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WEIGHTED SHARING OF Q-SHIFT DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS SHARING A SMALL FUNCTION

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Abstract. In this article, with the notion of weighted sharing we study the uniqueness problems of q-shift difference-differential polynomials of meromorphic functions sharing a small function $a(z)$ with weight l. Our result improves and generalize a recent result of Renukadevi S. Dyavanal and Ashwini M. Hattikal.

1. Introduction and main results

Let f be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory: $T(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $m(r,f)$, (see [17]). The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty, r \notin E$, where E is a set of positive real number of finite linear measure, not necessarily the same at each occurrence. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ provided that $T(r,a) = S(r,f)$. Suppose that $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with same counting multiplicities (ignoring multiplicities), then we say that f and g share $a(z)$ CM(IM).

Definition 1.[13] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, then we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f-a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g-a$ with multiplicity $m(\le k)$; and z_0 is a zero of $f-a$ with multiplicity $m(> k)$ if and only if it is a zero of $g-a$ with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a,l) to mean that f, g share the value 'a' with weight l. Clearly if f, g share (a,l), then f, g share (a,p) for all integer p, $0 \le p < k$. Also, we note that f, g share a value 'a' IM or CM, if and only if f, g share $(a,0)$ or

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(a, ∞), respectively.

Definition 2.[2] We denote and define order of $f(z)$ by

$$\rho(f) = \limsup_{r\to\infty} \frac{logT(r,f)}{logr}$$

If a non-constant meromorphic function $f(z)$ is of zero order, then $\rho(f) = 0$.

Recently difference polynomials in the complex plane $\mathbb{C}$ become a subject of great interest among the researcher around the world. With the development of difference analogue of Nevanlinna theory [see [3], [4], [5], [6]], a large number of papers have focused on value distribution and uniqueness of difference polynomials.

In 2014, X.M.Li, H.X.Yi and W.L.Li [7] proved the following theorem on uniqueness of difference polynomials of meromorphic functions sharing a small function.

Theorem 1. Let f and g be two transcendental meromorphic function of finite order, let $\alpha \not\equiv 0$ be an entire function such that $\rho(\alpha) < \rho(f)$, let η be a non-zero complex number and let n and m be two positive integers such that $n \geq m+12$ and $m \geq 2$. Suppose $f^n(z)(f^m(z)-1)f(z+\eta)-\alpha(z)$ and $g^n(z)(g^m(z)-1)g(z+\eta)-\alpha(z)$ share $0, \infty$ CM. Then $f(z) = tg(z)$, where t is a constant satisfying $t^m = 1$.

Further, K.Y. Zhang and H.X.Yi [19] extended the result of X.M.Li, H.X.Yi and W.L.Li [7] and proved the theorem on uniqueness of product of differential-difference polynomials of entire functions as in the following theorem.

Theorem 2. Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order, $\alpha(z) \not\equiv 0$ be a common small function with respect to f and g, $c_j\,(j = 1, 2, ..., d)$ be distinct finite complex numbers and n, m, d and $v_j\,(j = 1, 2, ..., d)$ are non-negative integers. If $n \geq 4k - m + \sigma + 9$ and the differential-difference polynomial $(f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{v_j})^{(k)}$ and $(g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{v_j})^{(k)}$ share $\alpha(z)$ CM, then $f \equiv g$.

In 2015, F.H.Liu and H.X.Yi [9] improved the previous results by considering uniqueness problems on product of difference polynomials of meromorphic functions.

Theorem 3. Let $f(z)$ and $g(z)$ be non-constant meromorphic functions satisfying $\rho(f) < \infty, \rho(g) < \infty$. $f(z)$ and $g(z)$ share ∞ IM. $\alpha(z) \not\equiv 0$ is an entire function satisfying $\rho(\alpha) < \rho(f)$. m, n, s, $\mu_j\,(j = 1, 2, ..., s)$ are non-negative integers, $\sigma = \sum_{j=1}^{s} \mu_j$, $c_j\,(j = 1, 2, ..., s)$ are non-zero complex constants. $F(z) = f^n(f^m - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}$, $G(z) = g^n(g^m - 1)\prod_{j=1}^{s} g(z + c_j)^{\mu_j}$ share α, ∞ CM. If $n \geq m + 2s + 3\sigma + 7$ we get $f(z) = tg(z)$, where t is a constant satisfying $t^m = 1$.

Recently, R. S. Dyavanal and A. M. Hattikal [2] investigated the uniqueness of difference polynomials of meromorphic functions sharing a small function $a(z)$ with counting multiplicity.

Theorem 4. Let f and g be two non-constant meromorphic functions of zero order and $a(z)$ is a small function with respect to both f and g. Let $n \geq m + 3\lambda + 2d + 7$ be a positive integer, where m, d, $\lambda\,(= \sum_{j=1}^{d} s_j\ for\, j = 1, 2, ..., d)$ are finite positive integers such that $d < \lambda$. Let q_j, $c_j\,(j = 1, 2, ..., d)$ are distinct non-zero complex constants. If

$$f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(q_j z + c_j)^{s_j}$$

and

$$g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}$$

share $a(z)$ CM, f and g share ∞ IM, then

(1) if $m \geq 2$, then either $f = tg$ for a constant t such that $t^d = 1$ where $d = GCD(n+m+\lambda, n+m+\lambda-1, ..., n+m+\lambda-i, ..., n+\lambda)$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n(w_1-1)^m \prod_{j=1}^{d} w_1(q_j z + c_j)^{s_j} - w_2^n(w_2-1)^m \prod_{j=1}^{d} w_2(q_j z + c_j)^{s_j}$$

(2) if $m = 1$, then $f = tg$ for a constant t such that $t^d = 1$ where $d = GCD(n+\lambda, n+1+\lambda)$.

In this paper, we define a $q-$shift difference product of meromorphic function $f(z)$ as follows.

$$F(z) = (f^n(z)P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k)} \tag{1}$$

$$F_1(z) = f^n(z)P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j} \tag{2}$$

where q_j, c_j $(j = 1, 2, ..., d)$ are distinct non-zero complex constants, n, d, k, λ, s_j $(j = 1, 2, ..., d)$ be positive integers. $\lambda = \sum_{j=1}^{d} s_j$. Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_0$ is a non-zero polynomial of degree m and $\Gamma_0 = m_1 + m_2$, where m_1 is the number of the simple zero of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$.

Here, we used the idea of weighted sharing values to extend the above results for meromorphic functions.

Theorem 5. Let f and g be two non-constant meromorphic functions of zero order and $a(z)$ is a small function with respect to both f and g. If F and G share $(a(z), l)$, where l, n are positive integers; f and g share ∞ IM with the conditions of n as below

(i) $n > 3k + 2\Gamma_0 - m + kd + \lambda + 3d + 7$, when $l \geq 2$

(ii) $n > 4k + \frac{5\Gamma_0}{2} - m + \frac{3kd}{2} + \frac{3\lambda}{2} + \frac{7d}{2} + 8$, when $l = 1$

(iii) $n > 9k + 5\Gamma_0 - m + 4kd + 4\lambda + 6d + 13$, when $l = 0$ then one of the following cases hold:

1) $f \equiv tg$ for a constant t such that $t^l = 1$, where $l = GCD\{n + \lambda_0 + \lambda, n + \lambda_1 + \lambda, ..., n + \lambda_m + \lambda\}$ and

$$\lambda_i = \begin{cases} i, & a_i \neq 0 \\ m, & a_i = 0 \end{cases} \qquad i = 0, 1, ..., m.$$

2) f and g satisfy the algebraic equation $R(f,\, g) \equiv 0$, where

$$R(w_1,\, w_2) = w_1^n P(w_1) \prod_{j=1}^{d} w_1(z+c_j)^{s_j} - w_2^n P(w_2) \prod_{j=1}^{d} w_2(z+c_j)^{s_j}.$$

Remark 1. When $k = 0$ and $\Gamma_0 = m_1 + m_2 = m$ in Theorem 5, then Theorem 5 improves and generalize Theorem 3 and Theorem 4.

Remark 2. When $k = 0$, $\Gamma_0 = m_1 + m_2 = m$, $\lambda = 1$ and $d = 1$ in Theorem 5, then Theorem 5 reduces to Theorem 1.

Corollary. Let f and g be two non-constant entire functions of zero order and $a(z)$ is a small function with respect to both f and g. If F and G share $(a(z), l)$, where l, n are positive integers; f and g share ∞ IM with the conditions of n as below

(i) $n \geq 2k - m + 2\Gamma_0 + \lambda + 5$, when $l \geq 2$

(ii) $n \geq \frac{5k}{2} + \frac{5\Gamma_0}{2} + \frac{3\lambda}{2} - m + \frac{11}{2}$, when $l = 1$

(iii) $n \geq 5k + 5\Gamma_0 + 4\lambda - m + 8$, when $l = 0$ then conclusion of Theorem 5 holds.

2. **Some Lemmas**

Lemma 1.[18] Let $f(z)$ be a non-constant meromorphic function, and $a_n(\neq 0), a_{n-1}, ..., a_0$ be small functions with respect to f. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f)$$

Lemma 2.[16] Let $f(z)$ be a non-constant meromorphic function of zero order, and let c and q be two non-zero complex numbers. Then

$$T(r, f(qz + c)) = T(r, f(z)) + S(r, f),$$

on a set of logarithmic density 1.

Lemma 3.[8] Let f be a meromorphic function with zero order and c and q be two non-zero complex numbers. Then

$$N\left(r, \frac{1}{f(qz+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f) \quad N(r, f(qz+c)) \leq N(r, f) + S(r, f)$$
$$\overline{N}\left(r, \frac{1}{f(qz+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f) \quad \overline{N}(r, (f(qz+c)) \leq N(r, f) + S(r, f)$$

outside of a possible exceptional set E with finite logarithmic measure.

Lemma 4.[10] Let $f(z)$ be a non-constant meromorphic function and p, k be positive integers. Then

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f), \tag{3}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f). \tag{4}$$

Lemma 5.[1] Let F, G be two nonconstant meromorphic functions sharing $(1, 2)$, $(\infty, 0)$ and $H \not\equiv 0$. Then

(i) $T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G) -$

$m(r,1;G) - N_E^{(3}(r,1;F) - \overline{N}_L(r,1;G) + S(r,F) + S(r,G)$;

(ii) $T(r,G) \leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) - m(r,1;F) - N_E^{(3}(r,1;G) - \overline{N}_L(r,1;F) + S(r,F) + S(r,G)$.

Lemma 6.[12] Let F, G be two nonconstant meromorphic functions sharing $(1,1)$, $(\infty,0)$ and $H \not\equiv 0$. Then

(i) $T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$;

(ii) $T(r,G) \leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \frac{3}{2}\overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$.

Lemma 7.[12] Let F, G be two nonconstant meromorphic functions sharing $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$. Then

(i) $T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$;

(ii) $T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + 3\overline{N}(r,\infty;G) + \overline{N}(r,0;F) + 2\overline{N}(r,0;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$.

Lemma 8. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, let n, k be two positive integers with $n > k+\Gamma_0-m+2\lambda+d+2$ and $a(z)\,(\not\equiv 0,\infty)$ be a small function with respect to f and g and let $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$, where $a_0, a_1, ..., a_{m-1}, a_m$ are complex constants. If

$$(f^n(z)P(f)\prod_{j=1}^{d} f(q_j z+c_j)^{s_j})^{(k)}(g^n(z)P(g)\prod_{j=1}^{d} g(q_j z+c_j)^{s_j})^{(k)} \equiv a^2,$$

f and g share ∞ IM, then $P(z)$ is reduced to a nonzero monomial, that is $P(z) = a_i z^i \neq 0$ for some $i = 0,1,2,...,m$.

Proof. If $P(z)$ is not reduced to a nonzero monomial, then without loss of generality, we assume that $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$, where $a_0\,(\neq 0), a_1, ..., a_{m-1}, a_m\,(\neq 0)$ are complex constants. By hypothesis of Lemma 8, we know that either both f and g are transcendental meromorphic functions or they are both rational functions. since f and g share ∞ IM, the poles of f and g are finite. Similarly f and g has finitely many zeros.

Case 1. If f and g are transcendental meromorphic functions. Let $f = he^{\beta}$, where β is a non-constant entire function and $h(z)$ is a nonzero rational function. Thus, by induction on k, we get

$$\begin{aligned}(a_i f^{i+n}\prod_{j=1}^{d} f(q_j z+c_j)^{s_j})^{(k)} = {} & P_i(\beta^{'},\beta^{''},...,\beta^{(k)},\sum s_j\beta^{'}(q_j z+c_j),...,\\ & \sum s_j\beta^{(k)}(q_j z+c_j), h, h^{'},...,h^{(k)},\sum s_j h(q_j z+c_j),\sum s_j h^{'}(q_j z+c_j),...,\\ & \sum s_j h^{(k)}(q_j z+c_j))e^{(i+n)\beta(z)+\sum_{j=1}^{d} s_j\beta(q_j z+c_j)}\end{aligned}$$

where, $P_i\,(i = 1,2,...,m)$ are difference-differential polynomials with coefficients as rational functions in $h(z)$ and $\sum s_j h(z+c_j)$ or its derivatives.

Notice that
$P_0(\beta^{'},\beta^{''},...,\beta^{(k)},\sum s_j\beta^{'}(q_jz+c_j),...,\sum s_j\beta^{(k)}(q_jz+c_j),h,h^{'},...,h^{(k)},$
$\sum s_jh(q_jz+c_j),\sum s_jh^{'}(q_jz+c_j),...,\sum s_jh^{(k)}(q_jz+c_j)),...,P_m(\beta^{'},\beta^{''},...,\beta^{(k)},$
$\sum s_j\beta^{'}(q_jz+c_j),...,\sum s_j\beta^{(k)}(q_jz+c_j),h,h^{'},...,h^{(k)},\sum s_jh(q_jz+c_j),$
$\sum s_jh^{'}(q_jz+c_j),...,\sum s_jh^{(k)}(q_jz+c_j))\not\equiv 0.$

Since $\beta(z)$ is an entire function,

$$T(r,\beta^{'}(z))=m(r,\beta^{'}(z))=m\left(r,\frac{(e^{\beta(z)})^{'}}{e^{\beta(z)}}\right)=S(r,f).$$

Thus, we obtain

$$T(r,\beta^{(k)}(z))\le T(r,\beta^{'})+S(r,f)=S(r,f)\, for\ j=1,2,...,k,$$

and

$$\begin{aligned}T(r,\sum s_j\beta^{'}(q_jz+c_j))&=m(r,\sum s_j\beta^{'}(q_jz+c_j))+N(r,\sum s_j\beta^{'}(q_jz+c_j))\\&=m(r,\sum s_j\beta^{'}(q_jz+c_j))\\&=m\left(r,\frac{(e^{\sum s_j\beta(q_jz+c_j)^{'}})}{e^{\sum s_j\beta(q_jz+c_j)}}\right)=S(r,f).\end{aligned}$$

Therefore

$$T(r,\sum s_j\beta^{(k)}(q_jz+c_j))\le T(r,\sum s_j\beta^{'}(q_jz+c_j))+S(r,f)=S(r,f)\, for\ j=1,2,...,k,$$

which is a contradiction.

Case 2. If f and g are rational functions, then a is a nonzero constant, thus f and g have no zeros and no poles, which is impossible. Since f and g are not constants.

The above two Cases imply that $P(z)$ is reduced to a nonzero monomial, namely, $P(z)=a_iz^i\not\equiv 0$ for some $i\in\{0,1,...,m\}$.

3. Proof of Theorem 5.

Let $F^*=\frac{F}{a(z)}$ and $G^*=\frac{G}{a(z)}$. From the hypothesis we have $F(z)$ and $G(z)$ share $(a(z),l)$ and f, g share ∞ IM. It follows that F^* and G^* share 1CM and ∞ IM. We now discuss the following two cases separately.

Case 1. We assume that $H\not\equiv 0$. Now we consider the following three subcases.

Subcase 1. Suppose that $l\ge 2$. Then using Lemma 5 we obtain

$$\begin{aligned}T(r,F)&\le T(r,F^*)+S(r,F)\\&\le N_2(r,0;F^*)+N_2(r,0;G^*)+\overline{N}(r,\infty;F^*)+\overline{N}(r,\infty;G^*)+\overline{N}_*(r,\infty;F^*,G^*)\\&-m(r,1;G^*)-N_E^{(3}(r,1;F^*)-\overline{N}_L(r,1;G^*)+S(r,F^*)+S(r,G^*)\\&\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)\\&+S(r,F)+S(r,G).\end{aligned}\tag{5}$$

Noting that

$$\begin{aligned}\overline{N}_*(r,\infty;F^*,G^*) &= \overline{N}_L(r,\infty;F)+\overline{N}_L(r,\infty;G)\\ &\leq \overline{N}(r,\infty;F)=\overline{N}(r,\infty;G),\end{aligned} \tag{6}$$

we obtain from (5) that

$$\begin{aligned}T(r,F) \leq\ & N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,F)\\ &+S(r,G).\end{aligned} \tag{7}$$

By using (3) and (4), we have

$$\begin{aligned}T(r,F) \leq\ & T(r,F)-T(r,F_1)+N_{k+2}(r,0;F_1)+k\overline{N}(r,\infty;G_1)+N_{k+2}(r,0;G_1)\\ &+2\overline{N}(r,\infty;F_1)+\overline{N}(r,\infty;G_1)+S(r,f)+S(r,g)\end{aligned}$$

$$\begin{aligned}T(r,F_1) \leq\ & N_{k+2}(r,0;f^nP(f)\prod_{j=1}^{d}f(q_jz+c_j)^{s_j})+k\overline{N}(r,\infty;g^nP(g)\prod_{j=1}^{d}g(q_jz+c_j)^{s_j})\\ &+N_{k+2}(r,0;g^nP(g)\prod_{j=1}^{d}g(q_jz+c_j)^{s_j})+2\overline{N}(r,\infty;f^nP(f)\prod_{j=1}^{d}f(q_jz+c_j)^{s_j})\\ &+\overline{N}(r,\infty;g^nP(g)\prod_{j=1}^{d}g(q_jz+c_j)^{s_j})+S(r,f)+S(r,g)\end{aligned}$$

$$\begin{aligned}(n+m+\lambda)T(r,f) \leq\ & (k+\Gamma_0+2d+\lambda+4)T(r,f)+(2k+\Gamma_0+\lambda+kd+d\\ &+3)T(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{8}$$

Similarly, we have for $T(r,g)$,

$$\begin{aligned}(n+m+\lambda)T(r,g) \leq\ & (k+\Gamma_0+2d+\lambda+4)T(r,g)+(2k+\Gamma_0+\lambda+kd+d\\ &+3)T(r,f)+S(r,f)+S(r,g)\end{aligned} \tag{9}$$

from (8) and (9), we have

$$\begin{aligned}(n+m+\lambda)[T(r,f)+T(r,g)] \leq\ & (3k+2\Gamma_0+2\lambda+kd+3d+7)[T(r,f)+T(r,g)]\\ &+S(r,f)+S(r,g),\end{aligned}$$

a contradiction with the fact that $n > 3k+2\Gamma_0-m+kd+\lambda+3d+7$.

Subcase 2. Let $l=1$. Then using (6) and Lemma 6 we obtain

$$\begin{aligned}T(r,F) \leq\ & T(r,F^*)+S(r,F)\\ \leq\ & N_2(r,0;F^*)+N_2(r,0;G^*)+\frac{3}{2}\overline{N}(r,\infty;F^*)+\overline{N}(r,\infty;G^*)+\overline{N}_*(r,\infty;F^*,G^*)\\ &+\frac{1}{2}\overline{N}(r,0;F^*)+S(r,F^*)+S(r,G^*)\\ \leq\ & N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)\\ &+\frac{1}{2}\overline{N}(r,0;F)+S(r,F)+S(r,G)\\ \leq\ & N_2(r,0;F)+N_2(r,0;G)+\frac{5}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F)\\ &+S(r,F)+S(r,G).\end{aligned} \tag{10}$$

Using (10), (3) and (4), we have

$$\begin{aligned}T(r,F) \le\ & T(r,F) - T(r,F_1) + N_{k+2}(r,0;F_1) + k\overline{N}(r,\infty;G_1) + N_{k+2}(r,0;G_1)\\ & + \frac{5}{2}\overline{N}(r,\infty;F_1) + \overline{N}(r,\infty;G_1) + \frac{1}{2}[k\overline{N}(r,\infty;F_1) + N_{k+1}(r,0;F_1)]\\ & + S(r,f) + S(r,g)\end{aligned}$$

$$\begin{aligned}T(r,F_1) \le\ & N_{k+2}(r,0;f^nP(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}) + k\overline{N}(r,\infty;g^nP(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})\\ & + N_{k+2}(r,0;g^nP(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}) + \frac{5}{2}\overline{N}(r,\infty;f^nP(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})\\ & + \overline{N}(r,\infty;g^nP(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}) + \frac{1}{2}[k\overline{N}(r,\infty;f^nP(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})\\ & + N_{k+1}(r,0;f^nP(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})] + S(r,f) + S(r,g)\end{aligned}$$

$$\begin{aligned}(n+m+\lambda)T(r,f) \le\ & (2k + \frac{3\Gamma_0}{2} + \frac{3\lambda}{2} + \frac{kd}{2} + \frac{5d}{2} + 5)T(r,f) + (2k + \Gamma_0 + d + \lambda\\ & + kd + 3)T(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{11}$$

Similarly, we have for $T(r,g)$,

$$\begin{aligned}(n+m+\lambda)T(r,g) \le\ & (2k + \frac{3\Gamma_0}{2} + \frac{3\lambda}{2} + \frac{kd}{2} + \frac{5}{2}d + 5)T(r,g) + (2k + \Gamma_0 + d + \lambda\\ & + kd + 3)T(r,f) + S(r,f) + S(r,g)\end{aligned} \tag{12}$$

from (11) and (12), we have

$$\begin{aligned}(n+m+\lambda)[T(r,f)+T(r,g)] \le\ & (4k + \frac{5\Gamma_0}{2} + \frac{5\lambda}{2} + \frac{3kd}{2} + \frac{7d}{2} + 8)[T(r,f)+T(r,g)]\\ & + S(r,f) + S(r,g),\end{aligned}$$

a contradiction with the fact that $n > 4k + \frac{5\Gamma_0}{2} - m + \frac{3kd}{2} + \frac{3\lambda}{2} + \frac{7d}{2} + 8$.

Subcase 3. Let $l = 0$. Then using (6) and Lemma 7 we obtain

$$\begin{aligned}T(r,F) \le\ & T(r,F^*) + S(r,F)\\ \le\ & N_2(r,0;F^*) + N_2(r,0;G^*) + 3\overline{N}(r,\infty;F^*) + 2\overline{N}(r,\infty;G^*) + \overline{N}_*(r,\infty;F^*,G^*)\\ & + 2\overline{N}(r,0;F^*) + \overline{N}(r,0;G^*) + S(r,F^*) + S(r,G^*)\\ \le\ & N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G)\\ & + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G).\\ \le\ & N_2(r,0;F) + N_2(r,0;G) + 4\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G)\\ & + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G).\end{aligned} \tag{13}$$

Using (13), (3) and (4), we have

$$\begin{aligned}T(r,F) \leq\ & T(r,F) - T(r,F_1) + N_{k+2}(r,0;F_1) + k\overline{N}(r,\infty;G_1) + N_{k+2}(r,0;G_1) + 4\overline{N}(r,\infty;F_1)\\ & + 2\overline{N}(r,\infty;G_1) + 2[k\overline{N}(r,\infty;F_1) + N_{k+1}(r,0;F_1)] + k\overline{N}(r,\infty;G_1) + N_{k+1}(r,0;G_1)\\ & + S(r,F) + S(r,G)\end{aligned}$$

$$\begin{aligned}T(r,F_1) \leq\ & N_{k+2}(r,0;f^nP(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}) + k\overline{N}(r,\infty;g^nP(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})\\ & + N_{k+2}(r,0;g^nP(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}) + 4\overline{N}(r,\infty;f^nP(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})\\ & + 2\overline{N}(r,\infty;g^nP(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}) + 2[k\overline{N}(r,\infty;f^nP(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})\\ & + N_{k+1}(r,0;f^nP(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})] + k\overline{N}(r,\infty;g^nP(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})\\ & + N_{k+1}(r,0;g^nP(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}) + S(r,f) + S(r,g)\end{aligned}$$

$$\begin{aligned}(n+m+\lambda)T(r,f) \leq\ & (5k+3\Gamma_0+3\lambda+2kd+4d+8)T(r,f) + (4k+2\Gamma_0+2kd\\ & + 2\lambda+2d+5)T(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{14}$$

Similarly, we have for $T(r,g)$,

$$\begin{aligned}(n+m+\lambda)T(r,g) \leq\ & (5k+3\Gamma_0+3\lambda+2kd+4d+8)T(r,g) + (4k+2\Gamma_0+2kd\\ & + 2\lambda+2d+5)T(r,f) + S(r,f) + S(r,g)\end{aligned} \tag{15}$$

from (14) and (15), we have

$$\begin{aligned}(n+m+\lambda)[T(r,f)+T(r,g)] \leq\ & (9k+5\Gamma_0+4kd+5\lambda+6d+13)[T(r,f)+T(r,g)]\\ & + S(r,f) + S(r,g),\end{aligned}$$

a contradiction with the fact that $n > 9k+5\Gamma_0-m+4kd+4\lambda+6d+13$.

Case 2. We now assume that $H \equiv 0$. Then

$$\left(\frac{F^{*\prime\prime}}{F^{*\prime}} - \frac{2F^{*\prime}}{F^*-1}\right) - \left(\frac{G^{*\prime\prime}}{G^{*\prime}} - \frac{2G^{*\prime}}{G^*-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F^*-1} = \frac{A}{G^*-1} + B, \tag{16}$$

where $A(\neq 0)$ and B are constants. From (16) it is obvious that F^*, G^* share the value 1CM and hence they share the value 1 with weight 2, and therefore, $n > 3k+2\Gamma_0-m+kd+\lambda+3d+7$. We now discuss the following three subcases separately.

Subcase 4. Suppose that $B \neq 0$ and $A = B$. Then from (16) we obtain

$$\frac{1}{F^* - 1} = \frac{BG^*}{G^* - 1}. \tag{17}$$

If $B = -1$, then from (17) we obtain

$$F^* G^* = 1,$$

i.e.,

$$(f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k)} (g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j})^{(k)} = a^2(z),$$

which is a contradiction by Lemma 8.

If $B \neq -1$, from (17), we have $\frac{1}{F^*} = \frac{BG^*}{(1+B)G^* - 1}$ and so $\overline{N}\left(r, \frac{1}{1+B}; G^*\right) = \overline{N}(r, 0; F^*)$.

Using (3), (4) and the Second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned} T(r, G) &\leq T(r, G^*) + S(r, G) \\ &\leq \overline{N}(r, 0; G^*) + \overline{N}\left(r, \frac{1}{1+B}; G^*\right) + \overline{N}(r, \infty; G^*) + S(r, G) \\ &\leq \overline{N}(r, 0; F^*) + \overline{N}(r, 0; G^*) + \overline{N}(r, \infty; G^*) + S(r, G) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G). \end{aligned} \tag{18}$$

Using (18), Lemma 4 we have

$$\begin{aligned} T(r, G) \leq{}& k\overline{N}(r, \infty; F_1) + N_{k+1}(r, 0; F_1) + T(r, G) - T(r, G_1) + N_{k+1}(r, 0; G_1) \\ &+ \overline{N}(r, \infty; G_1) + S(r, g) \end{aligned}$$

$$\begin{aligned} (n + m + \lambda)T(r, g) \leq{}& (2k + \Gamma_0 + kd + \lambda + 1)T(r, f) + (k + \Gamma_0 + d + \lambda + 2)T(r, g) \\ &+ S(r, f) + S(r, g). \end{aligned}$$

Similarly,

$$\begin{aligned} (n + m + \lambda)T(r, f) \leq{}& (2k + \Gamma_0 + kd + \lambda + 1)T(r, g) + (k + \Gamma_0 + d + \lambda + 2)T(r, f) \\ &+ S(r, f) + S(r, g). \end{aligned}$$

Thus we obtain

$$(n + m - 3k - 2\Gamma_0 - kd - d - \lambda - 3)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g),$$

a contradiction with $n > 3k + 2\Gamma_0 - m + kd + \lambda + 3d + 7$.

Subcase 5. Let $B \neq 0$ and $A \neq B$. Then from (17) we get $F^* = \frac{(B+1)G^* - (B-A+1)}{BG^* + (A-B)}$ and so $\overline{N}\left(r, \frac{B-A+1}{B+1}; G^*\right) = \overline{N}(r, 0; F^*)$. Proceeding in a manner similar to Subcase 4 we can arrive at a contradiction.

Subcase 6. Let $B = 0$ and $A \neq 0$. Then from (17) we get $F^* = \frac{G+A-1}{A}$ and $G = AF - (A - 1)$. If $A \neq 1$, it follows that $\overline{N}\left(r, \frac{A-1}{A}; F^*\right) = \overline{N}(r, 0; G^*)$ and $\overline{N}(r, 1 - A; G^*) = \overline{N}(r, 0; F^*)$. Using the similar arguments as in Subcase 4 we obtain a contradiction. Thus $A = 1$ which implies $F^* = G^*$, and therefore,

$$(f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j))^{(k)} \equiv (g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j))^{(k)}$$

Integrating k times, we get

$$f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j) \equiv g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j) + R(z),$$

where $R(z)$ is a polynomial of degree atmost $k-1$. If $R(z) \not\equiv 0$, above equation can be written as

$$\frac{f^n(z)P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j}}{R(z)} = \frac{g^n(z)P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}}{R(z)} + 1.$$

By the second fundamental theorem, we have

$$T\left(r, \frac{f^n(z)P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j}}{R}\right) \leq \overline{N}\left(r, \frac{f^n(z)P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j}}{R(z)}\right)$$
$$+ \overline{N}\left(r, \frac{R(z)}{f^n(z)P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j}}\right) + \overline{N}\left(r, \frac{R(z)}{g^n(z)P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}}\right)$$
$$+ S(r, f)$$

$(n+m+\lambda)T(r,f) \leq (2+\Gamma_0+2d)T(r,f) + (\Gamma_0+d+1)T(r,g) + S(r,f) + S(r,g)$.

Similarly, we get

$(n+m+\lambda)T(r,g) \leq (2+\Gamma_0+2d)T(r,g) + (\Gamma_0+d+1)T(r,f) + S(r,f) + S(r,g)$.

So

$(n+m+\lambda)[T(r,f)+T(r,g)] \leq (2\Gamma_0+3d+3)[T(r,f)+T(r,g)] + S(r,f) + S(r,g)$.

Which is contradiction to $n > 2\Gamma_0 - m + 3d - \lambda + 3$. Thus, we get $R(z) \equiv 0$ and hence

$$f^n(z)P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j} = g^n(z)P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}.$$

$$f^n(a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0) \prod_{j=1}^{d} f(q_j z + c_j) = g^n(a_m g^m + a_{m-1} g^{m-1} + ...$$
$$+ a_1 g + a_0) \prod_{j=1}^{d} g(q_j z + c_j)$$

Let $h = \frac{f}{g}$, If h is constant then substituting $f = gh$ and

$\prod_{j=1}^{d} f(q_j z + c_j)^{s_j} = \prod_{j=1}^{d} g(q_j z + c_j)_j^s h(q_j z + c_j)^{s_j}$ in above equation, we deduce

$$\prod_{j=1}^{d} g(z + c_j)^{s_j} [a_m g^{n+m}(h^{n+m+\lambda} - 1) + a_{m-1} g^{n+m-1}(h^{n+m+\lambda-1} - 1) + ... + a_0 g^n(h^{n+\lambda} - 1)] \equiv 0, \tag{19}$$

where a_m is a non-zero complex constant and $\prod_{j=1}^{d} g(q_j z + c_j)^{s_j} \not\equiv 0$, Since g is nonconstant meromorphic function. Then, from (19), we have

$$a_m g^{n+m}(h^{n+m+\lambda}-1)+a_{m-1}g^{n+m-1}(h^{n+m+\lambda-1}-1)+...+a_0 g^n(h^{n+\lambda}-1) \equiv 0. \quad (20)$$

If $a_m(\neq 0)$ and $a_{m-1} = a_{m-2} = ... = a_0 = 0$, then from (20) and g is nonconstant meromorphic function, we get $h^{n+m+\lambda} = 1$.

If $a_m(\neq 0)$ and there exists $a_i \neq 0\,(i \in \{0, 1, 2, ..., m-1\})$. Suppose that $h^{n+m+\lambda} \neq 1$, from (20), we have $T(r, g) = S(r, g)$ which is contradiction with transcendental function g. Then $h^{n+m+\lambda} = 1$. Similar to this discussion, we can see that $h^{n+j+\lambda} = 1$ when $a_j \neq 0$ for some $j = 0, 1, 2, ..., m$.

Thus, we have $f \equiv tg$ for a constant t such that $t^l = 1$, where $l = GCD\{n + \lambda_0 + \lambda, n + \lambda_1 + \lambda, ..., n + \lambda_m + \lambda\}$ and $\lambda_i\,(i = 0, 1, 2, ..., m)$ is stated as in Theorem 5.

Set $h = \frac{f}{g}$, If h is not a constant, from (20), we find that f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n P(w_1) \prod_{j=1}^{d} w_1(z + c_j)^{s_j} - w_2^n P(w_2) \prod_{j=1}^{d} w_2(z + c_j)^{s_j}.$$

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