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STABILITY RESULTS FOR JUNGCK AND JUNGCK MANN ITERATION PROCESSES USING CONTRACTIVE CONDITION OF INTEGRAL TYPE

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Abstract. In this paper, we present some stability results for Jungck and Jungck-Mann iteration processes in metric space and normed space by using a contractive condition of integral type. Our results generalize and improve those of Bosede [2] and Olatinwo [9].

1. Introduction

Let (E, d) be a complete metric space and $T : E \longrightarrow E$ a self mapping of E. Suppose that $F_p = \{p \in E, \quad Tp = p\}$ is the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset E$ be the sequence generated by an iteration procedure involving T which defined by:

$$x_{n+1} = f(T, x_n), n = 0, 1, 2..., \tag{1}$$

where $x_0 \in E$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. If in (1):

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2..., \tag{2}$$

we have the Picard iteration process.
Also, if in (1)

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2..., \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in $[0, 1]$, then we have Mann iteration process.
Jungck [5] introduced the following iteration process.
Let S and T be two operators on an arbitrary set Y with values in E such that $TY \subset SY$, SY is a complete subspace of E. For an arbitrary $x_o \in Y$, the sequence $\{Sx_{n+1}\}_{n=o}^{\infty}$ defined by:

$$Sx_{n+1} = Tx_n, \quad n = 0, 1, 2..., \tag{4}$$

called the Jungck iteration process.
We remark that if $Y = E$ and $S = id_E$, then (4) becomes the Picard iteration

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process.
Singh et al. [17] used the following iteration to obtain some stability results:

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n, \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$, the last process is called Jungck-Mann iteration process.
If $Y = E$ and $S = id_E$, then (5) becomes the Picad-Mann iteration.
Singh et al. [17] established some stability results for Jungck and Jungck- Mann iteration by using the two following contractive definitions both of which generalize results of Osilike [13]:

$$d(Tx,Ty) \leq \alpha d(Sx,Sy), \tag{6}$$

$$d(Tx,Ty) \leq \alpha d(Sx,Sy) + Ld(Sx,Tx), \tag{7}$$

where $T,S : Y \longrightarrow E$, $0 \leq \alpha < 1$ and L is an arbitrary positive number.
It is clear that the condition (6) implies (7).
In 2008, Olatinwo [9] established some stability and strong convergence results for Jungck-Ishikawa iteration process in normed space, where he used the two following contractive conditions:

(1) there exist a real number $\alpha \in [0,1)$ and a monotone increasing function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\phi(0) = 0$, such that for all $x,y \in E$,

$$\|Tx - Ty\| \leq \alpha\|Sx - Sy\| + \phi(\|Sx - Tx\|), \tag{8}$$

(2) there exist $M \geq 0$, $\alpha \in [0,1)$ and a monotone increasing function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\phi(0) = 1$, such that:

$$\forall x,y \in E, \quad \|Tx - Ty\| \leq \frac{\alpha\|Sx - Sy\| + \varphi(\|Sx - Tx\|)}{1 + M\|Sx - Tx\|}, \tag{9}$$

Recently, Bosede [**?**] proved some stability results of Jungck-Mann, Jungck-Krasnoselskij and Jungck iteration processes in an arbitrary Banach space, he used the following contractive definition:
Let $(E, \|.\|)$ be a Banach space and Y an arbitrary set. Suppose that $S,T : Y \longrightarrow E$ are two non self mappings such that $TY) \subset SY$ and SY is a complete subspace of E. Suppose also that $z \in Y$ is a coincidence point of S and T, with $p = Sz = Tz$ and that there exist a constant $\alpha \in [0,1)$ and a monotone increasing function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\phi(0) = 1$, such that:

$$\forall x,y \in Y, \quad \|Tx - Ty\| \leq \alpha\|Sx - Sy\|\psi(\|Sx - Tx\|), \tag{10}$$

In a complete metric space setting, the condition (10) becomes:

$$d(Tx,Ty) \leq \alpha d(Sx,Sy)\psi(d(Sx,Tx)). \tag{11}$$

2. Preliminaries

More recently, Olatinwo [9] obtained some stability results for Picard, Mann-Picard iteration processes in complete metric space, he used the following contractive definition:
For a self mapping $T : E \longrightarrow E$, there exist a real number $\alpha \in [0,1)$ and monotone

increasing functions $\nu, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\psi(0) = 0$ and for all x, y in E, we have:

$$\int_0^{d(Tx,Ty)} \varphi(t)d\nu(t) \leq \alpha \int_0^{d(x,y)} \varphi(t)d\nu(t) + \psi(\int_0^{d(x,Tx)} \varphi(t)d\nu(t)), \tag{12}$$

where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)d\nu(t) > 0$.

We will consider the following contractive condition.

Let non self mappings $T, S : Y \subset E \longrightarrow E, T(Y) \subset SY$, with SY is a complete subspace of E and S is injective. There exist a real number α in $[0, 1)$, and a monotone increasing functions:

$\nu, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\psi(0) = 0$, such that for all $x, y \in E$

$$\int_0^{d(Tx,Ty)} \varphi(t)d\nu(t) \leq \alpha \int_0^{d(Sx,Sy)} \varphi(t)d\nu(t) + \psi(\int_0^{d(Sx,Tx)} \varphi(t)d\nu(t)), \tag{13}$$

where φ is a Lebesgue-Stieltjes integrable function such for all $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$.

In normed space the condition (13) becomes:

$$\int_0^{\|Tx-Ty\|} \varphi(t)d\nu(t) \leq \alpha \int_0^{\|Sx-Sy\|} \varphi(t)d\nu(t) + \psi(\int_0^{\|Sx-Tx\|} \varphi(t)d\nu(t)). \tag{14}$$

Remark

(1) If in (13), we have:

$$\forall t \geq 0, \varphi(t) = 1, \nu(t) = t, \psi(t) = 0,$$

we obtain the condition (6).

(2) If in (14),we have

$$\forall t \geq 0 \ \ \varphi(t) = 1, \nu(t) = t,$$

we obtain the condition (8).

(3) If in (13), we have

$$\forall t \geq 0 \ \ \varphi(t) = 1, \nu(t) = t \ \ and \ \ \psi(t) = Lt \ (L \geq 0)$$

we obtain the condition (7).

(4) If in (13), we have:

$$Y = E \ \text{ and } \ S = id_E \text{ (the identity operator)},$$

we obtain the contractive condition (12).

(5) If in (14), for all $t \geq 0, \varphi(t) = 1,\ \ \nu(t) = t$ and $\phi(t) = \dfrac{\psi(t)}{\|Sx - Sy\|} + 1$, such that for all $x \neq y, \|Sx - Sy\| \neq 0$ and $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an increasing function such that $\phi(0) = 1$, then we obtain the condition (10).

In 2005, Singh [17] introduced the following definition of iteration process stability.

Definition Let S, T be two operators on an arbitrary set Y with values in E such that $TY \subset SY$, SY is a compleat subspace of E and z a coincidence point of S and T, that is, $Sz = Tz = p$. Then for $x_0 \in Y$ the sequence $\{Sx_n\}_0^{\infty}$ generated by the iteration procedure (1) converges to p. Let $\{Sx_n\}_0^{\infty}$ be an arbitrary sequence, and set $\varepsilon_n = d(Sx_{n+1}, f(T, x_n)), n = 0, 1, ...$ Then, the iteration procedure (1) will be called (S, T)-stable if and only if

$$\lim_{n\to+\infty} \varepsilon_n = 0 \text{ implies that } \lim_{n\to+\infty} Sy_n = p.$$

This definition reduces to that of the stability of iteration procedure due to Harder and Hicks [3] when $Y = E$ and $S = id_E$ (identity operator).

Lemma [1] If δ is a real number such that $0 \le \delta < 1$, and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying:

$$u_{n+1} \le \delta u_n + \varepsilon_n, n = 0, 1, ...$$

we have:

$$\lim_{n\to\infty} \varepsilon_n = 0.$$

Lemma [9] Let (E, d) be a complete metric space and $\varphi, \nu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative, and such that for each
$\varepsilon > 0, \int_0^{\varepsilon} \varphi(t)d\nu(t) > 0.$
Suppose that $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty} \subset E$ and $\{a_n\}_{n=0}^{\infty} \subset (0, 1)$ are two sequences such that:

$$|d(u_n, v_n) - \int_0^{d(u_n,v_n)} \varphi(t)d\nu(t)| \le a_n,$$

with $\lim_{n\to\infty} a_n = 0$, then

$$d(u_n, v_n) - a_n \le \int_0^{d(u_n;v_n)} \varphi(t)d\nu(t) \le d(u_n, v_n) + a_n.$$

3. Main results

Theorem 1 Let (E, d) be a complete metric space, Y an arbitrary set of E and $x_0 \in Y$. Suppose that $S, T : Y \to E$ are two non self mappings such that $TY \subseteq SY$, SY is a complete subspace of E and S is an injective operator. Suppose that z is a coincidence point of S and T, i,e., $p := Tz = Sz$. Suppose also that S and T satisfy the contractive condition (13). Let $\nu, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be monotone increasing functions such that $\psi(0) = 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is Lebesgue integrable function such for all $\varepsilon > 0, \int_0^{\varepsilon} \varphi(t)dt > 0$, then the Jungck iteration (4) is (S, T) stable.

Proof Let $\{Sy_n\}_0^{\infty}$ an arbitrary sequence in E, and define:

$$\varepsilon_n = d(Sy_{n+1}, Ty_n).$$

Assume $\lim_{n\to\infty} \varepsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} Sy_n = p$.
By using condition (13), lemma 2 and the triangle inequality, we get:

$$\begin{aligned}\int_0^{d(Sy_{n+1},p)} \varphi(t)dt &\le d(Sy_{n+1}, p) + a_n \\ &\le d(Sy_{n+1}, Ty_n) + d(Ty_n, p) + a_n \\ &\le \int_0^{\varepsilon_n} \varphi(t)dt) + \int_0^{d(Ty_n,Tz)} \varphi(t)dt + 3a_n\end{aligned}$$

$$\leq \int_0^{\varepsilon_n} \varphi(t)dt + \alpha \int_0^{d(Sy_n, Sz)} \varphi(t)dt + \psi(\int_0^{d(Sz,Tz)} \varphi(t)dt) + 3a_n,$$

therefore

$$\int_0^{d(Sy_{n+1},p)} \varphi(t)dt \leq \alpha \int_0^{d(Sy_n,p)} \varphi(t)dt + \int_0^{\varepsilon_n} \varphi(t)dt + 3a_n,$$

by using lemma 2, putting:

$u_n = \int_0^{d(Sy_n,p)} \varphi(t)dt, \varepsilon'_n = \int_0^{\varepsilon_n} \varphi(t)dt + 3a_n \longrightarrow 0$, we obtain:

$$\int_0^{d(Sy_n,p)} \varphi(t)dt \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

from condition on φ, we get

$$\lim_{n \longrightarrow +\infty} d(Sy_n, p) \longrightarrow 0.$$

Conversely, suppose that $\lim_{n\to\infty} Sy_n = p$, we prove $\varepsilon_n \longrightarrow 0$, where

$$\varepsilon_n = d(Sy_{n+1}, Ty_n).$$

Then, by the contractive condition (13), Lemma 2 and the triangle inequality again, we have:

$$\int_0^{\varepsilon_n} \varphi(t)d\nu(t) \leq d(Sy_{n+1}, Ty_n) + a_n$$

$$\leq d(Sy_{n+1}, p) + d(Ty_n, p) + a_n$$

$$\leq \int_0^{d(Sy_{n+1},p)} \varphi(t)d\nu(t) + \alpha \int_0^{d(Sy_n,p)} \varphi(t)d\nu(t) + \psi(\int_0^{d(Sz,Tz)} \varphi(t)d\nu(t)) + 3a_n$$

$$\leq \int_0^{d(Sy_{n+1},p)} \varphi(t)d\nu(t) + \alpha \int_0^{d(Sy_n,p)} \varphi(t)d\nu(t) + 3a_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

we get

$$\int_0^{\varepsilon_n} \varphi(t)d\nu(t) \longrightarrow 0,$$

but for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)d\nu(t) > 0$, then

$$\varepsilon_n \longrightarrow 0.$$

Remark If in Theorem 3 $Y = E$ and $S = id_E$ (the identity map of E), we obtain theorem 3.1 of Olatinwo [9], it is also a generalization and extension of some results obtained in [**?**, 15, 17].

Corollary Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a self mapping satisfying (12). Suppose p a fixed point of T and $\{x_n\}$ defined in Theorem 3. Then the Picard iteration is T-stable.

Theorem Let $(E, \|.\|)$ be a normed space and Y an arbitrary set and let $S, T : Y \to E$ be two non self mappings such that $TY \subseteq SY$, SY is a complete subspace of E and S is an injective operator. Suppose that they have a coincidence point z and satisfy contractive condition (14). For any $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Mann iteration process defined by (5), where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $0 < \gamma \leq \alpha_n (n = 0, 1, ...)$.

Let $\nu, \phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, be monotone increasing functions such that $\psi(0) = 0$ and

$\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a Lebesgue-Stieltjes integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$.

Then, the Jungck-Mann iteration process (5) is (S,T)-stable.

Proof Suppose that $\{Sy_n\}_{n=0}^{\infty}$ an arbitrary sequence in E and

$$\varepsilon_n = \|Sy_{n+1} - (1-\alpha_n)Sy_n - \alpha_n Ty_n\|, \quad n = 0,1,....$$

Assume $\lim_{n\to\infty} \varepsilon_n = 0$ and we shall establish that:

$$\lim_{n\to\infty} y_n = p.$$

Let $\{a_n\}_{n=0}^{\infty} \subset (0,1)$, then by Lemma 2, using (14) and the triangle inequality we get:

$$\begin{aligned}
\int_0^{\|(Sy_{n+1}-p\|} \varphi(t)dt &\leq \|Sy_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\| + \|(1-\alpha_n)y_n - \alpha_n Ty_n - p\| + a_n \\
&\leq \varepsilon_n + (1-\alpha_n)\|Sy_n - p\| + \alpha_n \|Ty_n - p\| + a_n \\
&\leq \varepsilon_n + (1-\alpha_n)\int_0^{\|Sy_n - p\|} \varphi(t)dt + \alpha_n \int_0^{\|(Ty_n - Tz\|} \varphi(t)dt + 2a_n \\
&\leq \varepsilon_n + 2a_n + (1 - \alpha_n(1-\alpha))\int_0^{\|Sy_n - p\|} \varphi(t)dt \\
&\leq \varepsilon_n + 3a_n + (1 - \gamma(1-\alpha))\int_0^{\|Sy_n - p\|} \varphi(t)dt,
\end{aligned}$$

by using lemma (2.2), where

$$\varepsilon'_n = \varepsilon_n + 2a_n \longrightarrow 0$$

and

$$u_n = \int_0^{\|Sy_n - p\|} \varphi(t)dt, 0 \leq \delta = 1 - \gamma(1-\alpha) < 1.$$

We obtain

$$Sy_n \longrightarrow p.$$

Conversely, suppose that $\lim_{n\to\infty} Sy_n = p$, we will show $\varepsilon_n \longrightarrow 0$.

$$\begin{aligned}
\int_0^{\varepsilon_n} \varphi(t)d\nu(t) &= \int_0^{\|Sy_{n+1} - (1-\alpha_n)Sy_n - \alpha_n Ty_n\|} \varphi(t)d\nu(t) \\
&\leq \|Sy_{n+1} - p\| + (1-\alpha_n)\|Sy_n - p\| + \alpha_n\|Ty_n - p\| + a_n \\
&\leq \|Sy_{n+1} - p\| + (1-\alpha_n)\int_0^{\|Sy_n - p\|} + \alpha_n \int_0^{\|Ty_n - Tz\|} \varphi(t)d\nu(t) + 2a_n \\
&\leq \|Sy_{n+1} - p\| + (1-\alpha_n)\int_0^{\|Sy_n - p\|} + \alpha\alpha_n \int_0^{\|Sy_n - Sz\|} \varphi(t)d\nu(t) + 2a_n \\
&\leq \|Sy_{n+1} - p\| + +2a_n + (1 - \alpha_n(1-\alpha))\int_0^{\|Sy_n - p\|} \varphi(t)d\nu(t).
\end{aligned}$$

From lemma 2, we obtain

$$\int_0^{\varepsilon_n} \varphi(t)d\nu(t) \longrightarrow 0 \text{ as } n \to 0,$$

applying the condition on φ, we obtain $\varepsilon_n \longrightarrow 0$.

Remark If in Theorem 3 $Y = E$ and $S = id_E$ (the identity mapping of E), we obtain theorem 3.2 of Olatinwo [9].

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