16)\left(4n^2+6n+2\right)+8(2n+2)+2\left(3^{2n+2}-1\right)}{(2n+2)!}\right)(\ln\phi)^{2n+2} x^{2n+2}\right] > 0,
\end{aligned}$$

$$\begin{aligned}
D_2(x) &= \tfrac{4}{5}(\ln\phi)^2 x^2 \operatorname{cosFh} x - \tfrac{4}{5}(\ln\phi)\, x \operatorname{sinFh} x - (\ln\phi)\, x \tfrac{1}{5}(\operatorname{sinFh} 3x + \operatorname{sinFh} x)\\
&= \tfrac{8}{5\sqrt{5}}(\ln\phi)^2 x^2 \cosh((\ln\phi)\, x) - \tfrac{8}{5\sqrt{5}}(\ln\phi)\, x \sinh((\ln\phi)\, x)\\
&- \tfrac{2}{5\sqrt{5}}(\ln\phi)\, x\,(\sinh((\ln\phi)\, 3x) + \sinh((\ln\phi)\, x))\\
&+ \tfrac{2}{5\sqrt{5}}(\cosh((\ln\phi)\, 3x) - \cosh((\ln\phi)\, x))\\
&= \tfrac{1}{5\sqrt{5}}\left[\sum_{n=0}^{\infty} \tfrac{8(\ln\phi)^{2n+2}}{(2n)!} x^{2n+2} - \sum_{n=0}^{\infty} \tfrac{8(\ln\phi)^{2n+2}}{(2n+1)!} x^{2n+2}\right]\\
&- \tfrac{1}{5\sqrt{5}}\left[\sum_{n=0}^{\infty} \tfrac{2\left(3^{2n+1}+1\right)(\ln\phi)^{2n+2}}{(2n+1)!} x^{2n+2} + \sum_{n=1}^{\infty} \tfrac{2\left(3^{2n}-1\right)(\ln\phi)^{2n}}{(2n)!} x^{2n}\right]\\
&= \tfrac{1}{5\sqrt{5}}\left[\sum_{n=0}^{\infty}\left(\tfrac{8\left(4n^2+6n+2\right)-8(2n+2)-2(2n+2)\left(3^{2n+1}+1\right)+2\left(3^{2n+2}-1\right)}{(2n+2)!}\right)(\ln\phi)^{2n+2} x^{2n+2}\right]\\
&= \tfrac{1}{5\sqrt{5}}\left[\sum_{n=2}^{\infty}\left(\tfrac{8\left(4n^2+6n+2\right)-8(2n+2)-2(2n+2)\left(3^{2n+1}+1\right)+2\left(3^{2n+2}-1\right)}{(2n+2)!}\right)(\ln\phi)^{2n+2} x^{2n+2}\right] < 0,
\end{aligned}$$

$$\begin{aligned}
D_3(x) &= \left(\tfrac{461646424}{5}\right)(\ln\phi)^2 x^2 \operatorname{cosFh} x - (115411608)(\ln\phi)\, x \left[\tfrac{1}{5}(\operatorname{sinFh} 3x + \operatorname{sinFh} x)\right]\\
&+ (115411609)\left[\tfrac{1}{5}(\operatorname{cosFh} 3x - \operatorname{cosFh} x)\right] - \left(\tfrac{461646428}{5}\right)(\ln\phi)\, x \operatorname{sinFh} x
\end{aligned}$$

$$
\begin{aligned}
&= \left(\tfrac{923292848}{5\sqrt{5}}\right)(\ln\phi)^2 x^2 \cosh\left((\ln\phi)x\right) - \tfrac{230823216}{5\sqrt{5}}(\ln\phi)x\left[\sinh\left((\ln\phi)3x\right)\right] \\
&- \tfrac{230823216}{5\sqrt{5}}(\ln\phi)x\left[\sinh\left((\ln\phi)x\right)\right] + \tfrac{230823218}{5\sqrt{5}}\left[\cosh\left((\ln\phi)3x\right)\right] \\
&- \tfrac{230823218}{5\sqrt{5}}\left[\cosh\left((\ln\phi)x\right)\right] - \tfrac{923292856}{5\sqrt{5}}(\ln\phi)x\sinh\left((\ln\phi)x\right) \\
&= \tfrac{1}{5\sqrt{5}}\left[\sum_{n=0}^{\infty}\tfrac{(923292848)(\ln\phi)^{2n+2}}{(2n)!}x^{2n+2}\right] \\
&+ \tfrac{1}{5\sqrt{5}}\left[\sum_{n=0}^{\infty}\tfrac{(-230823216)\left(3^{2n+1}+1\right)(\ln\phi)^{2n+2}}{(2n+1)!}x^{2n+2}\right] \\
&+ \tfrac{1}{5\sqrt{5}}\left[\sum_{n=1}^{\infty}\tfrac{(230823218)\left(3^{2n}-1\right)(\ln\phi)^{2n}}{(2n)!}x^{2n} + \sum_{n=0}^{\infty}\tfrac{(-923292856)(\ln\phi)^{2n+2}}{(2n+1)!}x^{2n+2}\right] \\
&= \tfrac{1}{5\sqrt{5}}\left[\sum_{n=0}^{\infty}\left(\tfrac{(923292848)}{(2n)!} + \tfrac{(-230823216)\left(3^{2n+1}+1\right)}{(2n+1)!}\right)(\ln\phi)^{2n+2}x^{2n+2}\right] \\
&+ \tfrac{1}{5\sqrt{5}}\left[\sum_{n=0}^{\infty}\left(\tfrac{(230823218)\left(3^{2n+2}-1\right)+(-923292856)(2n+2)}{(2n+2)!}\right)(\ln\phi)^{2n+2}x^{2n+2}\right] \\
&= \tfrac{1}{5\sqrt{5}}\left[\sum_{n=2}^{\infty}\left(\tfrac{(923292848)\left(4n^2+6n+2\right)+(-230823216)(2n+2)\left(3^{2n+1}+1\right)}{(2n+2)!}\right)(\ln\phi)^{2n+2}x^{2n+2}\right] \\
&+ \tfrac{1}{5\sqrt{5}}\left[\sum_{n=2}^{\infty}\left(\tfrac{(230823218)\left(3^{2n+2}-1\right)+(-923292856)(2n+2)}{(2n+2)!}\right)(\ln\phi)^{2n+2}x^{2n+2}\right] \\
&< 0.
\end{aligned}
$$

□

Lemma 3. *For $x \in (0, +\infty)$*

$$\left(\tfrac{\operatorname{sinFh} x}{x}\right)^3 > \operatorname{cosFh} x. \tag{3.7}$$

Proof. The upper bound of $\operatorname{cosFh} x$ hods true if the function $f(x) = \operatorname{sinFh}^3 x - x^3 \operatorname{cosFh} x$ is positive on $(0, \infty)$. Since $f''(x) = 6(\ln\phi)^2 \operatorname{sinFh} x \operatorname{cosFh}^2 x + 3(\ln\phi)^2 \operatorname{sinFh}^3 x - 6x \operatorname{cosFh} x - 6x^2 \ln\phi \operatorname{sinFh} x - x^3 (\ln\phi)^2 \operatorname{cosFh} x > \dot{0}$. The function $f'(x) = 3\operatorname{sinFh}^2 x \ln\phi \operatorname{cosFh} x - 3x^2 \operatorname{cosFh} x - x^3 \ln\phi \operatorname{sinFh} x$ is increasing. Therefore $f'(x) > f'(0) = 0$ and $f(x) > f(0) = 0$. □

Lemma 4. *Let $x \in [9, \infty)$ and $\alpha > 0$. For*

$$H(\alpha) = \frac{\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + \left(\frac{\operatorname{tanFh} x}{x}\right)^{\alpha}}{\left(\frac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\frac{x}{\operatorname{tanFh} x}\right)^{\alpha}} \tag{3.8}$$

if $H'(\alpha) > 0$, $H(\alpha)$ is increasing.

Proof. Let $x \in [9, \infty)$ and $\alpha > 0$,

$$H'(\alpha) = \left(\frac{\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + \left(\frac{\operatorname{tanFh} x}{x}\right)^{\alpha}}{\left(\frac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\frac{x}{\operatorname{tanFh} x}\right)^{\alpha}} \right)' = \frac{I(\alpha)}{\left(\left(\frac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\frac{x}{\operatorname{tanFh} x}\right)^{\alpha}\right)^2}$$

$$I(\alpha) = \left[\left(\frac{\operatorname{tanFh} x}{x}\right)^{\alpha}\left(\frac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\frac{x}{\operatorname{tanFh} x}\right)^{\alpha}\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + 2\right]\log\left(\frac{\operatorname{tanFh} x}{x}\right)$$
$$+ 2\left[\left(\frac{\operatorname{tanFh} x}{x}\right)^{\alpha}\left(\frac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\frac{x}{\operatorname{tanFh} x}\right)^{\alpha}\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + 2\right]\log\left(\frac{\operatorname{sinFh} x}{x}\right)$$

$$I(\alpha) = \left[\left(\left(\frac{\operatorname{tanFh} x}{x}\right)^{\alpha}\left(\frac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\frac{x}{\operatorname{tanFh} x}\right)^{\alpha}\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + 2\right)\right]\log\left[\left(\frac{\operatorname{sinFh} x}{x}\right)^{3}\left(\frac{1}{\operatorname{cosFh} x}\right)\right]$$

The first term is,

$$\left(\frac{\operatorname{tanFh} x}{x}\right)^{\alpha}\left(\frac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\frac{x}{\operatorname{tanFh} x}\right)^{\alpha}\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + 2 \geq 4$$
$$\left(\frac{\operatorname{tanFh} x}{x}\right)^{\alpha}\left(\frac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\frac{x}{\operatorname{tanFh} x}\right)^{\alpha}\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} \geq 2$$

The second term (derived from inequalities (3.7))

$$\left(\frac{\operatorname{sinFh} x}{x}\right)^{3}\frac{1}{\operatorname{cosFh} x} > 1$$
$$\log\left[\left(\frac{\operatorname{sinFh} x}{x}\right)^{3}\left(\frac{1}{\operatorname{cosFh} x}\right)\right] > \log 1 = 0$$

makes $I(\alpha) > 0$ and $H'(\alpha) > 0$. The $H(\alpha)$ is increasing. □

Proof. (Theorem 1) Let $H(x) = \frac{\left(\left(\frac{\operatorname{sinFh} x}{x}\right)^{p} - (0{,}43041)^{p}\right)}{\left((\operatorname{cosFh} x)^{p} - (0{,}89443)^{p}\right)} = \frac{\left(f(x) - f\left(\left(\frac{285}{40612}^{+}\right)\right)\right)}{\left(g(x) - g\left(\frac{285}{40612}^{+}\right)\right)}$, where $f(x) = \left(\frac{\operatorname{sinFh} x}{x}\right)^{p}$, and $g(x) = (\operatorname{cosFh} x)^{p}$. Then

$$k(x) \triangleq \frac{f'(x)}{g'(x)} = \left(\frac{\operatorname{sinFh} x}{x \operatorname{cosFh} x}\right)^{p-1}\frac{(\ln\phi)x\operatorname{cosFh} x - \operatorname{sinFh} x}{(\ln\phi)x^2\operatorname{sinFh} x},$$
$$k'(x) = \left(\frac{\sinh x}{x\operatorname{cosFh} x}\right)^{p-1}\frac{u(x)}{(\ln\phi)x^3\operatorname{sinFh}^2 x\operatorname{cosFh} x},$$

where

$$u(x) = (p-1)\left((\ln\phi)\tfrac{4}{5}x - \operatorname{sinFh} x\operatorname{cosFh} x\right)\left((\ln\phi)x\operatorname{cosFh} x - \operatorname{sinFh} x\right) \tag{3.9}$$
$$+ \operatorname{cosFh} x\left(-\tfrac{4}{5}(\ln\phi)^2x^2 - (\ln\phi)x\operatorname{sinFh} x\operatorname{cosFh} x + 2\operatorname{sinFh}^2 x\right)$$
$$= (p-1)\left((\ln\phi)\tfrac{4}{5}x - \operatorname{sinFh} x\operatorname{cosFh} x\right)\left((\ln\phi)x\operatorname{cosFh} x - \operatorname{sinFh} x\right)$$
$$- \tfrac{4}{5}(\ln\phi)^2x^2\operatorname{cosFh} x - (\ln\phi)x\operatorname{sinFh} x\operatorname{cosFh}^2 x + 2\operatorname{sinFh}^2 x\operatorname{cosFh} x$$
$$= (p-1)D_2(x) - \tfrac{4}{5}(\ln\phi)^2x^2\operatorname{cosFh} x - (\ln\phi)x\operatorname{sinFh} x\operatorname{cosFh}^2 x$$
$$+ 2\operatorname{sinFh}^2 x\operatorname{cosFh} x.$$

We obtain results in the following two cases.

(i) When $p \geq 115411608$, by (3.9), (3.5) and (3.6) we have

$$u(x) < \left(\tfrac{461646424}{5}\right)(\ln\phi)^2x^2\operatorname{cosFh} x - \left(\tfrac{461646428}{5}\right)(\ln\phi)x\operatorname{sinFh} x$$
$$- (115411608)(\ln\phi)x\operatorname{sinFh} x\operatorname{cosFh}^2 x + (115411609)\operatorname{sinFh}^2 x\operatorname{cosFh} x$$
$$= D_3(x) < 0.$$

So $k'(x) < 0$ and $k(x) = \frac{f'(x)}{g'(x)}$ is decreasing on $\left(\frac{285}{40612}, +\infty\right)$. This leads to that $H(x) = \frac{\left(f(x) - f\left(\frac{285}{40612}^{+}\right)\right)}{\left(g(x) - g\left(\frac{285}{40612}^{+}\right)\right)}$ is decreasing on $\left(\frac{285}{40612}, +\infty\right)$ by Lemma 1. At the same time, using power series expansions, we have that $\lim_{x \to \frac{285}{40612}^{+}} H(x) = (0,4812)^{p-1}(0,00036506)$, and rewriting $H(x)$ as $\frac{\left(\frac{\tanh \mathrm{Fh}\, x}{x}\right)^{p} - \left(\frac{0,43041}{\mathrm{cosFh}\, x}\right)^{p}}{1 - \left(\frac{0,89443}{\mathrm{cosFh}\, x}\right)^{p}}$, we see that $\lim_{x \to +\infty} H(x) = 0$. So the proof (i) in Theorem 9 is complete.

(ii) When $p < 0$, by (3.9), (3.5) and (3.4) we obtain

$$\begin{aligned}
u(x) &> -(\ln\phi)^2 \tfrac{4}{5} x^2 \,\mathrm{cosFh}\, x + \tfrac{4}{5}(\ln\phi)\, x \,\mathrm{sinFh}\, x + (\ln\phi)\, x \,\mathrm{sinFh}\, x \,\mathrm{cosFh}^2\, x \\
&\quad - \mathrm{sinFh}^2\, x \,\mathrm{cosFh}\, x - \tfrac{4}{5}(\ln\phi)^2 x^2 \,\mathrm{cosFh}\, x - (\ln\phi)\, x \,\mathrm{sinFh}\, x \,\mathrm{cosFh}^2\, x \\
&\quad + 2\,\mathrm{sinFh}^2\, x \,\mathrm{cosFh}\, x \\
&= -(\ln\phi)^2 \tfrac{8}{5} x^2 \,\mathrm{cosFh}\, x + \tfrac{4}{5}(\ln\phi)\, x \,\mathrm{sinFh}\, x + \mathrm{sinFh}^2\, x \,\mathrm{cosFh}\, x \\
&= D_1(x) > 0.
\end{aligned}$$

So $k'(x) > 0$ and $k(x) = \frac{f'(x)}{g'(x)}$ is increasing on $\left(\frac{285}{40612}, +\infty\right)$ and the function $H(x)$ is increasing on $\left(\frac{285}{40612}, +\infty\right)$ by Lemma 1. At the same time, $\lim_{x \to \frac{285}{40612}^{+}} H(x) = (0,4812)^{p-1}(0,00036506)$, but $\lim_{x \to \frac{\pi}{2}^{-}} H(x) = +\infty$. So the proof of (ii) in Theorem 9 is complete. □

When letting $p = -\alpha < 0$ in (3.2), one can obtain a result on (3.3).

Theorem 10. *(Fibonacci Huygens-type inequalities) Let $x > \frac{285}{40612}$. Then one has the following.*

(i) *When $p \geq 115411608$, the double inequality*

$$\begin{aligned}
&((0,43041)^p - \lambda(0,89443)^p)\left(\tfrac{x}{\mathrm{sinFh}\, x}\right)^p + \lambda\left(\tfrac{x}{\mathrm{tanFh}\, x}\right)^p < 1 \\
&\quad < ((0,43041)^p - \eta(0,89443)^p)\left(\tfrac{x}{\mathrm{sinFh}\, x}\right)^p + \eta\left(\tfrac{x}{\mathrm{tanFh}\, x}\right)^p
\end{aligned} \tag{3.10}$$

holds if and only if $\eta \geq (0,4812)^{p-1}(0,00036506)$ and $\lambda \leq 0$.

(ii) *When $p < 0$, the inequality*

$$1 < ((0,43041)^p - \eta(0,89443)^p)\left(\tfrac{x}{\mathrm{sinFh}\, x}\right)^p + \eta\left(\tfrac{x}{\mathrm{tanFh}\, x}\right)^p \tag{3.11}$$

holds if and only if $\eta \leq (0,4812)^{p-1}(0,00036506)$.

Let $p = -\alpha$, $\alpha > 0$, then inequality (4.2) is equivalent to

$$1 < \left(\left(\frac{1}{0,43041}\right)^{\alpha} - \eta\left(\frac{1}{0,89443}\right)^{\alpha}\right)\left(\tfrac{\mathrm{sinFh}\, x}{x}\right)^{\alpha} + \eta\left(\tfrac{\mathrm{tanFh}\, x}{x}\right)^{\alpha} \tag{3.12}$$

and holds if and only if $\eta \leq \frac{(0,00036506)}{(0,4812)^{\alpha+1}}$.

Proof. Let $x > \frac{285}{40612}$.

(i) For $p \geq 115411608$ when the inequalities (3.1) is multiplied by $\left(\frac{x}{\mathrm{sinFh}\, x}\right)^p$. Where $\left(\frac{x}{\mathrm{sinFh}\, x}\right)^p > 0$,

$$\begin{aligned}
&((0,43041)^p - \lambda(0,89443)^p)\left(\tfrac{x}{\mathrm{sinFh}\, x}\right)^p + \lambda(\mathrm{cosFh}\, x)^p\left(\tfrac{x}{\mathrm{sinFh}\, x}\right)^p < 1 \\
&\quad < ((0,43041)^p - \eta(0,89443)^p)\left(\tfrac{x}{\mathrm{sinFh}\, x}\right)^p + \eta(\mathrm{cosFh}\, x)^p\left(\tfrac{x}{\mathrm{sinFh}\, x}\right)^p \\
&\qquad ((0,43041)^p - \lambda(0,89443)^p)\left(\tfrac{x}{\mathrm{sinFh}\, x}\right)^p + \lambda\left(\tfrac{x}{\mathrm{tanFh}\, x}\right)^p < 1
\end{aligned}$$

$$< \left((0,43041)^p - \eta\,(0,89443)^p\right)\left(\tfrac{x}{\operatorname{sinFh} x}\right)^p + \eta\left(\tfrac{x}{\operatorname{tanFh} x}\right)^p$$

are obtained.

(ii) For $p < 0$ when the inequalities (3.2) is multiplied by $\left(\frac{x}{\operatorname{sinFh} x}\right)^p$. Where $\left(\frac{x}{\operatorname{sinFh} x}\right)^p > 0$,

$$1 < \left((0,43041)^p - \eta\,(0,89443)^p\right)\left(\tfrac{x}{\operatorname{sinFh} x}\right)^p + \eta\left(\tfrac{x}{\operatorname{tanFh} x}\right)^p$$

are obtained.

The proof of inequalities (3.12) is basically obtained from inequalities (3.3). □

Corollary 3. *One has that*

$$1 < \left(\left(\frac{1}{0,43041}\right) - \eta\left(\frac{1}{0,89443}\right)\right)\left(\tfrac{\operatorname{sinFh} x}{x}\right) + \eta\left(\tfrac{\operatorname{tanFh} x}{x}\right) \tag{3.13}$$

holds for all $x \in \left(\frac{285}{40612}, +\infty\right)$ *and* $\alpha > 0$ *if and only if* $\eta \leq \frac{(0,00036506)}{(0,4812)^{\alpha+1}}$.

Proof. Specifically when α is set to one ($\alpha = 1$) on inequalities (3.12),

$$1 < \left(\left(\frac{1}{0,43041}\right) - \eta\left(\frac{1}{0,89443}\right)\right)\left(\tfrac{\operatorname{sinFh} x}{x}\right) + \eta\left(\tfrac{\operatorname{tanFh} x}{x}\right)$$

is obtained. □

Corollary 4. *One has that*

$$\Big((0,89443)\,(0,4812)^{\alpha+1} - (1,\,5713)\Big)\left(\tfrac{\operatorname{sinFh} x}{x}\right) + (1,4054)\left(\tfrac{\operatorname{tanFh} x}{x}\right) > (3849,7)\,(0,4812)^{\alpha+1} \tag{3.14}$$

holds for all $x \in \left(\frac{285}{40612}, +\infty\right)$ *and* $\alpha > 0$ *if and only if* $\eta \geq \frac{(0,00036506)}{(0,4812)^{\alpha+1}}$.

Proof. When letting $\eta = \frac{(0,00036506)}{(0,4812)^{\alpha+1}} \leq \frac{(0,00036506)}{(0,4812)^{\alpha+1}}$ in (3.13),

$$\left(\left(\frac{1}{0,43041}\right) - \left(\frac{0,00036506}{(0,4812)^{\alpha+1}}\right)\left(\frac{1}{0,89443}\right)\right)\left(\tfrac{\operatorname{sinFh} x}{x}\right) + \left(\frac{0,00036506}{(0,4812)^{\alpha+1}}\right)\left(\tfrac{\operatorname{tanFh} x}{x}\right) > 1$$

$$\Big((0,89443)\,(0,4812)^{\alpha+1} - (1,\,5713)\Big)\left(\tfrac{\operatorname{sinFh} x}{x}\right) + (1,4054)\left(\tfrac{\operatorname{tanFh} x}{x}\right) > (3849,7)\,(0,4812)^{\alpha+1}$$

□

We obtain the following results on Wilker-type inequalities.

Theorem 11. *(Fibonacci Wilker-type inequalities) Let* $x \in [9, +\infty)$ *and* $\alpha > 0$. *Then the inequality*

$$\left(\tfrac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + \left(\tfrac{\operatorname{tanFh} x}{x}\right)^{\alpha} > \left(\tfrac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\tfrac{x}{\operatorname{tanFh} x}\right)^{\alpha} \tag{3.14}$$

holds.

Proof. Let $x \in [9, +\infty)$ and $\alpha > 0$. From Lemma 4, $H(\alpha)$ is increasing in conclusive,

$$H(\alpha) > H(0)$$

$$\frac{\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + \left(\frac{\operatorname{tanFh} x}{x}\right)^{\alpha}}{\left(\frac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\frac{x}{\operatorname{tanFh} x}\right)^{\alpha}} > 1$$

$$\left(\tfrac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + \left(\tfrac{\operatorname{tanFh} x}{x}\right)^{\alpha} > \left(\tfrac{x}{\operatorname{sinFh} x}\right)^{2\alpha} + \left(\tfrac{x}{\operatorname{tanFh} x}\right)^{\alpha}$$

are obtained and the proof is completed. □

Corollary 5. *(First Wilker-Type Inequalities) One has that*

$$\left(\frac{\operatorname{sinFh} x}{x}\right)^2 + \frac{\operatorname{tanFh} x}{x} \geq \frac{14373}{1000} \tag{3.15}$$

holds for all $x \in [9, +\infty)$.

Proof. Let $x \in [9, +\infty)$. The inequality, (3.15) can be rewritten as

$$\operatorname{sinFh}^2 x \operatorname{cosFh} x + x \operatorname{sinFh} x > \frac{14373}{1000} x^2 \operatorname{cosFh} x$$

Let

$$g(x) = \operatorname{sinFh}^2 x \operatorname{cosFh} x + x \operatorname{sinFh} x - \frac{14373}{1000} x^2 \operatorname{cosFh} x$$

$$h(x) = 2(\ln\phi) \operatorname{sinFh} x \operatorname{cosFh}^2 x + \left((\ln\phi) - \frac{28746}{1000}\right) x \operatorname{cosFh} x + \left(1 - \frac{14373}{2000}(\ln\phi) x^2\right) \operatorname{sinFh} x$$

Direct calculation yields

$$\begin{aligned} g'(x) &= 2(\ln\phi) \operatorname{sinFh} x \operatorname{cosFh}^2 x + (\ln\phi) \operatorname{sinFh}^3 x + \operatorname{sinFh} x + x(\ln\phi) \operatorname{cosFh} x \\ &- \frac{28746}{1000} x \operatorname{cosFh} x - \frac{14373}{1000} x^2 (\ln\phi) \operatorname{sinFh} x \\ &= \left(-\frac{14373}{2000} x^2 + \operatorname{sinFh}^2 x\right)(\ln\phi) \operatorname{sinFh} x + 2(\ln\phi) \operatorname{sinFh} x \operatorname{cosFh}^2 x \\ &+ \left((\ln\phi) - \frac{28746}{1000}\right) x \operatorname{cosFh} x + \left(1 - \frac{14373}{2000}(\ln\phi) x^2\right) \operatorname{sinFh} x \\ &= \left(-\frac{14373}{2000} x^2 + \operatorname{sinFh}^2 x\right)(\ln\phi) \operatorname{sinFh} x + h(x), \end{aligned}$$

$$\begin{aligned} h'(x) &= 2(\ln\phi)^2 \operatorname{cosFh}^3 x + 4(\ln\phi)^2 \operatorname{sinFh}^2 x \operatorname{cosFh} x + \left((\ln\phi) - \frac{28746}{1000}\right) \operatorname{cosFh} x \\ &+ \left((\ln\phi) - \frac{28746}{1000}\right) x (\ln\phi) \operatorname{sinFh} x - \frac{14373}{1000} x (\ln\phi) \operatorname{sinFh} x \\ &+ \left(1 - \frac{14373}{2000}(\ln\phi) x^2\right)(\ln\phi) \operatorname{cosFh} x \end{aligned}$$

Since $x > \operatorname{sinFh} x$ for $x \geq 9$, we have $h'(x) > 0$, $h(x)$ is increasing. From $h(9) > 0$, we obtain $h(x) > 0$, and then $g'(x) = \left(-x^2 + \operatorname{sinFh}^2 x\right)(\ln\phi) \operatorname{sinFh} x + h(x) > 0$, the function $g(x)$ is increasing. From $g(9) > 0$, we get $g(x) > 0$ for $x \in [9, +\infty)$. The proof of inequality (3.15) is complete. □

Corollary 6. *(Second Wilker-Type Inequalities) One has that*

$$\left(\frac{x}{\operatorname{sinFh} x}\right)^2 + \left(\frac{x}{\operatorname{tanFh} x}\right) \geq \frac{90732}{10000} \tag{3.16}$$

holds for all $x \in [9, +\infty)$.

Proof. Define a function

$$f : [9, +\infty) \to \mathbb{R}$$

by

$$f(x) = \left(\frac{x}{\operatorname{sinFh} x}\right)^2 + \left(\frac{x}{\operatorname{tanFh} x}\right)$$

Then, upon differentiating $f(x)$ with respect to x, we get

$$f'(x) = \frac{1}{\operatorname{sinFh}^3 x}\left[\operatorname{cosFh} x \operatorname{sinFh}^2 x - 2\ln\phi x^2 \operatorname{cosFh} x + x \operatorname{sinFh} x \left(-\frac{4}{5}\ln\phi + 2\right)\right]$$

Next by applying Lemma 3

$$\begin{aligned}
f'(x) &= \tfrac{x^2}{\operatorname{sinFh}^3 x}\left[\operatorname{cosFh} x\left(\tfrac{\operatorname{sinFh}^2 x}{x^2}-2\ln\phi\right)+\tfrac{\operatorname{sinFh} x}{x}\left(-\tfrac{4}{5}\ln\phi+2\right)\right]\\
&= \tfrac{x^2}{\operatorname{sinFh}^3 x}\left[\left(\operatorname{cosFh} x-\tfrac{\operatorname{sinFh}^3 x}{x^3}\right)\left(\tfrac{\operatorname{sinFh}^2 x}{x^2}-2\ln\phi\right)\right]\\
&+ \tfrac{x^2}{\operatorname{sinFh}^3 x}\left[\tfrac{\operatorname{sinFh}^3 x}{x^3}\left(\tfrac{\operatorname{sinFh}^2 x}{x^2}-2\ln\phi\right)+\tfrac{\operatorname{sinFh} x}{x}\left(-\tfrac{4}{5}\ln\phi+2\right)\right]\\
&= \tfrac{x^2}{\operatorname{sinFh}^3 x}\left[\left(\operatorname{cosFh} x-\tfrac{\operatorname{sinFh}^3 x}{x^3}\right)\left(\tfrac{\operatorname{sinFh}^2 x}{x^2}-2\ln\phi\right)\right]\\
&+ \tfrac{x^2}{\operatorname{sinFh}^3 x}\left[\tfrac{\operatorname{sinFh} x}{x}\left(\tfrac{\operatorname{sinFh}^4 x}{x^4}-2\ln\phi\tfrac{\operatorname{sinFh}^2 x}{x^2}-\tfrac{4}{5}\ln\phi+2\right)\right]\\
&> 0
\end{aligned}$$

This means that $f(x)$ is strictly increasing on the open interval $[9,+\infty)$. Consequently, we can deduce from the following observation:

$$\lim_{x\to 9} f(x)=\tfrac{90732}{10000}$$

that

$$f(x)\geq \tfrac{90732}{10000},\quad [9,+\infty)$$

which leads us to the inequality (3.16) asserted by Corollary 6. □

Corollary 7. *One has that*

$$\left(\tfrac{\operatorname{sinFh} x}{x}\right)^2+\left(\tfrac{\operatorname{tanFh} x}{x}\right)>\left(\tfrac{x}{\operatorname{sinFh} x}\right)^2+\left(\tfrac{x}{\operatorname{tanFh} x}\right)\geq\tfrac{90732}{10000} \tag{3.17}$$

holds for all $x\in[9,+\infty)$.

Proof. Proof is obviously obtained from inequalities (3.15) and (3.16). □

From Theorem 9, we can obtain some results on Sandor-Bencze-type inequalities.

Theorem 12. *(Fibonacci Sandor-Bencze-type inequalities) Let* $x\in[9,+\infty)$. *Then the following are considered*

(i) When $\alpha\geq\frac{49}{624}$, *one has*

$$\left(\tfrac{\operatorname{sinFh} x}{x}\right)^\alpha<\tfrac{(\operatorname{cosFh} x)^\alpha+\sqrt{(\operatorname{cosFh} x)^{2\alpha}+8}}{4}. \tag{3.18}$$

(ii) When $\alpha>0$, *one has*

$$\left(\tfrac{x}{\operatorname{sinFh} x}\right)^\alpha<\tfrac{1+\sqrt{8(\operatorname{cosFh} x)^{2\alpha}+1}}{4(\operatorname{cosFh} x)^\alpha}<\left(\tfrac{1}{\operatorname{cosFh} x}\right)^\alpha+1. \tag{3.19}$$

Proof. Let $x\geq 9$. (i) For $\alpha\geq\frac{49}{624}$, from inequalities

$$\left(\tfrac{x}{\operatorname{sinFh} x}\right)^{2\alpha}+\left(\tfrac{x}{\operatorname{tanFh} x}\right)^\alpha>2 \tag{3.20}$$

$$\left(\tfrac{x}{\operatorname{sinFh} x}\right)^{2\alpha}+(\operatorname{cosFh} x)^\alpha\left(\tfrac{x}{\operatorname{sinFh} x}\right)^\alpha-2>0$$

$$\left(\left(\tfrac{x}{\operatorname{sinFh} x}\right)^\alpha+\tfrac{(\operatorname{cosFh} x)^\alpha+\sqrt{(\operatorname{cosFh} x)^{2\alpha}+8}}{2}\right)\left(\left(\tfrac{x}{\operatorname{sinFh} x}\right)^\alpha+\tfrac{(\operatorname{cosFh} x)^\alpha-\sqrt{(\operatorname{cosFh} x)^{2\alpha}+8}}{2}\right)>0$$

$$\left(\left(\tfrac{x}{\operatorname{sinFh} x}\right)^\alpha+\tfrac{(\operatorname{cosFh} x)^\alpha-\sqrt{(\operatorname{cosFh} x)^{2\alpha}+8}}{2}\right)>0$$

is derived. There for, the proof (i)

$$\left(\tfrac{x}{\operatorname{sinFh} x}\right)^\alpha>\tfrac{-(\operatorname{cosFh} x)^\alpha+\sqrt{(\operatorname{cosFh} x)^{2\alpha}+8}}{2}=\tfrac{4}{(\operatorname{cosFh} x)^\alpha+\sqrt{(\operatorname{cosFh} x)^{2\alpha}+8}}$$

$$\left(\frac{\operatorname{sinFh} x}{x}\right)^{\alpha} < \frac{(\operatorname{cosFh} x)^{\alpha} + \sqrt{(\operatorname{cosFh} x)^{2\alpha} + 8}}{4}$$

is completed.

(*ii*) For $\alpha > 0$, from inequalities (3.14) and (3.20),

$$\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + \left(\frac{\operatorname{tanFh} x}{x}\right)^{\alpha} > 2$$

Then, form

$$\left(\frac{\operatorname{sinFh} x}{x}\right)^{2\alpha} + (\operatorname{cosFh} x)^{-\alpha} \left(\frac{\operatorname{sinFh} x}{x}\right)^{\alpha} - 2 > 0$$

is derived.

$$\left(\left(\frac{\operatorname{sinFh} x}{x}\right)^{\alpha} + \frac{(\operatorname{cosFh} x)^{-\alpha} + \sqrt{(\operatorname{cosFh} x)^{-2\alpha} + 8}}{2}\right)\left(\left(\frac{\operatorname{sinFh} x}{x}\right)^{\alpha} + \frac{(\operatorname{cosFh} x)^{-\alpha} - \sqrt{(\operatorname{cosFh} x)^{-2\alpha} + 8}}{2}\right) > 0$$

the inequalities

$$\left(\left(\frac{\operatorname{sinFh} x}{x}\right)^{\alpha} + \frac{(\operatorname{cosFh} x)^{-\alpha} - \sqrt{(\operatorname{cosFh} x)^{-2\alpha} + 8}}{2}\right) > 0$$

is obtained. Then the of proof of the inequalities with first two terms,

$$\left(\frac{\operatorname{sinFh} x}{x}\right)^{\alpha} > \frac{-(\operatorname{cosFh} x)^{-\alpha} + \sqrt{(\operatorname{cosFh} x)^{-2\alpha} + 8}}{2} = \frac{4(\operatorname{cosFh} x)^{\alpha}}{1 + \sqrt{1 + 8(\operatorname{cosFh} x)^{2\alpha}}}$$

$$\left(\frac{x}{\operatorname{sinFh} x}\right)^{\alpha} < \frac{1 + \sqrt{1 + 8(\operatorname{cosFh} x)^{2\alpha}}}{4(\operatorname{cosFh} x)^{\alpha}}$$

is obtained. For the second inequalities (the last two terms), Let $x \geq 9$ and $\alpha > 0$, the inequalities,

$$0 < (\operatorname{cosFh} x)^{\alpha}$$

is obtained. Then the inequalities

$$0 < 1 + 3(\operatorname{cosFh} x)^{\alpha} + (\cosh x)^{2\alpha}$$

$$8(\operatorname{cosFh} x)^{2\alpha} + 1 < (3 + 4(\operatorname{cosFh} x)^{\alpha})^{2}$$

$$\frac{1 + \sqrt{8(\operatorname{cosFh} x)^{2\alpha} + 1}}{4} < 1 + (\operatorname{cosFh} x)^{\alpha}$$

$$\frac{1 + \sqrt{8(\operatorname{cosFh} x)^{2\alpha} + 1}}{4(\operatorname{cosFh} x)^{\alpha}} < \left(\frac{1}{\operatorname{cosFh} x}\right)^{\alpha} + 1$$

are obtained. This proves (*ii*). □

Theorem 13. *(Fibonacci Carlson-type inequalities) Let $x > 1$. Then the following are considered*

(*i*) *When $p \geq 115411608$, the double inequality*

$$\frac{\left(2\sqrt{2\sqrt{5}x - 4}\right)^{p}}{\left((0{,}43041)^{p-1}(-2{,}8348)\right)\left(\sqrt{10}\right)^{p} + \left((0{,}4812)^{p-1}(0{,}00036506)\right)\left(\sqrt{2\sqrt{5}x + 4}\right)^{p}} \tag{3.21}$$

$$< \left(\operatorname{cosFh}^{-1} x\right)^{p} < \frac{\left(2\sqrt{2\sqrt{5}x - 4}\right)^{p}}{\left(2\sqrt{5}x + 4\right)^{p/6}}$$

(*ii*) *When $p < 0$, the left inequality of* (3.21) *holds too.*

Proof. Let $\operatorname{cosFh}^{-1} x = t$ for $x > 1$, then $x = \operatorname{cosFh} t$ for $t > 0$, and

$$\operatorname{cosFh} 2x = \frac{\sqrt{5}}{2}\left(2\operatorname{cosFh}^{2} x - \frac{4}{5}\right)$$

$$\sqrt{\frac{2}{\sqrt{5}}\operatorname{cosFh} 2x + \frac{4}{5}} = \sqrt{2}\operatorname{cosFh} x$$

replacing x with $\frac{t}{2}$ then the equalities,

$$\sqrt{\tfrac{2}{\sqrt{5}}\operatorname{cosFh} t + \tfrac{4}{5}} = \sqrt{2}\operatorname{cosFh}\tfrac{t}{2}$$

$$\frac{\sqrt{2\sqrt{5}x+4}}{\sqrt{10}} = \operatorname{cosFh}\tfrac{t}{2} \tag{3.22}$$

are obtained.

$$\operatorname{cosFh} 2x = \tfrac{\sqrt{5}}{2}\left(\tfrac{4}{5} + 2\operatorname{sinFh}^2 x\right)$$

$$\sqrt{\tfrac{2}{\sqrt{5}}\operatorname{cosFh} 2x - \tfrac{4}{5}} = \sqrt{2}\operatorname{sinFh}^2 x$$

when replacing x with $\frac{t}{2}$,

$$\sqrt{\tfrac{2}{\sqrt{5}}\operatorname{cosFh} t - \tfrac{4}{5}} = \sqrt{2}\operatorname{sinFh}\tfrac{t}{2}$$

$$\frac{\sqrt{2\sqrt{5}x-4}}{\sqrt{10}} = \operatorname{sinFh}\tfrac{t}{2} \tag{3.23}$$

equalities are derived.

(i) For $x > 1$ and $p \geq 115411608$, specifying $\Big(\eta = (0,4812)^{p-1}(0,00036506)\Big)$ in inequalities (3.1),

$$\left(\tfrac{\operatorname{sinFh} x}{x}\right)^p < \Big((0,43041)^{p-1}(-2,8348)\Big) + \Big((0,4812)^{p-1}(0,00036506)\Big)(\operatorname{cosFh} x)^p$$

holds. Replacing x with $\frac{t}{2}$,

$$\left(\frac{\operatorname{sinFh}\frac{t}{2}}{\frac{t}{2}}\right)^p < \Big((0,43041)^{p-1}(-2,8348)\Big) + \Big((0,4812)^{p-1}(0,00036506)\Big)\left(\operatorname{cosFh}\tfrac{t}{2}\right)^p$$

holds. Using the replacement at inequalities (3.22) and (3.23),

$$\frac{\left(2\sqrt{2\sqrt{5}x-4}\right)^p}{\left((0,43041)^{p-1}(-2,8348)\right)\left(\sqrt{10}\right)^p + \left((0,4812)^{p-1}(0,00036506)\right)\left(\sqrt{2\sqrt{5}x+4}\right)^p} < \left(\operatorname{cosFh}^{-1} x\right)^p \tag{3.24}$$

holds. For $\lambda \leq 0$, $((0,43041)^p - \lambda\,(0,89443)^p) \geq 0$. Then from inequalities (3.1)

$$\lambda\,(\operatorname{cosFh} x)^p < \left(\tfrac{\operatorname{sinFh} x}{x}\right)^p$$

holds. Replacing x with $\frac{t}{2}$,

$$\lambda\left(\operatorname{cosFh}\tfrac{t}{2}\right)^p < \left(\frac{\operatorname{sinFh}\frac{t}{2}}{\frac{t}{2}}\right)^p$$

is obtained and using the replacements preformed in inequalities (3.22) and (3.23),

$$\lambda\frac{\left(\sqrt{2\sqrt{5}x+4}\right)^p}{\left(\sqrt{10}\right)^p} < \left(\frac{2\sqrt{2\sqrt{5}x-4}}{\sqrt{10}\left(\operatorname{cosFh}^{-1} x\right)}\right)^p$$

$$\left(\operatorname{cosFh}^{-1} x\right)^p < \frac{1}{\lambda}\frac{\left(2\sqrt{2\sqrt{5}x-4}\right)^p}{\left(\sqrt{2\sqrt{5}x+4}\right)^p} < \frac{\left(2\sqrt{2\sqrt{5}x-4}\right)^p}{\left(\sqrt[6]{2\sqrt{5}x+4}\right)^p}$$

$$\left(\operatorname{cosFh}^{-1} x\right)^p < \frac{\left(2\sqrt{2\sqrt{5}x-4}\right)^p}{\left(2\sqrt{5}x+4\right)^{p/6}} \tag{3.25}$$

are obtained. The inequalities (3.21) is obtained from (3.24) and (3.25).

(ii) Similarly for $p < 0$, The left part of (3.21) is obtained. $\square$

Theorem 14. *(Fibonacci Shafer-Fink-type inequalities) Let $t>0$, $p\geq 115411608$ or $p<0$ Then the inequality*

$$\frac{t^p}{\left((0{,}43041)^{p-1}(-2{,}8348)\right)+\left((0{,}4812)^{p-1}(0{,}00036506)\right)\left(\sqrt{\frac{4}{5}+t^2}\right)^p}<\left(\operatorname{sinFh}^{-1}t\right)^p \tag{3.26}$$

holds.

Proof. Let $\operatorname{sinFh}^{-1}t=x$, for $t>0$ then $\operatorname{sinFh}x=t$, and

$$\operatorname{cosFh}^2 x-\operatorname{sinFh}^2 x=\frac{4}{5}$$

$$\operatorname{cosFh}x=\sqrt{\frac{4}{5}+t^2} \tag{3.27}$$

holds.

Let $x>\frac{285}{40612}$, $t>0$, $p\geq 115411608$ or $p<0$. When η is specified to $\left(\eta=(0,4812)^{p-1}(0,00036506)\right)$ at Theorem 9,

$$\left(\frac{\operatorname{sinFh}x}{x}\right)^p<\left((0,43041)^{p-1}(-2,8348)\right)+\left((0,4812)^{p-1}(0,00036506)\right)(\operatorname{cosFh}x)^p$$

is obtained replacing $\operatorname{sinFh}^{-1}t$ with x and performing the replacement at (3.22),

$$\frac{t^p}{\left(\operatorname{sinFh}^{-1}t\right)^p}<\left((0,43041)^{p-1}(-2,8348)\right)+\left((0,4812)^{p-1}(0,00036506)\right)\left(\sqrt{\frac{4}{5}+t^2}\right)^p$$

$$\frac{t^p}{\left((0{,}43041)^{p-1}(-2{,}8348)\right)+\left((0{,}4812)^{p-1}(0{,}00036506)\right)\left(\sqrt{\frac{4}{5}+t^2}\right)^p}<\left(\operatorname{sinFh}^{-1}t\right)^p$$

holds. ☐

Theorem 14 can deduce to the following result.

Corollary 8. *Let $x>0$*

$$\frac{x^{-1}}{(-15{,}302)+(0{,}0015766)\left(\sqrt{\frac{4}{5}+x^2}\right)^{-1}}<(\operatorname{sinFh}x). \tag{3.28}$$

Proof. When p is specified to -1 ($p=-1$) and x is replaced with t in (3.26),

$$\frac{x^{-1}}{\left((0{,}43041)^{-2}(-2{,}8348)\right)+\left((0{,}4812)^{-2}(0{,}00036506)\right)\left(\sqrt{\frac{4}{5}+x^2}\right)^{-1}}<\left(\operatorname{sinFh}^{-1}x\right)^{-1}$$

is deduced to

$$\frac{x^{-1}}{(-15{,}302)+(0{,}0015766)\left(\sqrt{\frac{4}{5}+x^2}\right)^{-1}}<(\operatorname{sinFh}x)$$

and the proof is completed. ☐

Theorem 15. *Let $t>0$, $p\geq 115411608$ or $p<0$. Then the inequality*

$$\frac{2\left(\sqrt{2}\right)^p\left(\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}}-\frac{4}{5}\right)^{p/2}}{2.\left((0{,}43041)^{p-1}(-2{,}8348)\right)+\left(\sqrt{2}\right)^{2-p}\left((0{,}4812)^{p-1}(0{,}00036506)\right)\left(\left(\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}}+\frac{4}{5}\right)^{p/2}\right)}<\left(\operatorname{sinFh}^{-1}t\right)^p \tag{3.29}$$

holds.

Proof. Let $u \in \mathbb{R}$ and $t > 0$,

$$\cos\mathrm{Fh}\, u = \cos\mathrm{Fh}\, u$$
$$\sqrt{\tfrac{4}{5} + \sin\mathrm{Fh}^2\, u} = \tfrac{\sqrt{5}}{2}\left(\tfrac{4}{5} + 2\sin\mathrm{Fh}^2\, \tfrac{u}{2}\right)$$

replacing t with $\sin\mathrm{Fh}\, u$,

$$\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}} - \frac{4}{5} = 2\sin\mathrm{Fh}^2\, \frac{u}{2}$$
$$\sin\mathrm{Fh}\, \frac{u}{2} = \frac{\left(\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}} - \frac{4}{5}\right)^{1/2}}{\sqrt{2}} \tag{3.30}$$

holds.

$$\cos\mathrm{Fh}\, u = \cos\mathrm{Fh}\, u$$
$$\sqrt{\tfrac{4}{5} + \sin\mathrm{Fh}^2\, u} = \tfrac{\sqrt{5}}{2}\left(2\cos\mathrm{Fh}^2\, \tfrac{u}{2} - \tfrac{4}{5}\right)$$

replacing t with $\sin\mathrm{Fh}\, u$,

$$\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}} + \frac{4}{5} = 2\cos\mathrm{Fh}^2\, \frac{u}{2}$$
$$\cos\mathrm{Fh}\, \frac{u}{2} = \frac{\left(\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}} + \frac{4}{5}\right)^{1/2}}{\sqrt{2}} \tag{3.31}$$

holds.

Let $x > \frac{285}{40612}$, $p \geq 115411608$ or $p < 0$. When η is specified to $\left(\eta = (0,4812)^{p-1}\,(0,00036506)\right)$ at Theorem 9,

$$\left(\frac{\sin\mathrm{Fh}\, x}{x}\right)^p < \left((0,43041)^{p-1}\,(-2,8348)\right) + \left((0,4812)^{p-1}\,(0,00036506)\right)(\cos\mathrm{Fh}\, x)^p$$

replacing x with $\frac{u}{2}$,

$$\left(\frac{\sin\mathrm{Fh}\, \frac{u}{2}}{\frac{u}{2}}\right)^p < \left((0,43041)^{p-1}\,(-2,8348)\right) + \left((0,4812)^{p-1}\,(0,00036506)\right)\left(\cos\mathrm{Fh}\, \tfrac{u}{2}\right)^p$$

holds and replacing inequalities at (3.30) and (3.31) with $t = \sin\mathrm{Fh}\, u$,

$$\frac{(2)^p\left(\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}} - \frac{4}{5}\right)^{p/2}}{\left((0,43041)^{p-1}(-2,8348)\right)\left(\sqrt{2}\right)^p + \left((0,4812)^{p-1}(0,00036506)\right)\left(\left(\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}} + \frac{4}{5}\right)^{p/2}\right)} < \left(\sin\mathrm{Fh}^{-1}\, t\right)^p$$

holds. By dividing the nominator and denominator of the left inequalities by $\left(\sqrt{2}\right)^p > 0$ and multiplying by 2 scaler.

$$\frac{2\left(\sqrt{2}\right)^p\left(\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}} - \frac{4}{5}\right)^{p/2}}{2.\left((0,43041)^{p-1}(-2,8348)\right) + \left(\sqrt{2}\right)^{2-p}\left((0,4812)^{p-1}(0,00036506)\right)\left(\left(\frac{2\left(\sqrt{\frac{4}{5}+t^2}\right)}{\sqrt{5}} + \frac{4}{5}\right)^{p/2}\right)} < \left(\sin\mathrm{Fh}^{-1}\, t\right)^p$$

is obtained. $\square$

Theorem 15 can deduce to the following result.

Corollary 9. *Let $x > 0$. Then*

$$\frac{x^{-1}}{(-15{,}302)+(0{,}0015766)\left(\sqrt{\frac{4}{5}+x^2}\right)^{-1}} < \frac{2\left(\sqrt{2}\right)^{-1}\left(\frac{2\left(\sqrt{\frac{4}{5}+x^2}\right)}{\sqrt{5}}-\frac{4}{5}\right)^{-1/2}}{2.(-15{,}302)+\left(2\sqrt{2}\right)(0{,}0015766)\left(\left(\frac{2\left(\sqrt{\frac{4}{5}+x^2}\right)}{\sqrt{5}}+\frac{4}{5}\right)^{-1/2}\right)} \tag{3.32}$$
$$< \operatorname{sinFh} x.$$

Proof. For $t > 0$, When p is specified to -1 ($p = -1$) and x is replaced with t in (3.24),

$$\frac{2\left(\sqrt{2}\right)^{-1}\left(\frac{2\left(\sqrt{\frac{4}{5}+x^2}\right)}{\sqrt{5}}-\frac{4}{5}\right)^{-1/2}}{2.\left((0{,}43041)^{-2}(-2{,}8348)\right)+\left(2\sqrt{2}\right)\left((0{,}4812)^{-2}(0{,}00036506)\right)\left(\left(\frac{2\left(\sqrt{\frac{4}{5}+x^2}\right)}{\sqrt{5}}+\frac{4}{5}\right)^{-1/2}\right)} < (\operatorname{sinFh} x) \tag{3.33}$$

hold.

$$\begin{aligned}
0 &< \operatorname{sinFh} t \\
\frac{\operatorname{sinFh} t}{5} &< \operatorname{sinFh} t \\
(\operatorname{sinFh} t)^{-1} &< 2\left(\frac{\operatorname{sinFh} t}{5}\right)^{-1} \\
\frac{(\operatorname{sinFh} t)^{-1}}{(-15{,}302)+(0{,}0015766)(\operatorname{cosFh} t)^{-1}} &< \frac{2\left(\frac{\operatorname{sinFh} t}{5}\right)^{-1}}{2.\left((-15{,}302)+(0{,}0015766)\left(\frac{2(\operatorname{cosFh} t)}{5}\right)^{-1}\right)} \\
\frac{(\operatorname{sinFh} t)^{-1}}{(-15{,}302)+(0{,}0015766)(\operatorname{cosFh} t)^{-1}} &< \frac{2\left(\sqrt{2}\right)^{-1}\left(\sqrt{2}\right)^{-1}\frac{2\operatorname{sinFh} t}{5}}{2.(-15{,}302)+\left(2\sqrt{2}\right)(0{,}0015766)\left(\sqrt{2}\right)^{-1}\left(\frac{2(\operatorname{cosFh} t)}{5}\right)^{-1}}
\end{aligned}$$

replacing $\operatorname{sinFh} t$ with x and performing the replacement at (3.27),

$$\operatorname{cosFh} t = \sqrt{\frac{4}{5}+x^2}$$

hold.

$$\begin{aligned}
\frac{x^{-1}}{(-15{,}302)+(0{,}0015766)\left(\sqrt{\frac{4}{5}+x^2}\right)^{-1}} &< \frac{2\left(\sqrt{2}\right)^{-1}\left(\sqrt{2}\right)^{-1}\left(\frac{2x}{5}\right)^{-1}}{2.(-15{,}302)+\left(2\sqrt{2}\right)(0{,}0015766)\left(\sqrt{2}\right)^{-1}\left(\frac{2\sqrt{\frac{4}{5}+x^2}}{5}\right)^{-1}} \\
&< \frac{2\left(\sqrt{2}\right)^{-1}\left(\frac{2\left(\sqrt{\frac{4}{5}+x^2}\right)}{\sqrt{5}}-\frac{4}{5}\right)^{-1/2}}{2.(-15{,}302)+\left(2\sqrt{2}\right)(0{,}0015766)\left(\left(\frac{2\left(\sqrt{\frac{4}{5}+x^2}\right)}{\sqrt{5}}+\frac{4}{5}\right)^{-1/2}\right)}
\end{aligned}$$

$$\frac{x^{-1}}{(-15{,}302)+(0{,}0015766)\left(\sqrt{\frac{4}{5}+x^2}\right)^{-1}} < \frac{2\left(\sqrt{2}\right)^{-1}\left(\frac{2\left(\sqrt{\frac{4}{5}+x^2}\right)}{\sqrt{5}}-\frac{4}{5}\right)^{-1/2}}{2.(-15{,}302)+\left(2\sqrt{2}\right)(0{,}0015766)\left(\left(\frac{2\left(\sqrt{\frac{4}{5}+x^2}\right)}{\sqrt{5}}+\frac{4}{5}\right)^{-1/2}\right)} \tag{3.34}$$

The inequalities (3.32) obtained from (3.33) and (3.34). □

Theorem 16. *Let $x \geq \frac{285}{40612}$, $p \geq 115411608$ or $p < 0$. Then the inequality*

$$\frac{(\operatorname{sinFh} x)^p}{\left((0{,}43041)^{p-1}(-2{,}8348)\right)\left(1-\left(\frac{36506}{283480000}\right)\left(\frac{48120}{43041}\right)^{p-1}(\operatorname{cosFh} x)^p\right)} < (x)^p \tag{3.35}$$

holds.

Proof. Let $x \geq \frac{285}{40612}$, $p \geq 115411608$ or $p < 0$. When η is specified to $\left(\eta = (0,4812)^{p-1}\,(0,00036506)\right)$ at Theorem 9.

$$\left(\frac{\operatorname{sinFh} x}{x}\right)^p < \left((0,43041)^{p-1}\,(-2,8348)\right) + \left((0,4812)^{p-1}\,(0,00036506)\right)(\operatorname{cosFh} x)^p$$

$$\left(\frac{\operatorname{sinFh} x}{x}\right)^p < \left((0,43041)^{p-1}\,(-2,8348)\right)\left(1 + \frac{\left((0{,}4812)^{p-1}(0{,}00036506)\right)(\operatorname{cosFh} x)^p}{\left((0{,}43041)^{p-1}(-2{,}8348)\right)}\right)$$

$$\frac{(\operatorname{sinFh} x)^p}{\left((0{,}43041)^{p-1}(-2{,}8348)\right)\left(1-\left(\frac{36506}{283480000}\right)\left(\frac{48120}{43041}\right)^{p-1}(\operatorname{cosFh} x)^p\right)} < (x)^p$$

hold. □

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