Electronic Journal of Mathematical Analysis and Applications
Vol. 7(2) July 2019, pp. 332-344.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

UNICITY THEOREM FOR HIGHER ORDER DERIVATIVES OF MEROMORPHIC FUNCTIONS ON ANNULI

DILIP CHANDRA PRAMANIK AND JAYANTA ROY

Abstract. In this paper, we deal with the uniqueness problem for higher order derivatives of meromorphic functions on annuli. Our results generalize the result given by H. Y. Xu and H. Wang [18].

1. Introduction and Main Results

The purpose of this paper is to study the uniqueness of two meromorphic functions sharing five or more values. Thus, we always assume that the reader is familiar with the notations of the Nevanlinna theory, such as $T(r, f), m(r, f), N(r, f)$ and so on (see [6, 19, 20, 23]). We use $\mathbb{C}$ to denote the open complex plane, $\overline{\mathbb{C}}$ to denote the extended complex plane and $\mathbb{X}$ to denote the subset of $\mathbb{C}$.

In 1929, R. Nevanlinna first investigated the uniqueness of meromorphic functions in the whole complex plane and obtained the well-known theorem: the five IM theorem.

Theorem 1. [14] If f and g are two non-constant meromorphic functions that share five distinct complex values a_1, a_2, a_3, a_4, a_5 IM, then $f(z) \equiv g(z)$.

After five IM theorem, there are vast references on the uniqueness of meromorphic functions sharing values and sets in the whole complex plane [20]. It is an interesting topic how to extend some important uniqueness results in the complex plane to an angular domain or the unit disc. In the past several decades, the uniqueness of meromorphic functions in the value distribution attracted many investigations. For example, I. Lahiri, H.X. Yi, X.M. Li and A. Banerjee [20, 10, 12, 1] studied the uniqueness of meromorphic functions on the whole complex plane sharing one, two, three or some sets; M.L. Fang, H.F. Liu, Z.Q. Mao and H.Y.Xu [5, 13, 15] investigated the shared value of meromorphic functions in the unit disc; J.H. Zheng, Q.C. Zhang, T.B. Cao and W.C. Lin [21, 22, 3, 11] considered many uniqueness problem of meromorphic functions on the angular domain.

However, the whole complex plane, the unit disc and the angular domain can all be regarded as a simply-connected region; in other words, the theorems stated in the above references are only regarded as the uniqueness results in a simply-connected

2010 *Mathematics Subject Classification.* 30D30, 30D35.
Key words and phrases. Meromorphic function, Nevanlinna theory, Uniqueness, Annuli.
Submitted Nov. 14, 2018. Revised Jan. 3, 2019.

region. In fact, there exists many sub-regions in the whole complex plane, such as the annuli, the m-punctured complex plane, etc.

Recently, there have been some results focusing on the Nevanlinna theory of meromorphic functions on the annulus [8, 9, 7, 16]. The annulus can be regarded as the doubly-connected region. From the doubly-connected mapping theorem, we can get each doubly-connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \le r < R \le +\infty$. For two cases: $r = 0, R = \infty$, simultaneously, and $0 < r < R < +\infty$; the latter case, the homothety $z \mapsto \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$ in two cases. In 2005, Khrystiyanyn and Kondratyuk [8, 9] proposed the Nevanlinna theory on annuli. Now we will introduce the basic notations and definitions used in the uniqueness theory of meromorphic functions on annuli.

For a meromorphic function f on whole plane $\mathbb{C}$, the classical notations of the Nevanlinna theory are denoted as follows

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \ T(r, f) = N(r, f) + m(r, f),$$

where $\log^+ = \max\{\log x, 0\}$, and $n(t, f)$ is the counting function of poles of the function f in $\{z : |z| \le t\}$. Let f be a meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < r < R_0 \le +\infty$; the notations of the Nevanlinna theory on annuli will be introduced as follows. Let

$$N_1(r, f) = \int_{\frac{1}{r}}^1 \frac{n_1(t, f)}{t} dt, \ \ N_2(r, f) = \int_1^r \frac{n_2(t, f)}{t} dt,$$

$$m_0(r, f) = m(r, f) + m(\frac{1}{r}, f) - 2m(1, f), \ \ N_0(r, f) = N_1(r, f) + N_2(r, f),$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of poles of the function f in $\{z : t < |z| \le 1\}$ and $\{z : 1 < |z| \le t\}$ respectively. Similarly, for $a \in \overline{\mathbb{C}}$ we have

$$\begin{aligned} \overline{N}_0\big(r, \frac{1}{f-a}\big) &= \overline{N}_1\big(r, \frac{1}{f-a}\big) + \overline{N}_2\big(r, \frac{1}{f-a}\big) \\ &= \int_{\frac{1}{r}}^1 \frac{\overline{n_1}(t, \frac{1}{f-a})}{t} dt + \int_1^r \frac{\overline{n_2}(t, \frac{1}{f-a})}{t} dt \end{aligned}$$

in which each zero of the function $f - a$ is counted only once, where by zeros of $f - \infty$ we mean poles of f. In addition, we use $\overline{n}_1^{k)}(t, \frac{1}{f-a})$(or $\overline{n}_1^{(k}(t, \frac{1}{f-a})$) to denote the counting function of poles of the function $\frac{1}{f-a}$ with multiplicities $\le k$(or $> k$) in $\{z : t < |z| \le 1\}$, each point counted only once. Similarly, we have the notations $\overline{N}_1^{k)}(t, f)$, $\overline{N}_1^{(k}(t, f)$, $\overline{N}_2^{k)}(t, f)$, $\overline{N}_2^{(k}(t, f)$, $\overline{N}_0^{k)}(t, f)$, $\overline{N}_0^{(k}(t, f)$.

The Nevanlinna characteristic of f on the annulus $\mathbb{A}$ is defined by

$$T_0(r, f) = m_0(r, f) + N_0(r, f).$$

Definition 1. We write $E(a, f) = \{z \in \mathbb{A} : f(z) - a = 0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a,f)$. We use $\overline{E}_{k)}(a,f)$ to denote the set of zeros of $f-a$ with multiplicities no greater than k, in which each zero is counted only once.

In 2009 and 2011, Cao [2, 4] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets and obtained an analog of Nevanlinna's famous five-value theorem.

Theorem 2. Let f_1 and f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\leq+\infty$. Let a_j $(j=1,2,...,q)$ be q distinct complex numbers in $\overline{\mathbb{C}}$ and k_j $(j=1,2,...,q)$ be positive integers or ∞, such that

$$k_1\geq k_2\geq...\geq k_q$$

and

$$\overline{E}_{k_j)}(a_j,f_1)=\overline{E}_{k_j)}(a_j,f_2),\ for\ (j=1,2,...,q).$$

Then
(i) if $q=7$, then $f_1(z)\equiv f_2(z)$.
(ii) if $q=6$ and $k_3\geq 2$, then $f_1(z)\equiv f_2(z)$.
(iii) if $q=5,\ k_3\geq 3$ and $k_5\geq 2$, then $f_1(z)\equiv f_2(z)$.
(iv) if $q=5$ and $k_4\geq 4$, then $f_1(z)\equiv f_2(z)$.
(v) if $q=5$, $k_3\geq 5$ and $k_4\geq 3$, then $f_1(z)\equiv f_2(z)$.
(vi) if $q=5$, $k_3\geq 6$ and $k_4\geq 2$, then $f_1(z)\equiv f_2(z)$.
From the above theorem we can get the following theorem immediately,

Theorem 3. [2] Let f_1 and f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\leq+\infty$. Let a_j $(j=1,2,3,4,5)$ be five distinct complex numbers in $\overline{\mathbb{C}}$. If $\overline{E}(a_j,f_1)=\overline{E}(a_j,f_2)$ for $j=1,2,3,4,5$, then $f_1\equiv f_2$.

Definition 2. For $B\subset\mathbb{A}$ and $a\in\overline{\mathbb{C}}$, we denote by $\overline{N}_0^B(r,\frac{1}{f-a})$ the reduced counting function of those zeros of $f-a$ on $\mathbb{A}$, which belong to the set B.

In 2016 H. Y. Xu and H. Wang [18] investigated the uniqueness of meromorphic functions on annuli and proved the following results:

Theorem 4. Let f and g be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\leq+\infty$. Let $a_1,...,a_q$ $(q\geq 5)$ be q distinct complex numbers or ∞. Suppose that $k_1\geq k_2\geq ...\geq k_q$, m are positive integers or infinity; $1\leq m\leq q$ and δ_j (≥ 0) $(j=1,2,...,q)$ are such that

$$\big(1+\frac{1}{k_m}\big)\sum_{j=m}^{q}\frac{1}{1+k_j}+3+\sum_{j=1}^{q}\delta_j<(q-m-1)\big(1+\frac{1}{k_m}\big)+m.$$

Let $B_j=\overline{E}_{k_j)}(a_j,f)\setminus\overline{E}_{k_j)}(a_j,g)$ for $j=1,2,...,q$. If

$$\overline{N}_0^{B_j}(r,a_j;f)\leq\delta_jT_0(r,f)$$

and

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{q}\overline{N}_0^{k_j)}(r,a_j;f)}{\sum_{j=1}^{q}\overline{N}_0^{k_j)}(r,a_j;g)}>\frac{k_m}{(1+k_m)\sum_{j=m}^{q}\frac{k_j}{1+k_j}-2(1+k_m)+(m-2-\sum_{j=1}^{q}\delta_j)k_m},$$

then $f(z)\equiv g(z)$.

Now one can asked the following question:
Question : What can be said if n-th order derivatives of f and g share k distinct complex numbers ?
In the current paper, we seek an answer of the above question and prove the following theorems.
Theorem 5. Let f_1, f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$ where $1 < R_0 \leq \infty$. Let $a_j \in \mathbb{C} \cup \{\infty\}$ be distinct for $j = 1, 2, ..., k$ $(k \geq 5)$ and for a non-negative integer n, suppose $E(a_j, f_1^{(n)}) \subset E(a_j, f_2^{(n)})$, for $j = 1, 2, ..., k$ and $E(0, f_i) \subseteq E(0, f_i^{(n)})$ for $i = 1, 2$. If

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} N_0(r, a_j; f_1^{(n)})}{\sum_{j=1}^{k} N_0(r, a_j; f_2^{(n)})} > \frac{n+1}{k-(n+3)},$$

then $f_1^{(n)}(z) \equiv f_2^{(n)}(z)$.

Theorem 6. Let f_1, f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq \infty$ and $a_j \in \mathbb{C} \cup \{\infty\}$ be distinct for $j = 1, 2, ..., k$ $(k \geq 5)$. Suppose that $p_1 \geq p_2 \geq ... \geq p_k$ are positive integers or infinity and δ (≥ 0) is such that

$$\frac{1}{p_1} + \big(1 + \frac{1}{p_1}\big) \sum_{j=2}^{k} \frac{1}{1+p_j} + 1 + \delta < \frac{k-2}{n+1}\big(1 + \frac{1}{p_1}\big)$$

for a non-negative integer n. Let $A_j = \overline{E}_{p_j)}(a_j, f_1^{(n)}) \setminus \overline{E}_{p_j)}(a_j, f_2^{(n)})$ for $j = 1, 2, ..., k$ and $E(0, f_i) \subseteq E(0, f_i^{(n)})$ for $i = 1, 2$. If $\sum_{j=1}^{k} \overline{N_0}^{A_j}(r, a_j; f_1^{(n)}) \leq \delta T_0(r, f_1^{(n)})$ and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r, a_j; f_1^{(n)})}{\sum_{j=1}^{k} \overline{N_0}^{p_j)}(r, a_j; f_2^{(n)})} > \frac{(n+1)p_1}{(k-2)(1+p_1) - (n+1)(1+p_1)\sum_{j=2}^{k} \frac{1}{1+p_j} - (n+1)\{(1+\delta)p_1 + 1\}},$$

then $f_1^{(n)}(z) \equiv f_2^{(n)}(z)$.

Theorem 7. Let f_1 and f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, ..., a_k$ $(k \geq 5)$ be k distinct complex numbers or ∞. Suppose that $p_1 \geq p_2 \geq ... \geq p_k$, m $(1 \leq m \leq k)$ are positive integers or infinity and δ_j (≥ 0) $(j = 1, 2, ..., k)$ are such that

$$\big(1 + \frac{1}{p_m}\big) \sum_{j=m}^{k} \frac{1}{1+p_j} + (2 + \sum_{j=1}^{k} \delta_j - m) < \frac{\{k - 2 - (n+1)(m-1)\}}{n+1}\big(1 + \frac{1}{p_m}\big),$$

for a non-negative integer n. Let $A_j = \overline{E}_{p_j)}(a_j, f_1^{(n)}) \setminus \overline{E}_{p_j)}(a_j, f_2^{(n)})$ for $j = 1, 2, ..., k$. If

$$\overline{N}_0^{A_j}(r, a_j; f_1^{(n)}) \leq \delta_j T_0(r, f_1^{(n)})$$

and

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}\overline{N}_0^{p_j)}(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k}\overline{N}_0^{p_j)}(r,a_j;f_2^{(n)})}$$
$$>\quad\frac{(n+1)\frac{p_m}{1+p_m}}{k-2-(n+1)\Big(m-1-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}+(1+\sum_{j=1}^{k}\delta_j)\frac{p_m}{1+p_m}\Big)},$$

then $f_1^{(n)}(z)\equiv f_2^{(n)}(z)$.

2. Lemmas

In this section we state some lemmas needed to prove the theorems.

Lemma 1. [8] Let f be a nonconstant meromomorphic function on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<r<R_0\leq+\infty$. Then
(i) $T_0(r,f)=T_0\big(r,\frac{1}{f}\big)$,
(ii) $\max\big\{T_0(r,f_1.f_2),T_0\big(r,\frac{f_1}{f_2}\big),T_0(r,f_1+f_2)\big\}\leq T_0(r,f_1)+T_0(r,f_2)+O(1)$.

Lemma 2. [8] (The first fundamental theorem) Let f be a nonconstant meromomorphic function on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<r<R_0\leq+\infty$. Let $T_0(r,f)$ be Nevanlinna characteristic function. Then $T_0\big(r,\frac{1}{f-a}\big)=T_0(r,f)+O(1)$, for every fixed $a\in\mathbb{C}$.

In 2005, the lemma on the logarithmic derivative on the annulus $\mathbb{A}$ was obtained by Khrystiyanyn and Kondratyuk [9].

Lemma 3. [9] Let f be a nonconstant meromorphic function on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<r<R_0\leq+\infty$, and let $\lambda\geq0$. Then

$$m_0\big(r,\frac{f'}{f}\big)=S_1(r,f),$$

where (i) in the case $R_0=+\infty$,

$$S_1(r,*)=O(\log(rT_0(r,*)))$$

for $r\in(1,+\infty)$, except for the set Δ_r, such that $\int_{\Delta_r}r^{\lambda-1}dr<+\infty$;
(ii) if $R_0<+\infty$, then

$$S_1(r,*)=O\big(\log\big(\frac{T_0(r,*)}{R_0-r}\big)\big)$$

for $r\in(1,R_0)$, except for the set Δ'_r, such that $\int_{\Delta'_r}\frac{dr}{(R_0-r)^{\lambda-1}}<+\infty$.

Lemma 4. [2] (The second fundamental theorem) Let f be a nonconstant meromorphic function on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<r<R_0\leq+\infty$. Let $a_1,a_2,...,a_q$ be q distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Then

$$(q-2)T_0(r,f)<\sum_{j=1}^{q}\overline{N}_0\big(r,\frac{1}{f-a_j}\big)+S_1(r,f),$$

where $S_1(r,f)$ is stated as in above Lemma.

Lemma 5. [2] Let f be a non-constant meromorphic function on the annulus

$\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < r < R_0 \leq +\infty$. Let a be an arbitrary complex number and k be a positive integer. Then

$$(i)\ \overline{N}_0\big(r,a;f\big) \leq \frac{k}{k+1}\overline{N}_0^{k)}\big(r,a;f\big) + \frac{1}{k+1}N_0\big(r,a;f\big),$$

$$(ii)\ \overline{N}_0\big(r,a;f\big) \leq \frac{k}{k+1}\overline{N}_0^{k)}\big(r,a;f\big) + \frac{1}{k+1}T_0(r,f) + O(1).$$

Lemma 6. Let f be a transcendental meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq \infty$ and $a_1,\ a_2, ..., a_k$ be k (≥ 3) distinct complex numbers. If for a non-negative integer n, $E(0,f) \subseteq E(0,f^{(n)})$, then
$(k-2+o(1))T_0(r,f) \leq \sum_{j=1}^{k} \overline{N_0}(r,a_j;f^{(n)})$.

Proof. By the first fundamental theorem on annulus, we have

$$\begin{aligned} T_0(r,f) &= T_0(r,\frac{1}{f}) + O(1) \\ &\leq N_0(r,0;f) + m_0(r,\frac{f^{(n)}}{f}) + m_0(r,\frac{1}{f^{(n)}}) + O(1) \\ &\leq N_0(r,0;f) + T_0(r,f^{(n)}) - N_0(r,0;f^{(n)}) + S_1(r,f) \qquad (1) \end{aligned}$$

By the second fundamental theorem on annulus, we get

$$(k-1)T_0(r,f^n) \leq \overline{N_0}(r,\infty;f^{(n)}) + \sum_{j=1}^{k-1}\overline{N_0}(r,a_j;f^{(n)}) + \overline{N_0}(r,0;f^{(n)}) + S_1(r,f).$$

Without loss of generality, we may assume that $a_k = 0$. Otherwise a suitable linear transformation is taken. Then the above inequality reduces to

$$(k-1)T_0(r,f^n) \leq \overline{N_0}(r,\infty;f^{(n)}) + \sum_{j=1}^{k}\overline{N_0}(r,a_j;f^{(n)}) + S_1(r,f), \qquad (2)$$

Using (2) in (1), we obtain

$$\begin{aligned} (k-1)T_0(r,f) &\leq (k-1)N_0(r,0;f) + \overline{N_0}(r,\infty;f^{(n)}) \\ &+ \sum_{j=1}^{k}\overline{N_0}(r,a_j;f^{(n)}) - (k-1)N_0(r,0;f^{(n)}) + S_1(r,f) \end{aligned}$$

$$\begin{aligned} \Rightarrow (k-1)T_0(r,f) &\leq (k-1)N_0(r,0;f) + \overline{N_0}(r,\infty;f) \\ &+ \sum_{j=1}^{k}\overline{N_0}(r,a_j;f^{(n)}) - (k-1)N_0(r,0;f^{(n)}) + S_1(r,f). \quad (3) \end{aligned}$$

Since $E(0,f) \subseteq E(0,f^{(n)})$, we have from (3)

$$(k-1)T_0(r,f) \leq \overline{N_0}(r,\infty;f) + \sum_{j=1}^{k}\overline{N_0}(r,a_j;f^{(n)}) + S_1(r,f)$$

$\Rightarrow (k-2+o(1))T_0(r,f) \leq \sum_{j=1}^{k}\overline{N_0}(r,a_j;f^{(n)})$.
This complete the proof of the lemma.

To prove unicity theorem related to multiple values and derivatives of meromorphic functions on annuli, we need the following Xiong inequality of meromorphic functions on annuli.

Lemma 7. [17] Let f be a transcendental or admissible meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq \infty$. Let a be a finite complex number and $b_1,\ b_2, ..., b_q$ be q distinct finite nonzero complex numbers and k be a natural number. Then

$$\begin{aligned} qT_0(r,f) &< \overline{N_0}(r,f) + qN_0(r,a;f) + \sum_{j=1}^{q} N_0(r,b_j;f^{(n)}) \\ &- (q-1)N_0(r,0;f^{(n)}) - N_0(r,0;f^{(n+1)}) + S_1(r,f), \end{aligned}$$

where $S_1(r,f)$ is same as in Lemma 4.

3. Proof of the Main Theorems

Proof of Theorem 5.

By Xiong inequality of meromorphic functions on annuli, we have

$$\begin{aligned} (k-2)T_0(r,f_1) &< \overline{N_0}(r,f_1) + (k-2)N_0(r,0;f_1) + \sum_{j=1}^{k-2} N_0(r,a_j;f_1^{(n)}) \\ &- (k-3)N_0(r,0;f_1^{(n)}) + S_1(r,f_1). \end{aligned} \tag{4}$$

Similarly,

$$(k-2)T_0(r,f_2) < \overline{N_0}(r,f_2)+(k-2)N_0(r,0;f_2)+\sum_{j=1}^{k-2} N_0(r,a_j;f_2^{(n)})-(k-3)N_0(r,0;f_2^{(n)})+S_1(r,f_2). \tag{5}$$

Since, $E(0,f_i) \subseteq E(0,f_i^{(n)})$ for $i = 1,2$ we get from (4) and (5)

$$(k-2)T_0(r,f_1) < \overline{N_0}(r,f_1) + \sum_{j=1}^{k-2} N_0(r,a_j;f_1^{(n)}) + N_0(r,0;f_1^{(n)}) + S_1(r,f_1), \tag{6}$$

and

$$(k-2)T_0(r,f_2) < \overline{N_0}(r,f_2) + \sum_{j=1}^{k-2} N_0(r,a_j;f_2^{(n)}) + N_0(r,0;f_2^{(n)}) + S_1(r,f_2). \tag{7}$$

Without loss of generality let $a_k = \infty$, $a_{k-1} = 0$. First we take all a_j $(1 \leq j \leq k)$ are finite. Then from (6) and (7) we get,

$$(k-3)T_0(r,f_1) < \sum_{j=1}^{k-1} N_0(r,a_j;f_1^{(n)}) + S_1(r,f_1), \tag{8}$$

and

$$(k-3)T_0(r,f_2) < \sum_{j=1}^{k-1} N_0(r,a_j;f_2^{(n)}) + S_1(r,f_2). \tag{9}$$

Assume that $f_1^{(n)}(z) \not\equiv f_2^{(n)}(z)$. Therefore using (8) and (9)

$$\begin{aligned}\sum_{j=1}^{k-1} N_0(r,a_j;f_1^{(n)}) &< N_0(r,0;f_1^{(n)}-f_2^{(n)})\\ &\leq T_0(r,f_1^{(n)})+T_0(r,f_2^{(n)})+O(1)\\ &\leq (n+1)\{T_0(r,f_1)+T_0(r,f_2)\}+O(1)\\ &\leq \big\{\frac{n+1}{k-3}+O(1)\big\}\big[\sum_{j=1}^{k-1} N_0(r,a_j;f_1^{(n)})+\sum_{j=1}^{k-1} N_0(r,a_j;f_2^{(n)})\big]\end{aligned}$$

which gives, $\liminf_{r\to\infty} \frac{\sum_{j=1}^{k-1} N_0(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k-1} N_0(r,a_j;f_2^{(n)})} \leq \frac{n+1}{k-(n+4)}$

i;e.,

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} N_0(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k} N_0(r,a_j;f_2^{(n)})} \leq \frac{n+1}{k-(n+3)} \tag{10}$$

which is a contradiction.

Similarly, when $a_k=\infty$, we get (10). Hence $f_1^{(n)}(z) \equiv f_2^{(n)}(z)$. This complete the proof.

Corollary 1. From Theorem 5. it follows that $f_1(z) \equiv f_2(z)+p(z)$, where $p(z)$ is a polynomial of degree less than n.

Proof of Theorem 6.

By Lemma 6. we have

$$(k-2+o(1))T_0(r,f_1) < \sum_{j=1}^{k} \overline{N_0}(r,a_j;f_1^{(n)}) \tag{11}$$

and

$$(k-2+o(1))T_0(r,f_2) < \sum_{j=1}^{k} \overline{N_0}(r,a_j;f_2^{(n)}). \tag{12}$$

From (11) and Lemma 5. we have

$$\begin{aligned}(k-2+o(1))T_0(r,f_1) &\leq \sum_{j=1}^{k}\big\{\frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+\frac{1}{1+p_j}N_0(r,a_j;f_1^{(n)})\big\}\\ &\leq \sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+\left(\sum_{j=1}^{k}\frac{1}{1+p_j}\right)T_0(r,f_1^{(n)})\\ &\leq \sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+(n+1)\left(\sum_{j=1}^{k}\frac{1}{1+p_j}\right)T_0(r,f_1)\end{aligned}$$

i.e.,

$$\big\{k-2-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}+o(1)\big\}T_0(r,f_1) \leq \sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$

Similarly from (12) we get

$$\Big\{k-2-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}+o(1)\Big\}T_0(r,f_2)\le\sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})$$

Let $B_j=\overline{E}_{p_j)}(a_j,f_1^{(n)})\setminus A_j$ for $j=1,2,...,k$.
Now

$$\begin{aligned}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)}) &= \sum_{j=1}^{k}\overline{N_0}^{A_j}(r,a_j;f_1^{(n)})+\sum_{j=1}^{k}\overline{N_0}^{B_j}(r,a_j;f_1^{(n)})\\ &\le \delta T_0(r,f_1^{(n)})+N_0(r,0;f_1^{(n)}-f_2^{(n)})\\ &\le (1+\delta)(n+1)T_0(r,f_1)+(n+1)T_0(r,f_2)\end{aligned}$$

i.e.,

$$\begin{aligned}&\Big\{k-2-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}+o(1)\Big\}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})\\ \le\ & (1+\delta)(n+1)\sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+(n+1)\sum_{j=1}^{k}\frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})\end{aligned}$$

Since $1\ge\frac{p_1}{1+p_1}\ge\frac{p_2}{1+p_2}\ge...\ge\frac{p_k}{1+p_k}\ge\frac{1}{2}$, we get from above inequality

$$\begin{aligned}&\Big\{k-2-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}+o(1)\Big\}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})\\ \le\ & (1+\delta)(n+1)\frac{p_1}{1+p_1}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+(n+1)\frac{p_1}{1+p_1}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})\end{aligned}$$

i.e.,

$$\Big\{k-2-(n+1)\sum_{j=1}^{k}\frac{1}{1+p_j}-(1+\delta)(n+1)\frac{p_1}{1+p_1}+o(1)\Big\}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$

$$\le(n+1)\frac{p_1}{1+p_1}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)}).$$

Therefore

$$\begin{aligned}&\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})}\\ \le\ & \frac{(n+1)p_1}{(k-2)(1+p_1)-(n+1)(1+p_1)\sum_{j=1}^{k}\frac{1}{1+p_j}-(n+1)(1+\delta)p_1}\\ =\ & \frac{(n+1)p_1}{(k-2)(1+p_1)-(n+1)(1+p_1)\sum_{j=2}^{k}\frac{1}{1+p_j}-(n+1)\{(1+\delta)p_1+1\}},\end{aligned}$$

which is a contradiction.

Therefore $f_1^{(n)}(z)\equiv f_2^{(n)}(z)$. This complete the proof.

Proof of Theorem 7.
By Lemma 6. we have

$$(k-2+o(1))T_0(r,f_1)<\sum_{j=1}^{k}\overline{N_0}(r,a_j;f_1^{(n)}) \tag{13}$$

and

$$(k-2+o(1))T_0(r,f_2)<\sum_{j=1}^{k}\overline{N_0}(r,a_j;f_2^{(n)}). \tag{14}$$

From (13), Lemma 5. and using $1\geq\frac{p_1}{1+p_1}\geq\frac{p_2}{1+p_2}\geq...\geq\frac{p_k}{1+p_k}\geq\frac{1}{2}$, we have

$$\begin{aligned}(k-2+o(1))T_0(r,f_1) &\leq \sum_{j=1}^{k}\{\frac{p_j}{1+p_j}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+\frac{1}{1+p_j}N_0(r,a_j;f_1^{(n)})\}\\ &\leq \sum_{j=1}^{m-1}(\frac{p_j}{1+p_j}-\frac{p_m}{1+p_m})\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+\left(\sum_{j=1}^{k}\frac{1}{1+p_j}\right)T_0(r,f_1^{(n)})\\ &+ \sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+S_1(r,f_1)\\ &\leq \sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})\\ &+ (n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}\right)T_0(r,f_1)+S_1(r,f_1)\end{aligned}$$

i.e.,

$$\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}\right)\right\}T_0(r,f_1)\leq\sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$

Similarly from (14) we get

$$\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}\right)\right\}T_0(r,f_2)\leq\sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})$$

Let $B_j=\overline{E}_{p_j)}(a_j,f_1^{(n)})\setminus A_j$ for $j=1,2,...,k$.
Now

$$\begin{aligned}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)}) &= \sum_{j=1}^{k}\overline{N_0}^{A_j}(r,a_j;f_1^{(n)})+\sum_{j=1}^{k}\overline{N_0}^{B_j}(r,a_j;f_1^{(n)})\\ &\leq \sum_{j=1}^{k}\delta_jT_0(r,f_1^{(n)})+N_0(r,0;f_1^{(n)}-f_2^{(n)})\\ &\leq (1+\sum_{j=1}^{k}\delta_j)(n+1)T_0(r,f_1)+(n+1)T_0(r,f_2)\end{aligned}$$

i.e.,

$$\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}\right)\right\}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$\leq \quad (1+\sum_{j=1}^{k}\delta_j)(n+1)\sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+(n+1)\sum_{j=1}^{k}\frac{p_m}{1+p_m}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})$$

Therefore we get from above inequality

$$\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}\right)\right\}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$\leq \quad (1+\sum_{j=1}^{k}\delta_j)(n+1)\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})+(n+1)\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})$$

i.e.,

$$\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}+(1+\sum_{j=1}^{k}\delta_j)\frac{p_m}{1+p_m}\right)\right\}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})$$
$$\leq (n+1)\frac{p_m}{1+p_m}\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)}).$$

Therefore

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_1^{(n)})}{\sum_{j=1}^{k}\overline{N_0}^{p_j)}(r,a_j;f_2^{(n)})}$$
$$\leq \quad \frac{(n+1)\frac{p_m}{1+p_m}}{\left\{k-2-(n+1)\left((m-1)-\frac{(m-1)p_m}{1+p_m}+\sum_{j=m}^{k}\frac{1}{1+p_j}+(1+\sum_{j=1}^{k}\delta_j)\frac{p_m}{1+p_m}\right)\right\}}$$

which is a contradiction.

Therefore $f_1^{(n)}(z)\equiv f_2^{(n)}(z)$. This complete the proof.

Corollary 2. For $n=0$ Theorem 7., reduced to Theorem 4.

Corollary 3. For $m=1$ and $\delta=\sum_{j=1}^{k}\delta_j$ Theorem 7., reduced to Theorem 6.

Conclusion: The main aim of the paper is to study the uniqueness of meromorphic functions on annuli through the uniqueness of their higher order derivatives on the annular region. There are many ways available in the literature to study the uniqueness of meromorphic functions on simply connected domain. After the Nevanlinna's Theory of meromorphic functions on annuli many authors investigated the uniqueness of meromorphic functions on this type of regions. Using the theory of value sharing, we have established some uniqueness results of meromorphic functions on annular region. We hope that our results will help researchers of this field to carry further research work.

References

[1] A. Banerjee, Weighted sharing of a small function by a meromorphic function and its derivativer. Comput. Math. Appl., Vol. 53, 1750–1761, 2007.

[2] T.B. Cao, H.X. Yi, H.Y Xu, On the multiple values and uniqueness of meromorphic functions on annuli. Comput. Math. Appl., Vol. 58, 1457–1465, 2009.

[3] T.B. Cao, H.X Yi, On the uniqueness of meromorphic functions that share four values in one angular domain. J. Math. Anal. Appl., Vol. 358, 81–97, 2009.

[4] T.B. Cao, H.X. Yi, Uniqueness theorems of meromorphic functions sharing sets IM on annuli. Acta Math. Sin. (Chin. Ser.), Vol. 54, 623–632, 2011.

[5] M. L. Fang, On the uniqueness of admissible meromorphic functions in the unit disc. Sci. China, Vol. 42, 367–381, 1999.

[6] W. K. Hayman, Meromorphic function, Clarendon Press, Oxford, 1964.

[7] R. Korhonen, Nevanlinna theory in an annulus, value distribution theory and related topics. Adv. Complex Anal. Appl., Vol. 3, 167–179, 2004.

[8] A.Y Khrystiyanyn, A.A. Kondratyuk, On the Nevanlinna theory for meromorphic functions on annuli I. Mat. Stud., Vol. 23, 19–30, 2005.

[9] A.Y Khrystiyanyn, A.A. Kondratyuk, On the Nevanlinna theory for meromorphic functions on annuli II. Mat. Stud., Vol. 24, 57–68, 2005.

[10] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions. Nagoya Math. J., Vol. 161, 193–206, 2001.

[11] W.C Lin, S. Mori, K. Tohge, Uniqueness theorems in an angular domain. Tohoku Math. J., Vol. 58, 509–527, 2006.

[12] X.M. Li, and H. X. Yi, On a uniqueness theorem of meromorphic functions concerning weighted sharing of three values. Bull. Malays. Math. Sci. Soc., Vol. 33, 1–16, 2010.

[13] Z. Q. Mao, and H. F. Liu, Meromorphic functions in the unit disc that share values in an angular domain. J. Math. Anal. Appl., Vol. 359, 444–450, 2009.

[14] R. Nevanlinna, Eindentig keitss?tze in der theorie der meromorphen funktionen. Acta Math., Vol. 48, 367–391, 1926.

[15] H.Y. Xu,C. F. Yi, T.B. Cao, The uniqueness problem for meromorphic functions in the unit disc sharing values and a set in an angular domain. Math. Scand., Vol. 109, 240–252, 2011.

[16] H.Y Xu, Z.X. Xuan, The uniqueness of analytic functions on annuli sharing some values. Abstr. Appl. Anal., Vol. 2012, 1–13, 2012.

[17] J. Xu, L. Chen, A unicity theorem related to multiple values and derivatives of meromorphic functions on annuli. J. Math. Reseach. appl., Vol. 36, 467–476, 2016.

[18] H. Y. Xu, H. Wang, Nevannlinna‘s five value theorem on annuli, $\sum$ Mathematics, Vol. 4(1), pages 7, 2016.

[19] L. Yang, Value distributions theory, Springer-Verlag, Berlin, 1993.

[20] H. X. Yi and C. C. Yang, Uniqueness theory of meromorphic functions(in Chinese), Science Press, Beijing, 1995.

[21] J.H Zheng, On uniqueness of meromorphic functions with shared values in one angular domains. Complex Var. Elliptic Equ., Vol. 48, 777–785, 2003.

[22] Q.C Zhang, Meromorphic Functions Sharing Values in an Angular Domain. J.Math. Anal. Appl., Vol. 349, 100–112, 2009.

[23] J.H Zheng, Value Distribution of Meromorphic Functions; Tsinghua University Press Beijing, China, 2010.

Dilip Chandra Pramanik

Department of Mathematics, University of North Bengal, Raja Rammohunpur, Darjeeling-734013, West Bengal, India

E-mail address: dcpramanik.nbu2012@gmail.com

Jayanta Roy
Department of Mathematics, University of North Bengal, Raja Rammohunpur, Darjeeling-734013, West Bengal, India
E-mail address: jayantaroy983269@yahoo.com