Electronic Journal of Mathematical Analysis and Applications
Vol. 7(2) July 2019, pp. 309-331.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

DISLOCATED QUASI RECTANGULAR b-METRIC SPACES AND RELATED FIXED POINT THEOREMS

P. G. GOLHARE AND C.T. AAGE

Abstract. In this paper, we introduce the notion of dislocated quasi rectangular b-metric space. We extend the well-known Banach and Kannan fixed point theorems in this space. We also prove some well known fixed point theorems for ϕ-weak contraction in it. We give examples to support our results.

1. Introduction

Now a days fixed point theory is being extensively studied. Many researchers have been generalized concept of metric spaces and proved fixed point theorems for different types contraction mappings in these spaces. Initially, metric space was generalized by Wilson[14] by introducing the concept of quasi-metric space. Bakhtin[2] introduced the b-metric space which is generalizes the metric spaces and established basic fixed point theorems it. Hitzler et al.[11] put forth concept of dislocated metric spaces. R. George et al.[12] introduced notion of rectangular b-metric spaces as a generalization of both metric spaces and b-metric spaces. They also proved analogue of Banach contraction principle and Kannan type contraction in rectangular b-metric spaces. In the literature, many generalizations of metric spaces are found namely dislocated b-metric space, quasi b-metric space, dislocated quasi b-metric space etc. In this paper, we also introduce the new generalization of metric space, which we call dislocated quasi rectangular b-metric space. We establish analogues of some well known results in the literature in dislocated quasi rectangular b-metric spaces.

Bakhtin[2] defined the b-metric space as follows:

Definition 1 ([2]) Let X be a non-empty set and mapping $d : X \times X \to [0, \infty)$ satisfies:

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) there exists a real number $k \geq 1$ such that $d(x, y) \leq k[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

2010 *Mathematics Subject Classification.* 47H05, 47H10.
Key words and phrases. fixed points, b-metric spaces, quasi-b-metric spaces, dislocated quasi-b-metric spaces.
Submitted Nov. 27, 2018.

Then d is called b-metric on X and (X, d) is called a b-metric space with coefficient k.
Shah and Huassain[10] extended b-metric space to quasi-b-metric spaces and proved some fixed point theorems in it. Alghamdi, Husasain and Salimi[8] defined the term b-metric-like spaces or dislocated b-metric spaces to generalize metric-like spaces. Some of generalizations of metric spaces are mentioned below.
Definition 2([10]) Let X be a non-empty set. Let $d : X \times X \to [0, \infty)$ be a mapping and $k \geq 1$ be a constant such that:

(i) $d(x, y) = 0 = d(y, x)$ if and only if $x = y$ for all $x, y \in X$,
(ii) $d(x, y) \leq k[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then pair (X, d) is called quasi-b-metric space.
Definition 3 ([8]) Let X be a non-empty set. Let $d : X \times X \to [0, \infty)$ be a mapping and $k \geq 1$ be a constant such that:

(i) $d(x, y) = 0$ then $x = y$ for all $x, y \in X$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \leq k[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then pair (X, d) is called dislocated b-metric space.

Chakkrid and Cholatis[4] defined the concept of dislocated quasi-b-metric space as follows:
Definition 4([4]) Let X be a non-empty set. Let the mapping $d : X \times X \to [0, \infty)$ and constant $k \geq 1$ satisfy following conditions:

(i) $d(x, y) = 0 = d(y, x)$ then $x = y$ for all $x, y \in X$,
(ii) $d(x, y) \leq k[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then the pair (X, d) is called dislocated quasi-$b-$metric space or in short dqb-metric space. The constant k is called coefficient of space (X, d). It is clear that b-metric spaces, quasi-b-metric spaces and b-metric-like spaces are dqb-metric spaces but converse is not true.
Example 1([9]) Let $X = R^+$ and for $p > 1, d : X \times X \to [0, \infty)$ be defined as,

$$d(x, y) = |x - y|^p + |x|^p, \forall x, y \in X.$$

Then (X, d) is dqb-metric space with $k = 2^p > 1$. But (X, d) is not b-metric space and also not dislocated quasi metric space.
Example 2([4]) Let $X = R$ and suppose,

$$d(x, y) = |2x - y|^2 + |2x + y|^2,$$

then (X, d) is dqb-metric space with coefficient $k = 2$ but (X, d) is not a quasi-b-metric space. Also (X, d) is not dislocated quasi metric space.
Definition 5([1])Let X be a non-empty set and mapping $d : X \times X \to [0, \infty)$ satisfies:

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \leq [d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$.

Then d is called a rectangular metric on X and (X, d) is called a rectangular metric space.

R. George et al.([12]) defined rectangular b-metric space as follows:
Definition 6([12]) Let X be a non-empty set and mapping $d : X \times X \to [0, \infty)$ satisfies:

(i) $d(x,y)=0$ if and only if $x=y$ for all $x,y\in X$,
(ii) $d(x,y)=d(y,x)$ for all $x,y\in X$,
(iii) there exist a real number $s\geq 1$ such that $d(x,y)\leq s[d(x,u)+d(u,v)+d(v,y)]$ for all $x,y\in X$ and all distinct points $u,v\in X\setminus\{x,y\}$.

Then d is called a rectangular b-metric on X and (X,d) is called a rectangular b-metric space with coefficient s.
Example 3([12]) Let $X=\mathbb{N}$, define $d:X\times X\to[0,\infty)$ by

$$d(x,y)=\begin{cases}0, & \text{if } x=y,\\ 4\alpha, & \text{if } x,y\in\{1,2\}\text{and } x\neq y,\\ \alpha, & \text{if } x \text{ or } y\notin\{1,2\}\text{and } x\neq y,\end{cases}$$

where $\alpha>0$ is a constant. Then (X,d) is a rectangular b-metric space with coefficient $k=\frac{4}{3}>1$.
Example 4([6]) Let $A=\{0,2\}$,$B=\{\frac{1}{n}:n\in\mathbb{N}\}$ and $X=A\cup B$ define $d:X\times X\to[0,\infty)$ by

$$d(x,y)=\begin{cases}0, & \text{if } x=y,\\ 1, & \text{if } x\neq y \text{ and } \{x,y\}\subset A \text{ or } \{x,y\}\subset B,\\ y^2, & \text{if } x\in A \text{ and } y\in B,\\ x^2, & \text{if } x\in B \text{ and } y\in A,\end{cases}$$

then (X,d) is rectangular b-metric space with coefficient $k=3$. Now, we introduce the notion of dislocated rectangular b-metric space as follows:
Definition 7 Let X be a non-empty set and mapping $d:X\times X\to[0,\infty)$ satisfies:

(i) $d(x,y)=0$ then $x=y$ for all $x,y\in X$,
(ii) $d(x,y)=d(y,x)$ for all $x,y\in X$,
(iii) there exist a real number $k\geq 1$ such that $d(x,y)\leq k[d(x,u)+d(u,v)+d(v,y)]$ for all $x,y\in X$ and all distinct points $u,v\in X\setminus\{x,y\}$.

Then d is called a dislocated rectangular b-metric on X and (X,d) is called a dislocated rectangular b-metric space with coefficient k.

One can note that every rectangular b-metric space is dislocated rectangular b-metric space but converse need not be true as illustrated by following example.
Example 5 Let $X=\mathbb{N}$, define $d:X\times X\to[0,\infty)$ by

$$d(x,y)=\begin{cases}4\alpha, & \text{if } x,y\in\{1,2\},\\ \alpha, & \text{otherwise}\end{cases}$$

where $\alpha>0$ is a constant. Then (X,d) is a dislocated rectangular b-metric space with coefficient $k=2>1$. Note that $d(1,1)=4\alpha\neq 0$ and $d(2,2)=4\alpha\neq 0$. Therefore (X,d) is not a rectangular b-metric space.
Now, we define the notion of dislocated quasi rectangular b-metric space or in short dq-rectangular b-metric space as follows
Definition 8 Let X be a non-empty set and mapping $d:X\times X\to[0,\infty)$ satisfies:

(i) $d(x,y)=0=d(y,x)$ then $x=y$ for all $x,y\in X$,
(ii) there exist a real number $k\geq 1$ such that $d(x,y)\leq k[d(x,u)+d(u,v)+d(v,y)]$ for all $x,y\in X$ and all distinct points $u,v\in X\setminus\{x,y\}$.

Then d is called a dislocated quasi or dq-rectangular b-metric on X and (X,d) is called a dislocated quasi or dq-rectangular b-metric space with coefficient k.
Example 6 Let $X=\mathbb{N}$, define $d:X\times X\to[0,\infty)$ by

$$d(x,y)=\begin{cases}4\alpha, & \text{if } x=1, y=2,\\ 3\alpha, & \text{if } x=2, y=1,\\ \frac{\alpha}{2}, & \text{otherwise}\end{cases}$$

where $\alpha>0$ is a constant. Then (X,d) is a dislocated quasi rectangular b-metric space with coefficient $k=3>1$. Note that for any $x\in\mathbb{N}$, $d(x,x)=\frac{\alpha}{2}\neq 0$. Therefore (X,d) is not a rectangular b-metric space. Also $d(1,2)=4\alpha\neq 3\alpha=d(2,1)$.

We give some definitions regarding dislocated rectangular b-metric spaces with inspiration from M. Alghamdi et al.([8]). We define open ball of radius r about x in dislocated quasi rectangular b-metric space (X,d) as

$$B_r(x)=\Big\{y\in X:\max\{|d(x,y)-d(x,x)|,|d(y,x)-d(x,x)|\}<r\Big\}.$$

Definition 9 A subset G of a dislocated rectangular b-metric space (X,d) is said to be open if for every $x\in G$ there exists $r>0$ such that $B_r(x)\subset G$.
Definition 10 A subset F of a dislocated rectangular b-metric space (X,d) is said to be closed if its complement $X\setminus F$ is open.

Dislocated rectangular b-metric space (X,d) with coefficient $k>1$ is not necessarily Hausdorff. Indeed in Example 1, there does not exist $r_1,r_2>0$ such that $B_{r_1}(1)\cap B_{r_2}(2)=\emptyset$. In fact for every $r>0$, $1\in B_r(2)$ and $2\in B_r(1)$.
Definition 11 A sequence $\{x_n\}$ in a dislocated quasi rectangular b-metric space (X,d) is said to be convergent to $x\in X$ if and only if $\lim_{n\to\infty}d(x_n,x)=\lim_{n\to\infty}d(x,x_n)=d(x,x)$. In this case, we say that x is limit of sequence $\{x_n\}$.

This can also put as $\lim_{n\to\infty}|d(x_n,x)-d(x,x)|=0=\lim_{n\to\infty}|d(x,x_n)-d(x,x)|$. From this definition it is clear that given $\epsilon>0$ there exists $N\in\mathbb{N}$ such that $\max\{|d(x_n,x)-d(x,x)|,|d(x,x_n)-d(x,x)|\}<\epsilon$ for all $n\geq N$. We write this as $x_n\to x$ as $n\to\infty$.
Definition 12 Let (X,d_1) and (Y,d_2) be two dislocated quasi rectangular b-metric spaces. A mapping $T:X\to Y$ is said to be continuous at $u\in X$ if and only if given $\epsilon>0$ there exists $\delta>0$ such that $\max\{|d_2(Tx,Tu)-d_1(u,u)|,|d_2(Tu,Tx)-d_1(u,u)|\}<\epsilon$ whenever $\max\{|d_1(x,u)-d_1(u,u)|,|d_1(u,x)-d_1(u,u)|\}<\delta$.
Definition 13 A sequence $\{x_n\}$ in a dislocated quasi rectangular b-metric space (X,d) is called as Cauchy sequence if and only if $\lim_{n\to\infty}d(x_n,x_{n+i})$ and $\lim_{n\to\infty}d(x_{n+i},x_n)$ exists and is finite for all $i\in\mathbb{N}$.
Definition 14 A dislocated rectangular b-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

2. Main Results

Our first results is given below.
Theorem 1 Let (X,d) be a complete dislocated rectangular b-metric space with coefficient $k>1$. Let $T:X\to X$ be a mapping satisfying

$$d(Tx,Ty)\leq\alpha d(x,y),\tag{1}$$

for all $x,y\in X$, where $0\leq\alpha\leq\frac{1}{k}$. Then T has a unique fixed point in X.
Proof. We choose any arbitrary point $x_0\in X$. Now define sequence $\{x_n\}$ in X

such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, $x_{n-1} = x_n$, then x_{n-1} becomes fixed point of T and we have nothing to prove. Therefore, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From inequality (1), we have

$$d(x_{n-1}, x_n) = d(Tx_{n-2}, Tx_{n-1}) \leq \alpha d(x_{n-2}, x_{n-1}). \tag{2}$$

Applying inequality (2) repeatedly, we get,

$$d(x_{n-1}, x_n) \leq \alpha d(x_{n-2}, x_{n-1}) \leq \cdots \leq \alpha^{n-1} d(x_0, x_1). \tag{3}$$

Similarly,

$$d(x_n, x_{n-1}) \leq \alpha d(x_{n-1,x_{n-2}}) \leq \cdots \leq \alpha^{n-1} d(x_1, x_0). \tag{4}$$

We also assume that $x_0 \neq x_n$ for any $2 \leq n \in \mathbb{N}$. If not, then for some $n \geq 2$ in view of (3), we have

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) \\ d(x_0, x_1) &= d(x_n, x_{n+1}). \end{aligned}$$

It implies that

$$d(x_0, x_1) \leq \alpha^n d(x_0, x_1),$$

which is a contradiction unless $d(x_0, x_1) = 0$. Thus $x_0 = x_1$ and s x_0 turns out to be a fixed point of T. So, we assume that $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$. In view of (1), for any $n \in \mathbb{N}$, we can write

$$d(x_{n-1}, x_{n+1}) = d(Tx_{n-2}, Tx_n) \leq \alpha d(x_{n-2}, x_n). \tag{5}$$

Applying (1) repeatedly, we get

$$d(x_{n-1}, x_{n+1}) \leq \alpha^{n-1} d(x_0, x_2). \tag{6}$$

Similarly,

$$d(x_{n+1}, x_{n-1}) \leq \alpha^{n-1} d(x_2, x_0). \tag{7}$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence in X, equivalently, we will show

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n),$$

for all $n, m \in \mathbb{N}$.

Case (i): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequalities (3), (4) and rectangular inequality, we get

$$\begin{aligned}
d(x_n, x_{n+m}) &\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i})]\\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i})]\\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})]\\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) +] + \cdots\\
&+ k^{i-1}[d(x_{n-4+2i}, x_{n-3+2i}) + d(x_{n-3+2i}, x_{n-2+2i})] + k^{i-1}[d(x_{n-2+2i}, x_{n+2i})]\\
&\le k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)]\\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^{i-1}[\alpha^{n-4+2i} d(x_0, x_1) + \alpha^{n-3+2i} d(x_0, x_1)]\\
&+ k^{i-1}\alpha^{n-2+2i} d(x_0, x_2)\\
&\le k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1)\\
&+ k^{i-1}\alpha^{n-2+2i} d(x_0, x_2)\\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big] k\alpha^n d(x_0, x_1) + k^{i-1}\alpha^{n-2+2i} d(x_0, x_2)\\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big] k\alpha^n d(x_0, x_1) + \alpha^{n-2} d(x_0, x_2).
\end{aligned}$$

Letting $n \to \infty$ in last inequality above, we get

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0,$$

for all even $m \in \mathbb{N}$.

Case (ii): Suppose m is odd i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequalities (3), (4) and rectangular inequality, we get

$$\begin{aligned}
d(x_n, x_{n+m}) &\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i-1})]\\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i-1})]\\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})]\\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots\\
&+ k^i[d(x_{n+2i-2}, x_{n+2i-1})]\\
&\le k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)]\\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^i\alpha^{n+2i-2} d(x_0, x_1)\\
&\le k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1)\\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big] k\alpha^n d(x_0, x_1).
\end{aligned}$$

Letting $n \to \infty$ in last inequality above, we see that limit on the right hand side exist and is finite. Therefore, $\lim_{n\to\infty} d(x_n, x_{n+m})$ exists and is finite for all odd $m \in \mathbb{N}$. Thus from the case(i) and case(ii), it follows that $\lim_{n\to\infty} d(x_n, x_{n+m})$ exists and for all $m \in \mathbb{N}$

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0. \tag{8}$$

Now, we will prove that $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$ for all $m, n \in \mathbb{N}$ with $m > n$. We consider two cases:

Case (a): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be odd or even. Then

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2} d(x_2, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\leq (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

It gives that

$$\lim_{n\to\infty} d(x_{n+m}, x_n) = 0. \tag{9}$$

Case (b): Suppose m is odd i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be odd or even. Then

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i-1}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2} d(x_1, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\leq (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

It gives that $\lim_{n,m\to\infty} d(x_{n+m}, x_n) = 0$. Thus from case (a) and case (b), it follows that

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n),$$

for all $n, m \in \mathbb{N}$. Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete dislocated quasi rectangular b-metric space, there exists some $u \in X$ such that $x_n \to u$.

We will show that u is fixed point of T. For any given $n \in \mathbb{N}$, we can write

$$\begin{aligned} d(u, Tu) &\leq k[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\ &= k[d(u, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tu)] \\ &\leq k[d(u, x_n) + d(x_n, x_{n+1}) + \alpha d(x_n, u)]. \end{aligned}$$

Letting $n \to \infty$, using fact that $x_n \to u$ and (6), we get $d(u, Tu) = 0$. Also,

$$\begin{aligned} d(Tu, u) &\leq k[d(Tu, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, u)] \\ &= k[d(Tu, Tx_n) + d(x_{n+1}, x_n) + d(x_n, u)] \\ &\leq k[\alpha d(u, x_n) + d(x_{n+1}, x_n) + d(x_n, u)]. \end{aligned}$$

Letting $n \to \infty$, using fact that $x_n \to u$ and (6), we get $d(Tu, u) = 0$. Thus $d(u, Tu) = 0 = d(Tu, u)$. This gives that $Tu = u$. Hence u is fixed point of T in X. Now, we prove that u is unique fixed point of T in X. Suppose u' be another fixed point of T in X. In view of (1), we have

$$d(u, u') = d(Tu, Tu') \leq \alpha d(u, u') < d(u, u').$$

This a contradiction unless $d(u, u') = 0$. Similarly,

$$d(u', u) = d(Tu', Tu) \leq \alpha d(u', u) < d(u', u).$$

It is also contradiction unless $d(u', u) = 0$. Thus $d(u, u') = 0 = d(u', u)$. Hence $u = u'$. Thus uniqueness of u is established.

Example 7 Let $X = \mathbb{N}$, define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 4\alpha, & \text{if } x = 1, y = 2, \\ \frac{7}{2}\alpha, & \text{if } x = 2, y = 1, \\ \frac{\alpha}{4}, & \text{if } x = p^2, y = q^2 \text{ for some } p, q \in \mathbb{N}, \\ \alpha, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a dislocated quasi rectangular b-metric space with coefficient $k = 2 > 1$. If $T : \mathbb{N} \to \mathbb{N}$ is defined as follows:

$$Tx = \begin{cases} 1, & \text{if } x = p^2 \text{ for some } p \in \mathbb{N}, \\ x^2, & \text{otherwise,} \end{cases}$$

then T is Banach contraction in dislocated quasi rectangular b-metric space $(\mathbb{N}, d)$ and T has unique fixed point $x = 1 \in \mathbb{N}$.

Theorem 2 Let (X, d) be a complete dislocated rectangular b-metric space with coefficient $k > 1$. Let $T : X \to X$ be a mapping satisfying

$$d(Tx, Ty) \leq \gamma[d(x, Tx) + d(y, Ty)], \tag{10}$$

for all $x, y \in X$, where $0 \leq \gamma < \frac{1}{k}$. Then T has a unique fixed point in X.

Proof. We choose any arbitrary point $x_0 \in X$. Define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, $x_{n-1} = x_n$, then x_{n-1}

becomes fixed point of T. Therefore, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From inequality (10), we have

$$\begin{aligned} d(x_{n-1}, x_n) = d(Tx_{n-2}, Tx_{n-1}) &\leq \gamma[d(x_{n-2}, Tx_{n-2}) + d(x_{n-1}, Tx_{n-1})] \\ &= \gamma[d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)]. \end{aligned}$$

It gives that

$$d(x_{n-1}, x_n) \leq \frac{\gamma}{1-\gamma} d(x_{n-2}, x_{n-1}) = \alpha d(x_{n-2}, x_{n-1}), \tag{11}$$

where $\alpha = \frac{\gamma}{1-\gamma}$. Applying inequality (11) repeatedly, we get

$$d(x_{n-1}, x_n) \leq \alpha d(x_{n-2}, x_{n-1}) \leq \cdots \leq \alpha^{n-1} d(x_0, x_1). \tag{12}$$

We also assume that $x_0 \neq x_n$ for any $2 \leq n \in \mathbb{N}$. If not, then for some $n \geq 2$ in view of (12), we have $d(x_0, Tx_0) = d(x_n, Tx_n)$, which implies that $d(x_0, x_1) = d(x_n, x_{n+1})$ and hence $d(x_0, x_1) \leq \alpha^n d(x_0, x_1)$, which is a contradiction unless $d(x_0, x_1) = 0$. Thus $x_0 = x_1$ and x_0 turns out to be a fixed point of T. Hence, we assume that $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$. In view of (10), for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n-1}, x_{n+1}) = d(Tx_{n-2}, Tx_n) &\leq \gamma[d(x_{n-2}, Tx_{n-2}) + d(x_n, Tx_n)] \\ &= \gamma[d(x_{n-2}, x_{n-1}) + d(x_n, x_{n+1})] \\ &\leq \gamma[\alpha^{n-2} d(x_0, x_1) + \alpha^n d(x_0, x_1)] \\ &= \gamma \alpha^{n-2}[1 + \alpha^2] d(x_0, x_1) \\ &= \beta \alpha^{n-2} d(x_0, x_1), \end{aligned}$$

where $\beta = \gamma[1 + \alpha^2]$. Hence, we have

$$d(x_{n-1}, x_{n+1}) \leq \beta \alpha^{n-2} d(x_0, x_1). \tag{13}$$

In order to show $\{x_n\}$ is a Cauchy sequence in X, it is sufficient to show that

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n),$$

for all $n, m \in \mathbb{N}$. For this, we consider the following cases:

Case (i): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be even or odd. Then using inequalities (12), (13) and rectangular inequality, we get

$$\begin{aligned}
d(x_n, x_{n+2i}) &\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i})] \\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i})] \\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) +] + \cdots \\
&+ k^{i-1}[d(x_{n+2i-4}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-2})] + k^{i-1}[d(x_{n+2i-2}, x_{n+2i})] \\
&\le k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)] \\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^{i-1}[\alpha^{n+2i-4} d(x_0, x_1) + \alpha^{n+2i-3} d(x_0, x_1)] \\
&+ k^{i-1}\alpha^{n+2i-2}\beta d(x_0, x_2) \\
&\le k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) \\
&+ k^{i-1}\alpha^{n-3+2i} d(x_0, x_2) \\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big] k\alpha^{n-1} d(x_0, x_1) + k^{i-1}\alpha^{n-3+2i}\beta d(x_0, x_2) \\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big] k\alpha^{n-1} d(x_0, x_1) + \alpha^{n-3}\beta d(x_0, x_2).
\end{aligned}$$

Letting $n \to \infty$ in last inequality above, we get $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$, for all even $m \in \mathbb{N}$.

Case (ii): Suppose m is even i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequalities (12), (13) and rectangular inequality, we get

$$\begin{aligned}
d(x_n, x_{n+2i-1}) &\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i-1})] \\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i-1})] \\
&\le k[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots \\
&+ k^i[d(x_{n+2i}, x_{n+2i-1})] \\
&\le k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)] \\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^i\alpha^{n+2i} d(x_0, x_1) \\
&\le k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) \\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big] k\alpha^{n-1} d(x_0, x_1).
\end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$, for all odd $m \in \mathbb{N}$. Thus from case (i) and case (ii), it follows that for all $m, n \in \mathbb{N}$.

$$\lim_{n,m\to\infty} d(x_n, x_{n+m}) = 0. \tag{14}$$

Now, we prove that $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$ for all $m, n \in \mathbb{N}$. Again, we consider two cases:

Case (a): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be odd or even. Then

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2} d(x_2, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\leq (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

It gives that $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$.
Case (b): Suppose m is odd i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be odd or even. Then

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i-1}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2} d(x_1, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\leq (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

It gives that $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$. Thus taking account all cases, we have

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n)$$

for all $n, m \in \mathbb{N}$. Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete dislocated quasi rectangular b-metric space, there exists some $u \in X$ such that $x_n \to u$. We claim that u is fixed point of T. For any given $n \in \mathbb{N}$, we can write

$$\begin{aligned} d(u, Tu) &\leq k[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\ &= k[d(u, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tu)] \\ &\leq k\Big\{d(u, x_n) + d(x_n, x_{n+1}) + \gamma[d(x_n, Tx_n) + d(u, Tu)]\Big\} \\ &= k\Big\{d(u, x_n) + d(x_n, x_{n+1}) + \gamma[d(x_n, x_{n+1}) + d(u, Tu)]\Big\}, \end{aligned}$$

which gives that,

$$d(u, Tu) \leq \frac{1}{1-\gamma}\Big\{d(u, x_n) + d(x_n, x_{n+1}) + \gamma d(x_n, x_{n+1})\Big\}. \tag{15}$$

Letting $n \to \infty$, the sequence $x_n \to u$, we get $d(u, Tu) = 0$. Also,

$$\begin{aligned} d(Tu, u) &\leq k[d(Tu, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, u)] \\ &= k[d(Tu, Tx_n) + d(x_{n+1}, x_n) + d(x_n, u)] \\ &\leq k\Big\{\gamma[d(u, Tu) + d(x_n, Tx_n)] + d(x_{n+1}, x_n) + d(x_n, u)\Big\} \\ &= k\Big\{\gamma[d(u, Tu) + d(x_n, x_{n+1})] + d(x_{n+1}, x_n) + d(x_n, u)\Big\}. \end{aligned}$$

Letting $n \to \infty$, the sequence $x_n \to u$, we get $d(Tu, u) = 0$. Thus $d(u, Tu) = 0 = d(Tu, u)$. It gives that $Tu = u$. Hence u is fixed point of T in X.

Note that

$$d(u, u) = d(Tu, Tu) \leq \gamma[d(u, Tu) + d(u, Tu)] = 2\gamma d(u, u) < d(u, u), \tag{16}$$

which is a contradiction unless $d(u, u) = 0$. Thus if v is fixed point of T, then we have $d(v, v) = 0$. Suppose u' be another fixed point of T in X. In view of (10), we have

$$d(u, u') = d(Tu, Tu') \leq \gamma[d(u, Tu) + d(u', Tu')] = \gamma[d(u, u) + d(u', u')] = 0.$$

Also,

$$d(u', u) = d(Tu', Tu) \leq \gamma[d(u', Tu') + d(u, Tu)] = \gamma[d(u', u') + d(u, u)] = 0.$$

Thus $d(u, u') = d(u', u) = 0$ i.e. $u = u'$. u is a unique fixed point of T.

Example 8 Let $A = \{0, 2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $X = A \cup B$ define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 3, & \text{if } x, y \in A, \\ 1, & \text{if } x, y \in B, \\ \frac{1}{n}, & \text{if } x \in A \text{ and } y \in B, \\ 1 - \frac{1}{n}, & \text{if } x \in B \text{ and } y \in A, \end{cases}$$

then (X, d) is dislocated quasi rectangular b-metric space with coefficient $k = \frac{3}{2} > 1$. If $T : X \to X$ is defined as follows:

$$Tx = \begin{cases} 2, & \text{if } x \in A, \\ 0, & \text{if } x \in B, \end{cases}$$

then T is Kannan contraction in dislocated quasi rectangular b-metric space (X, d) and T has unique fixed point $x = 2 \in X$.

Now, we define $\Phi = \{\phi : [0, \infty) \to [0, \infty) |\, \phi$ is continuous, nondecreasing and $\phi(\alpha) = 0$ if and only if $\alpha = 0\}$.

Theorem 3 Let (X, d) be a complete dislocated quasi rectangular b-metric space with coefficient $k \geq 1$. Let $T : X \to X$ be a mapping satisfying

$$d(Tx, Ty) \leq \alpha d(x, y) - \phi(d(x, y)), \tag{17}$$

for all $x, y \in X$, where $0 \leq \alpha < \frac{1}{k}$ and $\phi \in \Phi$. Then T has a unique fixed point in X.

Proof. We choose any arbitrary point $x_0 \in X$. Define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, $x_{n-1} = x_n$, then x_{n-1} becomes fixed point of T. Therefore, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From inequality (17), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha d(x_{n-1}, x_n) - \phi(d(x_{n-1}, x_n)) \leq \alpha d(x_{n-1}, x_n).$$

Applying inequality (17) repeatedly, we get,

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) \leq \cdots \leq \alpha^n d(x_0, x_1). \tag{18}$$

Similarly,

$$d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}) \leq \cdots \leq \alpha^n d(x_1, x_0). \tag{19}$$

We also assume that $x_0 \neq x_n$ for any $2 \leq n \in \mathbb{N}$. If not, then for some $n \geq 2$ in view of (18), we have $d(x_0, Tx_0) = d(x_n, Tx_n)$ that is $d(x_0, x_1) = d(x_n, x_{n+1})$ and

$$d(x_0, x_1) \leq \alpha^n d(x_0, x_1),$$

which is a contradiction unless $d(x_0, x_1) = 0$ i.e. $x_0 = x_1$. Thus x_0 turns out to be a fixed point of T. Hence we assume that $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$. Now, in view of (17), for any $n \in \mathbb{N}$,

$$d(x_{n-1}, x_{n+1}) = d(Tx_{n-2}, Tx_n) \leq \alpha d(x_{n-2}, x_n) - \phi(d(x_{n-2}, x_n)) \leq \alpha d(x_{n-2}, x_n). \tag{20}$$

Applying (20) repeatedly, we get

$$d(x_{n-1}, x_{n+1}) \leq \alpha^{n-1} d(x_0, x_2). \tag{21}$$

Similarly,

$$d(x_{n+1}, x_{n-1}) \leq \alpha^{n-1} d(x_2, x_0). \tag{22}$$

In order to show, $\{x_n\}$ is a Cauchy sequence in X, it is sufficient to show, $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n)$ for all $n, m \in \mathbb{N}$. First, we will prove that $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$. So, we consider the following cases:

Case (i): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequalities (18), (21) and rectangular inequality, we get

$$
\begin{aligned}
d(x_n, x_{n+2i}) &\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i})] \\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i})] \\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) +] + \cdots \\
&+ k^{i-1}[d(x_{n+2i-4}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-2})] + k^{i-1}[d(x_{n+2i-2}, x_{n+2i})] \\
&\le k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)] \\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^{i-1}[\alpha^{n+2i-4} d(x_0, x_1) + \alpha^{n+2i-3} d(x_0, x_1)] \\
&+ k^{i-1}\alpha^{n+2i-2}\beta d(x_0, x_2) \\
&\le k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) \\
&+ k^{i-1}\alpha^{n-3+2i} d(x_0, x_2) \\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big]k\alpha^{n-1} d(x_0, x_1) + k^{i-1}\alpha^{n-3+2i}\beta d(x_0, x_2) \\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big]k\alpha^{n-1} d(x_0, x_1) + \alpha^{n-3}\beta d(x_0, x_2).
\end{aligned}
$$

Letting $n \to \infty$ in last inequality above, we get $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$, for all even $m \in \mathbb{N}$.

Case (ii): Suppose m is odd i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequality (22) and rectangular inequality, we get

$$
\begin{aligned}
d(x_n, x_{n+2i-1}) &\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i-1})] \\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i-1})] \\
&\le k[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots \\
&+ k^i[d(x_{n+2i}, x_{n+2i-1})] \\
&\le k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)] \\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^i\alpha^{n+2i} d(x_0, x_1) \\
&\le k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) \\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big]k\alpha^{n-1} d(x_0, x_1).
\end{aligned}
$$

Letting $n \to \infty$ in last inequality above, we get $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$, for all odd $m \in \mathbb{N}$. Taking account the Case (i) and Case (ii), it follows that, for all $m, n \in \mathbb{N}$,

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0. \tag{23}$$

Now, we will prove that $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$ for all $m, n \in \mathbb{N}$. so, we consider following two cases:

Case (a): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be odd or even. Using inequalities (19), (22) and rectangular inequality, we get

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2} d(x_2, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\leq (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$.

Case (b): Suppose m is odd i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be odd or even. Using inequality (19) and rectangular inequality, we get

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i-1}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2} d(x_1, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\leq (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

Letting $m, n \to \infty$, we get $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$. Taking account the Case (a) and Case (b), we have

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n),$$

for all $n, m \in \mathbb{N}$. Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete dislocated quasi rectangular b-metric space, there exists some $u \in X$ such that $x_n \to u$.

Now we show that u is fixed point of T. For any given $n \in \mathbb{N}$, we can write

$$\begin{aligned} d(u, Tu) &\leq k[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\ &= k[d(u, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tu)] \\ &\leq k\Big\{d(u, x_n) + d(x_n, x_{n+1}) + \alpha d(x_n, u) - \phi(d(x_n, u))\Big\}. \end{aligned}$$

Letting $n \to \infty$, using fact that $x_n \to u$ and (18), we get, $d(u, Tu) = 0$. Also,

$$\begin{aligned} d(Tu, u) &\leq k[d(Tu, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, u)] \\ &= k[d(Tu, Tx_n) + d(x_{n+1}, x_n) + d(x_n, u)] \\ &\leq k\Big\{\alpha d(u, x_n) - \phi(d(u, x_n)) + d(x_{n+1}, x_n) + d(x_n, u)\Big\}. \end{aligned}$$

Letting $n \to \infty$, using fact that $x_n \to u$ and (19), we get $d(Tu, u) = 0$. Thus $d(u, Tu) = 0 = d(Tu, u)$. It gives that $Tu = u$ i.e. u is fixed point of T in X.

Note that,

$$d(u, u) = d(Tu, Tu) \leq \alpha d(u, u) - \phi(d(u, u)) \leq \alpha d(u, u) < d(u, u), \tag{24}$$

which is a contradiction unless $d(u, u) = 0$. Thus in general, if v is fixed point of T, then $d(v, v) = 0$. Now, we will prove, u is unique fixed point of T in X. Soppose u' be another fixed point of T in X. In view of (17), we have

$$d(u, u') = d(Tu, Tu') \leq \alpha d(u, u') - \phi(d(u, u')) \leq \alpha d(u, u') < d(u, u').$$

It is a contradiction unless $d(u, u') = 0$. Also,

$$d(u', u) = d(Tu', Tu) \leq \alpha d(u', u) - \phi(d(u', u)) \leq \alpha d(u', u) < d(u', u).$$

It is a contradiction unless $d(u', u) = 0$. Thus $d(u, u') = d(u', u) = 0$ and $u = u'$. Hence u is a unique fixed point of T in X.

We define quasi-like contraction in dislocated quasi rectangular b-metric space (X, d) as follows:

Definition 15 A mapping $T : X \to X$ said to be a quasi-like contraction if,

$$d(Tx, Ty) \leq \alpha \max\{d(x, y), d(Tx, x), d(y, Ty)\}, \tag{25}$$

for all $x, y \in X$, where $0 \leq \alpha < \frac{1}{k}$.

Theorem 4 Let (X, d) be a complete dislocated quasi rectangular b-metric space with coefficient $k \geq 1$. Let $T : X \to X$ be a quasi-like contraction. Then T has unique fixed point in X.

Proof. We choose any arbitrary point $x_0 \in X$. Now define sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, $x_{n-1} = x_n$, then x_{n-1}

becomes fixed point of T and we have nothing to prove. Therefore, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From inequality (25), we have

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \\ &\leq \alpha \max\{d(x_0, x_1), d(Tx_0, x_0), d(x_1, Tx_1)\} \\ &= \alpha \max\{d(x_0, x_1), d(x_1, x_0), d(x_1, x_2)\} \\ &\leq \alpha \max\{d(x_0, x_1), d(x_1, x_0)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_2, x_1) &= d(Tx_1, Tx_0) \\ &\leq \alpha \max\{d(x_1, x_0), d(Tx_1, x_1), d(x_0, Tx_0)\} \\ &= \alpha \max\{d(x_1, x_0), d(x_2, x_1), d(x_0, x_1)\} \\ &\leq \alpha \max\{d(x_1, x_0), d(x_0, x_1)\}. \end{aligned}$$

Let $\eta = \max\{d(x_1, x_0), d(x_0, x_1)\}$. Then

$$d(x_1, x_2) \leq \alpha\eta \tag{26}$$

and

$$d(x_2, x_1) \leq \alpha\eta. \tag{27}$$

Now,

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \\ &\leq \alpha \max\{d(x_1, x_2), d(Tx_1, x_1), d(x_2, Tx_2)\} \\ &= \alpha \max\{d(x_1, x_2), d(x_2, x_1), d(x_2, x_3)\} \\ &\leq \alpha \max\{d(x_1, x_2), d(x_2, x_1)\} \\ &\leq \alpha^2\eta. \end{aligned} \tag{28}$$

Similarly,

$$\begin{aligned} d(x_3, x_2) &= d(Tx_2, Tx_1) \\ &\leq \alpha \max\{d(x_2, x_1), d(Tx_2, x_2), d(x_1, Tx_1)\} \\ &= \alpha \max\{d(x_2, x_1), d(x_3, x_2), d(x_1, x_2)\} \\ &\leq \alpha \max\{d(x_2, x_1), d(x_1, x_2)\} \\ &\leq \alpha^2\eta. \end{aligned} \tag{29}$$

Applying above inequalities (28) and (29), we get,

$$d(x_n, x_{n+1}) \leq \alpha^n\eta \tag{30}$$

and

$$d(x_{n+1}, x_n) \leq \alpha^n\eta. \tag{31}$$

We also assume that $x_0 \neq x_n$ for any $2 \leq n \in \mathbb{N}$. If not, then for some $n \geq 2$ in view of (30), we have

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) \\ d(x_0, x_1) &= d(x_n, x_{n+1}) \\ d(x_0, x_1) &\leq \alpha^n\eta. \end{aligned}$$

If $\eta = d(x_0, x_1)$, then we get $d(x_0, x_1) \leq \alpha^n d(x_0, x_1)$, which is a contradiction unless $d(x_0, x_1) = 0$. And hence $d(x_1, x_0) = 0$. This yields that $x_0 = x_1$. And thus x_0 turns out to be a fixed point of T. Similarly,

$$\begin{aligned} d(Tx_0, x_0) &= d(Tx_n, x_n) \\ d(x_1, x_0) &= d(x_{n+1}, x_n) \\ d(x_1, x_0) &\leq \alpha^n \eta. \end{aligned}$$

If $\eta = d(x_1, x_0)$, then we get $d(x_1, x_0) \leq \alpha^n d(x_1, x_0)$, which is a contradiction unless $d(x_1, x_0) = 0$. And hence $d(x_0, x_1) = 0$. This yields that $x_0 = x_1$ and thus x_0 turns out to be a fixed point of T. Hence we assume that $x_n \neq x_m$, for all $n \neq m \in \mathbb{N}$.

Let $\beta = \max\{d(x_2, x_0), d(x_0, x_2), \eta\}$. We claim that $d(x_n, x_{n+2}) \leq \alpha^n \beta$ and $d(x_{n+2}, x_n) \leq \alpha^n \beta$, for all $n \in \mathbb{N}$. We first prove $d(x_n, x_{n+2}) \leq \alpha^n \beta$. We proceed by induction. For $n = 1$,

$$\begin{aligned} d(x_1, x_3) &= d(Tx_0, Tx_2) \\ &\leq \alpha \max\{d(x_0, x_2), d(Tx_0, x_0), d(x_2, Tx_2)\} \\ &= \alpha \max\{d(x_0, x_2), d(x_1, x_0), d(x_2, x_3)\} \\ &\leq \alpha \max\{d(x_0, x_2), \eta, \alpha^2 \eta\} \\ &\leq \alpha \max\{d(x_0, x_2), \eta\} \\ &= \alpha\beta. \end{aligned}$$

Assume that $d(x_{n-1}, x_{n+1}) \leq \alpha^{n-1}\beta$. Now consider

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \alpha \max\{d(x_{n-1}, x_{n+1}), d(Tx_{n-1}, x_{n-1}), d(x_{n+1}, Tx_{n+1})\} \\ &= \alpha \max\{d(x_{n-1}, x_{n+1}), d(x_n, x_{n-1}), d(x_{n+1}, x_{n+2})\} \\ &\leq \alpha \max\{\alpha^{n-1}\beta, \alpha^{n-1}\eta, \alpha^{n+1}\eta\} \\ &\leq \alpha\alpha^{n-1}\beta \\ &= \alpha^n \beta. \end{aligned}$$

Thus for all $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+2}) \leq \alpha^n \beta. \tag{32}$$

Now, we prove that $d(x_{n+2}, x_n) \leq \alpha^n \beta$. Again we proceed by induction. For $n = 1$,

$$\begin{aligned} d(x_3, x_1) &= d(Tx_2, Tx_0) \\ &\leq \alpha \max\{d(x_2, x_0), d(Tx_2, x_2), d(x_0, Tx_0)\} \\ &= \alpha \max\{d(x_2, x_0), d(x_3, x_2), d(x_0, x_1)\} \\ &\leq \alpha \max\{d(x_2, x_0), \alpha^2 \eta, \eta\} \\ &\leq \alpha \max\{d(x_2, x_0), \eta\} \\ &= \alpha\beta. \end{aligned}$$

Assume that $d(x_{n+1}, x_{n-1}) \leq \alpha^{n-1}\beta$. Now, we consider

$$\begin{aligned} d(x_{n+2}, x_n) &= d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \alpha \max\{d(x_{n+1}, x_{n-1}), d(Tx_{n+1}, x_{n+1}), d(x_{n-1}, Tx_{n-1})\} \\ &= \alpha \max\{d(x_{n+1}, x_{n-1}), d(x_{n+2}, x_{n+1}), d(x_{n-1}, x_n)\} \\ &\leq \alpha \max\{\alpha^{n-1}\beta, \alpha^{n+1}\eta, \alpha^{n-1}\eta\} \\ &\leq \alpha\alpha^{n-1}\beta \\ &= \alpha^n\beta. \end{aligned}$$

Thus, for all $n \in \mathbb{N}$, we have

$$d(x_{n+2}, x_n) \leq \alpha^n\beta. \tag{33}$$

Now, we will prove, $\{x_n\}$ is a Cauchy sequence in X, we prove that $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n)$, for all $n, m \in \mathbb{N}$. First, we prove that $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$. For this, consider the following cases:

Case (i): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequalities (30), (32) and rectangular inequality, we get

$$\begin{aligned} d(x_n, x_{n+2i}) &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i})] \\ &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i})] \\ &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ &+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) +] + \cdots \\ &+ k^{i-1}[d(x_{n+2i-4}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-2})] + k^{i-1}[d(x_{n+2i-2}, x_{n+2i})] \\ &\leq k[\alpha^n\eta + \alpha^{n+1}\eta] + k^2[\alpha^{n+2}\eta + \alpha^{n+3}\eta] \\ &+ k^3[\alpha^{n+4}\eta + \alpha^{n+5}\eta] + \cdots + k^{i-1}[\alpha^{n+2i-4}\eta + \alpha^{n+2i-3}\eta] \\ &+ k^{i-1}\alpha^{n+2i-2}\beta \\ &\leq k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]\eta + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]\eta \\ &+ k^{i-1}\alpha^{n-3+2i}\beta \\ &\leq \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big]k\alpha^{n-1}\eta + k^{i-1}\alpha^{n-3+2i}\beta \\ &\leq \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big]k\alpha^{n-1}\eta + \alpha^{n-3}\beta. \end{aligned}$$

Letting $n \to \infty$ in last inequality above, we get $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$, for all even $m \in \mathbb{N}$.

Case (ii): m is odd i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequality (30) and rectangular inequality, we get

$$\begin{aligned}
d(x_n, x_{n+2i-1}) &\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i-1})] \\
&\le k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i-1})] \\
&\le k[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots \\
&+ k^i[d(x_{n+2i}, x_{n+2i-1})] \\
&\le k[\alpha^n\eta + \alpha^{n+1}\eta] + k^2[\alpha^{n+2}\eta + \alpha^{n+3}\eta] \\
&+ k^3[\alpha^{n+4}\eta + \alpha^{n+5}\eta] + \cdots + k^i\alpha^{n+2i}\eta \\
&\le k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]\eta + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]\eta \\
&\le \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big]k\alpha^{n-1}\eta.
\end{aligned}$$

Letting $n \to \infty$ in last inequality above, we get $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$, for all odd $m \in \mathbb{N}$. Thus from case (i) and case (ii), it follows that, for all $m, n \in \mathbb{N}$,

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0. \tag{34}$$

we prove that $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$ for all $m, n \in \mathbb{N}$. So, we consider two cases:

Case (a): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be odd or even. Using inequalities (31), (33) and rectangular inequality, we get

$$\begin{aligned}
d(x_{n+m}, x_n) &\le k^{i-2}d(x_{n+2i}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\le k^{i-2}\alpha^{n+2i-2}\beta + k^{i-2}[\alpha^{n+2i-3}\eta + \alpha^{n+2i-4}\eta] \\
&+ k^{i-3}[\alpha^{n+2i-4}\eta + \alpha^{n+2i-5}\eta] + \cdots \\
&+ k[\alpha^{n+1}\eta + \alpha^n\eta] \\
&= (k\alpha)^{2i-2}\alpha^n\beta + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\}\eta \\
&\le (k\alpha)^{2i-2}\alpha^n\beta + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\beta + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\beta + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\beta + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1}\eta.
\end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$.

Case (b): m is odd i.e. $m = 2i-1$ for some $i \in \mathbb{N}$ and n may be odd or even. Using inequality (31) and rectangular inequality, we get

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i-1}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2}\eta + k^{i-2}[\alpha^{n+2i-3}\eta + \alpha^{n+2i-4}\eta] \\
&+ k^{i-3}[\alpha^{n+2i-4}\eta + \alpha^{n+2i-5}\eta] + \cdots \\
&+ k[\alpha^{n+1}\eta + \alpha^n\eta] \\
&= (k\alpha)^{2i-2}\alpha^n\eta + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\}\eta \\
&\leq (k\alpha)^{2i-2}\alpha^n\eta + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\eta + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\eta + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\eta + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1}\eta.
\end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$. Thus, from case(a) and case(b), it follows that, for all $m, n \in \mathbb{N}$,

$$\lim_{n\to\infty} d(x_{n+m}, x_n) = 0. \tag{35}$$

It shows that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete dislocated quasi rectangular b-metric space, there exists some $u \in X$ such that $x_n \to u$. Now, we show that u is fixed point of T. For any given $n \in \mathbb{N}$, we can write

$$\begin{aligned}
d(u, Tu) &\leq k[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\
&= k[d(u, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tu)] \\
&\leq k\Big\{d(u, x_n) + d(x_n, x_{n+1}) + \alpha \max\{d(x_n, u), d(x_{n+1}, x_n), d(u, Tu)\}\Big\}.
\end{aligned}$$

Letting $n \to \infty$, using fact that $x_n \to u$, and inequalities (30), (31), we get,

$$d(u, Tu) \leq k\alpha d(u, Tu),$$

which is a contradiction unless $d(u, Tu) = 0$. Also,

$$\begin{aligned}
d(Tu, u) &\leq k[d(Tu, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, u)] \\
&= k[d(Tu, Tx_n) + d(x_{n+1}, x_n) + d(x_n, u)] \\
&\leq k\Big\{\alpha \max\{d(u, x_n), d(Tu, u), d(x_n, x_{n+1})\} + d(x_{n+1}, x_n) + d(x_n, u)\Big\}.
\end{aligned}$$

Letting $n \to \infty$, using fact that $x_n \to u$, and inequalities (30), (31), we get,

$$d(Tu, u) \leq k\alpha d(Tu, u),$$

which is a contradiction unless $d(Tu, u) = 0$. Hence, we get $d(u, Tu) = 0 = d(Tu, u)$. It gives that $Tu = u$. Hence u is fixed point of T in X.

Note that,

$$\begin{aligned} d(u,u) = d(Tu,Tu) &\leq \alpha \max\{d(u,u), d(Tu,u), d(u,Tu)\} \\ &= \alpha \max\{d(u,u), d(u,u), d(u,u)\} \\ &= \alpha d(u,u) < d(u,u), \end{aligned}$$

which is a contradiction unless $d(u,u) = 0$. Thus in general if v is fixed point of T then, $d(v,v) = 0$. Now, we prove that u is unique fixed point of T in X. Suppose, u' is another fixed point of T in X. In view of (25), we have

$$\begin{aligned} d(u,u') = d(Tu,Tu') &\leq \alpha \max\{d(u,u'), d(Tu,u), d(u',Tu')\} \\ &= \alpha \max\{d(u,u'), d(u,u), d(u',u')\} \\ &\leq \alpha d(u,u') < d(u,u'). \end{aligned}$$

It is a contradiction unless $d(u,u') = 0$. Also, consider

$$\begin{aligned} d(u',u) = d(Tu',Tu) &\leq \alpha \max\{d(u',u), d(Tu',u'), d(u,Tu)\} \\ &= \alpha \max\{d(u',u), d(u',u'), d(u,u)\} \\ &\leq \alpha d(u',u) < d(u',u). \end{aligned}$$

It is a contradiction unless $d(u',u) = 0$. Hence, $d(u,u') = d(u',u) = 0$ i.e. $u = u'$. So u is a unique fixed point of T in X.

References

[1] A. Branciari, A fixed point theorem of Banach-Caccippoli type on a class of generalised metric spaces, Publ. Math. Debrecen, 57, 31-37(2000).

[2] I. Bakhtin, The contraction mapping principle in quasi-metric spaces, Functional analysis, vol. 30, 26-37 (1989).

[3] C. Chen, J. Dong and C. Zhu, Some fixed point theorems in b-metric-like spaces,Fixed Point Theory and Applications, DOI 10.1186/s13663-015-0369-3 (2015).

[4] C. Klin-eam and C. Suanoom, Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions, Fixed Point Theory and Applications, DOI 10.1186/s13663-015-0325-2(2015).

[5] C. Suanoom, C. Klin-eam and S. Suantai, Dislocated quasi-b-metric spaces and fixed point theorems for cyclic weakly contractions, J. Nonlinear Sci. Appl. 9, 2779-2788(2016).

[6] J. Roshan, V. Parvaneh, Z. Kadelburg, N. Hussain, New fixed point results in b-rectangular metric spaces, Nonlinear Analysis: Modelling and Control, Vol. 21, No. 5, 614634(2016).

[7] M. Abbas, G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341, 416420(2008).

[8] M. Alghamdi, N. Hussain and P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, J. Inequal. Appl., 402 (2013).

[9] M. Rahman and M. Sarwar, Dislocated quasi b-metric space and fixed point theorems, Electronic Journal of Mathematical Analysis and Applications Vol. 4(2), 16-24(2016).

[10] M. Shah, N. Hussain, Nonlinear contractions in partially ordered quasi b-metric spaces. Commun. Korean Math. Soc. 27(1), 117-128 (2012).

[11] P. Hitzler and A. Seda, Dislocated topologies, Journal of Electrical Engineering, 3-7(2000).

[12] R. George, S. Radenovic, K. P. Reshma, S. Shukla, Rectangular b-metric space and contraction principles, J. Nonlinear Sci. Appl. 8, 1005-1013(2015).

[13] S. Banach, sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fundam. Math. 3, 133-181 (1922).

[14] W. A. Wilson, On quasi-metric spaces, American Journal of Mathematics,Vol. 53, No. 3 , 675-684(Jul., 1931).

P. G. Golhare
Department of Mathematics, Sant Dnyaneshwar Mahavidyalaya, Aurangabad, India
E-mail address: golhare@gmail.com

C.T.Aage
Department of Mathematics, KBC North Maharashtra University, Jalgaon, India
E-mail address: caage17@gmail.com