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COMMON FIXED POINTS IN COMPLEX VALUED A_b-METRIC SPACE

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Abstract. In this paper, we prove two common fixed point theorems for two self mappings in complex valued A_b-metric space. Our results generalize the common fixed point results in complex valued b-metric space by Aiman A. Mukheimer [16] which are already the generalizations of the results of Azam et al. [1] and S. Bhatt et al. [15].

1. Introduction

Azam et al. [1] introduced the concept of complex valued metric space and proved some fixed point results for a pair of mappings for a contraction condition satisfying a rational expression. In 2013, K. Rao et al.[14] introduced complex valued b-metric space as a generalization of complex valued metric space. Azam et al. [1] and S. Bhatt et al. [15] established common fixed point results in complex valued metric space and as generalizations of these results, Aiman A. Mukheimer [16] obtained common fixed point results in complex valued b-metric space. Recently K. Anthony Singh and M. R. Singh [17] introduced complex valued A_b-metric space as further generalization of complex valued metric space. Complex valued A_b-metric space can also be looked upon as an extension of A_b-metric space introduced by Manoj Ughade et al. [3].

The aim of this paper is to present two common fixed point results in complex valued A_b-metric space. Our results generalize the results of Aiman A. Mukheimer [16].

2. Preliminaries

In this section, we recall some properties of A-metric space, A_b-metric space, complex valued metric space, complex valued b-metric space and complex valued A_b-metric space.

Definition 2.1.[13] Let X be a nonempty set. A function $A : X^n \to [0, \infty)$ is called an $A-$ metric on X if for any $x_i, a \in X, i = 1, 2, 3, \ldots, n$,the following conditions hold:
(A1) $A(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) \geq 0$,

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(A2) $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$,
(A3) $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n)$

$$\begin{aligned} &\leq A(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\ &+ A(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\ &+ A(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \dots \\ &+ A(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) \\ &+ A(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a). \end{aligned}$$

The pair (X, A) is called an $A-$ *metric* space.
Definition 2.2.[3] Let X be a nonempty set and $b \geq 1$ be a given number. A function $A : X^n \to [0, \infty)$ is called an A_b-metric on X if for any $x_i, a \in X, i = 1, 2, 3, \dots, n$, the following conditions hold:
(A_b1) $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$,
(A_b2) $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$,
(A_b3) $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n)$

$$\begin{aligned} &\leq b[A(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\ &+ A(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\ &+ A(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \dots \\ &+ A(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) \\ &+ A(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)]. \end{aligned}$$

The pair (X, A) is called an A_b-metric space.
Remark 2.3. A_b-metric space is more general than $A-$metric space. Moreover, $A-$metric space is a special case of A_b-metric space with $b = 1$.
Example 2.4.[3] Let $X = [1, +\infty)$. Define $A_b : X^n \to [0, \infty)$ by

$$A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2 \tag{1}$$

for all $x_i \in X, i = 1, 2, 3, \dots, n$.
Then (X, A_b) is an A_b-metric space with $b = 2 > 1$.

The concept of complex valued metric space was initiated by Azam et al. [1]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.
It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied :
(C_1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(C_2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(C_3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(C_4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
Particularly, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C_2),(C_3) and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:
(1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \precsim bz$ for all $0 \precsim z \in \mathbb{C}$.
(2) If $z_1 \precsim z_2$, then $az_1 \precsim az_2$ for all $0 \leq a \in \mathbb{R}$.
(3) If $0 \precsim z_1 \precsim z_2$, then $| z_1 | \leq | z_2 |$.
(4) If $0 \precsim z_1 \precnsim z_2$, then $| z_1 | < | z_2 |$.
(5) If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 2.5.[1] Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$, the following conditions are satisfied:
(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.
The pair (X, d) is called a complex valued metric space.
Definition 2.6.[14] Let X be a nonempty set and let $s \geq 1$. A function $d : X\times X \to \mathbb{C}$ is called a complex valued $b-$metric on X if for all $x, y, z \in X$, the following conditions are satisfied:
(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.
The pair (X, d) is called a complex valued $b-$metric space.
Definition 2.7.[17] Let X be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $A : X^n \to \mathbb{C}$ satisfies for all $x_i, a \in X, i = 1, 2, 3, \ldots, n$:
(CA_b1) $0 \precsim A(x_1, x_2, x_3, \ldots, x_n)$,
(CA_b2) $A(x_1, x_2, x_3, \ldots, x_n) = 0 \Leftrightarrow x_1 = x_2 = x_3 = \cdots = x_n$,
(CA_b3) $A(x_1, x_2, x_3, \ldots, x_n)$

$$\begin{aligned} &\precsim b[A(x_1, x_1, x_1, \ldots, (x_1)_{n-1}, a) \\ &+ A(x_2, x_2, x_2, \ldots, (x_2)_{n-1}, a) + \ldots \\ &+ A(x_{n-1}, x_{n-1}, x_{n-1}, \ldots, (x_{n-1})_{n-1}, a) \\ &+ A(x_n, x_n, x_n, \ldots, (x_n)_{n-1}, a)]. \end{aligned}$$

Then A is called a complex valued A_b-metric on X and the pair (X, A) is called a complex valued A_b-metric space.
Example 2.8.[17] Let $X = \mathbb{R}$ and $A : X^n \to \mathbb{C}$ be such that

$$A(x_1, x_2, x_3, \ldots, x_n) = (\alpha + i\beta)A_*(x_1, x_2, x_3, \ldots, x_n), \tag{2}$$

where $\alpha, \beta \geq 0$ are constants and A_* is an A_b-metric on X. Then A is a complex valued A_b-metric on X. As a particular case , we have the following example of complex valued A_b-metric on X. The mapping $A : X^n \to \mathbb{C}$ defined by $A(x_1, x_2, x_3, \ldots, x_n) = (1 + i) \sum_{i=1}^{n} \sum_{i<j} \mid x_i - x_j \mid^2$ is a complex valued A_b-metric on $X = \mathbb{R}$ with $b = 2$.
Definition 2.9.[17] A complex valued A_b-metric space (X, A) is said to be symmetric if

$$A(x_1, x_1, x_1, \ldots, (x_1)_{n-1}, x_2) = A(x_2, x_2, x_2, \ldots, (x_2)_{n-1}, x_1) \tag{3}$$

for all $x_1, x_2 \in X$.
Definition 2.10.[17] Let (X, A) be a complex valued A_b-metric space.
(i) A sequence $\{x_p\}$ in X is said to be complex valued A_b-convergent to x if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $A(x_p, x_p, \ldots, x_p, x) \prec a$ or $A(x, x, \ldots, x, x_p) \prec a$ for all $p \geq k$ and is denoted by $\lim_{p\to\infty} x_p = x$ or $x_p \to x$ as $p \to \infty$.
(ii) A sequence $\{x_p\}$ in X is called complex valued A_b-Cauchy if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $A(x_p, x_p, \ldots, x_p, x_q) \prec a$ for each $p, q \geq k$.
(iii) If every complex valued A_b-Cauchy sequence is complex valued A_b-convergent in X, then (X, A) is said to be complex valued A_b-complete.
Lemma 2.11.[17] Let (X, A) be a complex valued A_b-metric space and let $\{x_p\}$ be

a sequence in X. Then $\{x_p\}$ is complex valued A_b-convergent to x if and only if $|A(x_p, x_p, \ldots, x_p, x)| \to 0$ as $p \to \infty$ or $|A(x, x, \ldots, x, x_p)| \to 0$ as $p \to \infty$.
Lemma 2.12.[17] Let (X, A) be a complex valued A_b-metric space and let $\{x_p\}$ be a sequence in X. Then $\{x_p\}$ is complex valued A_b-Cauchy sequence if and only if $|A(x_p, x_p, \ldots, x_p, x_q)| \to 0$ as $p, q \to \infty$.
Lemma 2.13.[17] Let (X, A) be a complex valued A_b-metric space. Then

$$A(x, x, \ldots, x, y) \precsim bA(y, y, \ldots, y, x) \tag{4}$$

for all $x, y \in X$.
Theorem 2.14.[16] Let (X, d) be a complete complex valued $b-$metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be mappings satisfying

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx)d(y, Ty)}{1 + d(x, y)} \tag{5}$$

for all $x, y \in X$ where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then S, T have a unique common fixed point in X.
Theorem 2.15.[16] Let (X, d) be a complete complex valued $b-$metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be mappings satisfying

$$d(Sx, Ty) \precsim \frac{a[d(x, Sx)d(x, Ty) + d(y, Ty)d(y, Sx)]}{d(x, Ty) + d(y, Sx)} \tag{6}$$

for all $x, y \in X$ where $sa \in [0, 1)$. Then S, T have a unique common fixed point in X.

3. Main Results

We now state and prove our main results. Our next Theorem is a generalization of Theorem 2.14. in complex valued A_b-metric space. But in order to compensate for the condition of symmetry in complex valued b-metric space which is required in the proof of the Theorem, we make our space symmetric.
Theorem 3.1. Let (X, A) be a complete complex valued A_b-metric space which is symmetric and let $f, g : X \to X$ be mappings satisfying

$$A(fx, fx, \ldots, fx, gy) \precsim \lambda A(x, x, \ldots, x, y) + \frac{\mu A(x, x, \ldots, x, fx)A(y, y, \ldots, y, gy)}{1 + A(x, x, \ldots, x, y)} \tag{7}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $b\lambda + \mu < 1$. Then f and g have a unique common fixed point in X.
Proof. Let $x_0 \in X$ be an arbitrary point. And let a sequence $\{x_p\}$ in X be defined as $x_{2p+1} = fx_{2p}$ and $x_{2p+2} = gx_{2p+1}$ for $p = 0, 1, 2, 3, \ldots$ Then we show that the sequence $\{x_p\}$ is complex valued A_b-Cauchy.

From (7), we have

$$\begin{aligned}
& A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}) \\
= \; & A(fx_{2p},fx_{2p},\ldots,fx_{2p},gx_{2p+1}) \\
\precsim \; & \lambda A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}) \\
& + \frac{\mu A(x_{2p},x_{2p},\ldots,x_{2p},fx_{2p})A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},gx_{2p+1})}{1+A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1})} \\
= \; & \lambda A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}) \\
& + \frac{\mu A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1})A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2})}{1+A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1})} \\
\Rightarrow \quad & | A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}) | \\
\leq \; & \lambda \, | A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}) | \\
& + \frac{\mu \, | A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}) \| A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}) |}{| 1+A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}) |} \\
\leq \; & \lambda \, | A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}) | + \mu \, | A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}) |
\end{aligned}$$

(since $| 1+A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}) | > | A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}) |$)

$$\Rightarrow | A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}) | \leq \frac{\lambda}{1-\mu} \, | A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1}) | \tag{8}$$

Similarly, using the symmetry of X, we obtain

$$| A(x_{2p+2},x_{2p+2},\ldots,x_{2p+2},x_{2p+3}) | \leq \frac{\lambda}{1-\mu} \, | A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2}) | \tag{9}$$

From (8) and (9), we have

$$| A(x_p,x_p,\ldots,x_p,x_{p+1}) | \leq \alpha \, | A(x_{p-1},x_{p-1},\ldots,x_{p-1},x_p) |, \forall p \in \mathbb{N} \tag{10}$$

where $\alpha = \frac{\lambda}{1-\mu} < 1$.
By repeatedly applying (10), we get

$$| A(x_p,x_p,\ldots,x_p,x_{p+1}) | \leq \alpha^p \, | A(x_0,x_0,\ldots,x_0,x_1) | \tag{11}$$

Using (CA_b3) and (11), we have for all $p, q \in \mathbb{N}$ with $p < q$

$$\begin{aligned}
& \mid A(x_p, x_p, \dots, x_p, x_q) \mid \\
\leq\ & (n-1)b \mid A(x_p, x_p, \dots, x_p, x_{p+1}) \mid + b \mid A(x_{p+1}, x_{p+1}, \dots, x_{p+1}, x_q) \mid \\
\leq\ & (n-1)b \mid A(x_p, x_p, \dots, x_p, x_{p+1}) \mid + (n-1)b^2 \mid A(x_{p+1}, x_{p+1}, \dots, x_{p+1}, x_{p+2}) \mid \\
& + b^2 \mid A(x_{p+2}, x_{p+2}, \dots, x_{p+2}, x_q) \mid \\
\leq\ & (n-1)b \mid A(x_p, x_p, \dots, x_p, x_{p+1}) \mid + (n-1)b^2 \mid A(x_{p+1}, x_{p+1}, \dots, x_{p+1}, x_{p+2}) \mid \\
& + (n-1)b^3 \mid A(x_{p+2}, x_{p+2}, \dots, x_{p+2}, x_{p+3}) + \dots \\
& + (n-1)b^{q-p-1} \mid A(x_{q-2}, x_{q-2}, \dots, x_{q-2}, x_{q-1}) \mid + b^{q-p-1} \mid A(x_{q-1}, x_{q-1}, \dots, x_{q-1}, x_q) \mid \\
\leq\ & [(n-1)b\alpha^p + (n-1)b^2\alpha^{p+1} + \dots \\
& + (n-1)b^{q-p-1}\alpha^{q-2} + b^{q-p-1}\alpha^{q-1}] \mid A(x_0, x_0, \dots, x_0, x_1) \mid \\
\leq\ & [(n-1)b\alpha^p + (n-1)b^2\alpha^{p+1} + \dots + (n-1)b^{q-p-1}\alpha^{q-2} + (n-1)b^{q-p}\alpha^{q-1}] \\
& \mid A(x_0, x_0, \dots, x_0, x_1) \mid \\
\leq\ & (n-1)[(b\alpha)^p + (b\alpha)^{p+1} + \dots + (b\alpha)^{q-2} + (b\alpha)^{q-1}] \mid A(x_0, x_0, \dots, x_0, x_1) \mid \\
\leq\ & (n-1)[(b\alpha)^p + (b\alpha)^{p+1} + (b\alpha)^{p+2} + \dots \infty] \mid A(x_0, x_0, \dots, x_0, x_1) \mid \\
=\ & \frac{(n-1)(b\alpha)^p}{1-(b\alpha)} \mid A(x_0, x_0, \dots, x_0, x_1) \mid \to 0
\end{aligned}$$

as $p, q \to \infty$ since $b\alpha = \frac{b\lambda}{1-\mu} < 1$ which follows from $b\lambda + \mu < 1$.
Therefore, $\mid A(x_p, x_p, \dots, x_p, x_q) \mid \to 0$ as $p, q \to \infty$ and hence the sequence $\{x_p\}$ is complex valued A_b-Cauchy.
Since X is complete, there exists $u \in X$ such that the sequence $\{x_p\}$ is complex valued A_b-convergent to u. We show that u is a common fixed point of f and g.
We have

$$\begin{aligned}
& A(fu, fu, \dots, fu, u) \\
\precsim\ & (n-1)bA(fu, fu, \dots, fu, x_{2p+2}) + bA(u, u, \dots, u, x_{2p+2}) \\
=\ & (n-1)bA(fu, fu, \dots, fu, gx_{2p+1}) + bA(u, u, \dots, u, x_{2p+2}) \\
\precsim\ & (n-1)b\lambda A(u, u, \dots, u, x_{2p+1}) \\
& + \frac{(n-1)b\mu A(u, u, \dots, u, fu)A(x_{2p+1}, x_{2p+1}, \dots, x_{2p+1}, gx_{2p+1})}{1 + A(u, u, \dots, u, x_{2p+1})} \\
& + bA(u, u, \dots, u, x_{2p+2}) \\
\Rightarrow\ & \mid A(fu, fu, \dots, fu, u) \mid \\
\leq\ & (n-1)b\lambda \mid A(u, u, \dots, u, x_{2p+1}) \mid \\
& + \frac{(n-1)b\mu \mid A(u, u, \dots, u, fu) \mid\mid A(x_{2p+1}, x_{2p+1}, \dots, x_{2p+1}, x_{2p+2}) \mid}{\mid 1 + A(u, u, \dots, u, x_{2p+1}) \mid} \\
& + b \mid A(u, u, \dots, u, x_{2p+2}) \mid \to 0 \ as\ p \to \infty
\end{aligned}$$

$\Rightarrow A(fu, fu, \dots, fu, u) = 0$
$\Rightarrow fu = u$.
Similarly, we can show that $gu = u$. Therefore, f and g have a common fixed point $u \in X$. Finally, to show the uniqueness of the common fixed point of f and g, let us

assume that $v \in X$ is another common fixed point of f and g. Then, we have

$$\begin{aligned} A(u,u,\ldots,u,v) &= A(fu,fu,\ldots,fu,gv) \\ &\precsim \lambda A(u,u,\ldots,u,v) + \frac{\mu A(u,u,\ldots,u,fu)A(v,v,\ldots,v,gv)}{1+A(u,u,\ldots,u,v)} \\ &= \lambda A(u,u,\ldots,u,v) \end{aligned}$$

$\Rightarrow \mid A(u,u,\ldots,u,v) \mid \leq \lambda \mid A(u,u,\ldots,u,v) \mid$
Since $\lambda < 1$, we have
$A(u,u,\ldots,u,v) = 0 \Rightarrow u = v$
which proves the uniqueness of the common fixed point of f and g.

Corollary 3.2. Let (X, A) be a complete complex valued A_b-metric space which is symmetric and let $f : X \to X$ be a mapping satisfying

$$A(fx,fx,\ldots,fx,fy) \precsim \lambda A(x,x,\ldots,x,y) + \frac{\mu A(x,x,\ldots,x,fx)A(y,y,\ldots,y,fy)}{1+A(x,x,\ldots,x,y)} \quad (12)$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $b\lambda + \mu < 1$. Then$f$ has a unique fixed point in X.

Proof. Follows from the proof of Theorem 3.1. by taking $g = f$.

Corollary 3.3. Let (X, A) be a complete complex valued A_b-metric space which is symmetric and let $f : X \to X$ be a mapping satisfying for some positive integer m

$$A(f^m x,f^m x,\ldots,f^m x,f^m y) \precsim \lambda A(x,x,\ldots,x,y) + \frac{\mu A(x,x,\ldots,x,f^m x)A(y,y,\ldots,y,f^m y)}{1+A(x,x,\ldots,x,y)} \quad (13)$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $b\lambda + \mu < 1$. Then$f$ has a unique fixed point in X.

Proof. From Corollary 3.2., we have f^m has a unique fixed point $u \in X$. And we have $f(f^m u) = fu \Rightarrow f^m(fu) = fu$.
This implies that fu is a fixed point of f^m.
Since u is the unique fixed point of f^m, we must have $f(u) = u$. Therefore, u is a fixed point of f. Further to show the uniqueness of the fixed point of f we see that a fixed point of f is also a fixed point of f^m since $fv = v \Rightarrow f^2 v = fv = v$ and so on, thus giving $f^m v = v$. And the uniqueness of the fixed point of f^m implies the fixed point of f is also unique.
Our next Theorem is a generalization of Theorem 2.15. in complex valued A_b-metric space.

Theorem 3.4. Let (X, A) be a complete complex valued A_b-metric space which is symmetric and let $f, g : X \to X$ be mappings satisfying

$$\begin{aligned} & A(fx,fx,\ldots,fx,gy) \quad (14) \\ \precsim \; & \frac{\alpha[A(x,x,\ldots,x,fx)A(x,x,\ldots,x,gy) + A(y,y,\ldots,y,gy)A(y,y,\ldots,y,fx)]}{A(x,x,\ldots,x,gy)+A(y,y,\ldots,y,fx)} \end{aligned}$$

for all $x, y \in X$ and $\alpha \in \left[0, \frac{1}{b}\right)$. Then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and let us define a sequence $\{x_p\}$ in X as

$$x_{2p+1} = fx_{2p} \text{ and } x_{2p+2} = gx_{2p+1}, \text{ for } p = 0, 1, 2, 3, \ldots$$

Then we show that the sequence $\{x_p\}$ is complex valued A_b-Cauchy.
From (14), we have
$A(x_{2p+1}, x_{2p+1}, \ldots, x_{2p+1}, x_{2p+2})$
$= A(fx_{2p}, fx_{2p}, \ldots, fx_{2p}, gx_{2p+1})$
$\precsim \frac{\alpha[A(x_{2p},\ldots,x_{2p},fx_{2p})A(x_{2p},\ldots,x_{2p},gx_{2p+1})+A(x_{2p+1},\ldots,x_{2p+1},gx_{2p+1})A(x_{2p+1},\ldots,x_{2p+1},fx_{2p})]}{A(x_{2p},x_{2p},\ldots,x_{2p},gx_{2p+1})+A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},fx_{2p})}$
$= \frac{\alpha[A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1})A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+2})+A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2})A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+1})]}{A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+2})+A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+1})}$
$= \frac{\alpha[A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+1})A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+2})]}{A(x_{2p},x_{2p},\ldots,x_{2p},x_{2p+2})}$
$= \alpha A(x_{2p}, x_{2p}, \ldots, x_{2p}, x_{2p+1})$.
And this implies

$$\left|A(x_{2p+1}, x_{2p+1}, \ldots, x_{2p+1}, x_{2p+2})\right| \leq \alpha \left|A(x_{2p}, x_{2p}, \ldots, x_{2p}, x_{2p+1})\right|. \tag{15}$$

Similarly, using the symmetry of X, we obtain

$$\left|A(x_{2p+2}, x_{2p+2}, \ldots, x_{2p+2}, x_{2p+3})\right| \leq \alpha \left|A(x_{2p+1}, x_{2p+1}, \ldots, x_{2p+1}, x_{2p+2})\right| \tag{16}$$

Combining (15) and (16), we get

$$\left|A(x_p, x_p, \ldots, x_p, x_{p+1})\right| \leq \alpha \left|A(x_{p-1}, x_{p-1}, \ldots, x_{p-1}, x_p)\right|, \ \forall\, p \in \mathbb{N}. \tag{17}$$

By repeatedly applying (17), we get

$$\left|A(x_p, x_p, \ldots, x_p, x_{p+1})\right| \leq \alpha^p \left|A(x_0, x_0, \ldots, x_0, x_1)\right|. \tag{18}$$

Using (CA$_b$3) and (18), we have, for all $p, q \in \mathbb{N}$ with $p < q$

$$\begin{aligned}
& |A(x_p, x_p, \ldots, x_p, x_q)| \\
\leq\ & (n-1)b|A(x_p, x_p, \ldots, x_p, x_{p+1})| + b|A(x_{p+1}, x_{p+1}, \ldots, x_{p+1}, x_q)| \\
\leq\ & (n-1)b|A(x_p, x_p, \ldots, x_p, x_{p+1})| + (n-1)b^2|A(x_{p+1}, x_{p+1}, \ldots, x_{p+1}, x_{p+2})| \\
& +b^2|A(x_{p+2}, x_{p+2}, \ldots, x_{p+2}, x_q)| \\
\leq\ & (n-1)b|A(x_p, x_p, \ldots, x_p, x_{p+1})| + (n-1)b^2|A(x_{p+1}, x_{p+1}, \ldots, x_{p+1}, x_{p+2})| \\
& +(n-1)b^3|A(x_{p+2}, x_{p+2}, \ldots, x_{p+2}, x_{p+3})| + \cdots \\
& +(n-1)b^{q-p-1}|A(x_{q-2}, x_{q-2}, \ldots, x_{q-2}, x_{q-1})| \\
& +b^{q-p-1}|A(x_{q-1}, x_{q-1}, \ldots, x_{q-1}, x_q)| \\
\leq\ & [(n-1)b\alpha^p + (n-1)b^2\alpha^{p+1} + \cdots + (n-1)b^{q-p-1}\alpha^{q-2} \\
& +b^{q-p-1}\alpha^{q-1}]|A(x_0, x_0, \ldots, x_0, x_1)| \\
\leq\ & [(n-1)b\alpha^p + (n-1)b^2\alpha^{p+1} + \cdots + (n-1)b^{q-p-1}\alpha^{q-2} \\
& +(n-1)b^{q-p}\alpha^{q-1}]|A(x_0, x_0, \ldots, x_0, x_1)| \\
\leq\ & (n-1)[(b\alpha)^p + (b\alpha)^{p+1} + \cdots + (b\alpha)^{q-2} + (b\alpha)^{q-1}]|A(x_0, x_0, \ldots, x_0, x_1)| \\
\leq\ & (n-1)[(b\alpha)^p + (b\alpha)^{p+1} + \cdots \infty]|A(x_0, x_0, \ldots, x_0, x_1)| \\
=\ & \frac{(n-1)(b\alpha)^p}{1-b\alpha}|A(x_0, x_0, \ldots, x_0, x_1)| \to 0 \text{ as } p, q \to \infty.
\end{aligned}$$

Therefore, $|A(x_p, x_p, \ldots, x_p, x_q)| \to 0$ as $p, q \to \infty$ and hence the sequence $\{x_p\}$ is complex valued A_b-Cauchy. Since X is complete, there exists $u \in X$ such that the sequence $\{x_p\}$ is complex valued A_b-convergent to u. We show that u is a common

fixed point of f and g.
Let us assume that $fu \neq u$ so that

$$|A(fu, fu, \ldots, fu, u)| > 0. \tag{19}$$

Then we have
$A(fu, fu, \ldots, fu, u)$
$\precsim (n-1)bA(fu, fu, \ldots, fu, x_{2p+2}) + bA(u, u, \ldots, u, x_{2p+2})$
$= (n-1)bA(fu, fu, \ldots, fu, gx_{2p+1}) + bA(u, u, \ldots, u, x_{2p+2})$
$\precsim \frac{(n-1)b\alpha A(u,u,\ldots,u,fu)A(u,u,\ldots,u,gx_{2p+1})+(n-1)b\alpha A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},gx_{2p+1})A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},fu)}{A(u,u,\ldots,u,gx_{2p+1})+A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},fu)}$
$+ bA(u, u, \ldots, u, x_{2p+2})$.
And this implies
$|A(fu, fu, \ldots, fu, u)|$
$\leq \frac{(n-1)b\alpha|A(u,u,\ldots,u,fu)||A(u,u,\ldots,u,x_{2p+2})|+(n-1)b\alpha|A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},x_{2p+2})||A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},fu)|}{|A(u,u,\ldots,u,x_{2p+2})+A(x_{2p+1},x_{2p+1},\ldots,x_{2p+1},fu)|}$
$+ b|A(u, u, \ldots, u, x_{2p+2})|$
$\to 0$ as $p \to \infty$.
Thus we have
$|A(fu, fu, \ldots, fu, u)| = 0,$ which contradicts (19).
Therefore, we must have $fu = u$.
Similarly, we can show that $gu = u$.
Therefore, f and g have a common fixed point $u \in X$.
Finally to show the uniqueness of the common fixed point of f and g, let $v \in X$ be another common fixed point of f and g. And let us assume that $u \neq v$ so that $A(u, u, \ldots, u, v) \neq 0$.
Then we have

$$\begin{aligned}
& A(u, u, \ldots, u, v) \\
= \; & A(fu, fu, \ldots, fu, gv) \\
\precsim \; & \frac{\alpha[A(u, u, \ldots, u, fu)A(u, u, \ldots, u, gv) + A(v, v, \ldots, v, gv)A(v, v, \ldots, v, fu)]}{A(u, u, \ldots, u, gv) + A(v, v, \ldots, v, fu)} \\
= \; & \frac{\alpha[A(u, u, \ldots, u, u)A(u, u, \ldots, u, v) + A(v, v, \ldots, v, v)A(v, v, \ldots, v, u)]}{A(u, u, \ldots, u, v) + A(v, v, \ldots, v, u)} \\
= \; & 0
\end{aligned}$$

$\Rightarrow A(u, u, \ldots, u, v) = 0,$ which is a contradiction.
Therefore, we must have $u = v$, which proves the uniqueness of the common fixed point of f and g.
Corollary 3.5. Let (X, A) be a complete complex valued A_b-metric space which is symmetric and let $f : X \to X$ be a mapping satisfying

$$\begin{aligned}
& A(fx, fx, \ldots, fx, fy) \\
\precsim \; & \frac{\alpha[A(x, x, \ldots, x, fx)A(x, x, \ldots, x, fy) + A(y, y, \ldots, y, fy)A(y, y, \ldots, y, fx)]}{A(x, x, \ldots, x, fy) + A(y, y, \ldots, y, fx)}
\end{aligned} \tag{20}$$

for all $x, y \in X$ and $\alpha \in \left[0, \frac{1}{b}\right)$. Then f has a unique fixed point in X.
Corollary 3.6. Let (X, A) be a complete complex valued A_b-metric space which is

symmetric and let $f : X \to X$ be a mapping satisfying for some positive integer m

$$A(f^m x, f^m x, \dots, f^m x, f^m y) \precsim \frac{\alpha[A(x,x,\dots,x,f^m x)A(x,x,\dots,x,f^m y) + A(y,y,\dots,y,f^m y)A(y,y,\dots,y,f^m x)]}{A(x,x,\dots,x,f^m y) + A(y,y,\dots,y,f^m x)} \tag{21}$$

for all $x, y \in X$ and $\alpha \in \left[0, \frac{1}{b}\right)$. Then f has a unique fixed point in X.

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