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OSCILLATION OF SECOND ORDER NONLINEAR IMPULSIVE DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we establish some new oscillation criteria for the second-order nonlinear impulsive delay dynamic equation

$$(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))^{\Delta}+p(t)f(x(\tau(t)))=0,\ t\neq\theta_k,$$
$$\Delta(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))_{|_{t=\theta_k}}+b_k h(x(\tau(\theta_k)))=0$$

on a time scale $\mathbb{T}$. Our results generalize and extend some pervious results [11, 18, 19, 21] and can be applied to some oscillation problems that not discussed before. These results extend the known results for the dynamic equations with and without impulses. Finally, we give some examples to show that impulses play a dominant part in the oscillations of dynamic equations on time scales and to illustrate our main results.

1. Introduction

The theory of time scales was introduced by Hilger [16] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus. A time scale is an arbitrary closed subset of the reals. When time scale equals to the reals or to the integers, it represents the classical theories of differential and difference equations. Many other interesting time scales exist, e.g., $\mathbb{T}=q^{\mathbb{N}_0}:=\{q^t: t\in\mathbb{N}_0 \text{ for } q>1\}$ (which has important applications in quantum theory), $\mathbb{T}=h\mathbb{N}$ with $h>0$, $\mathbb{T}=\mathbb{N}^2$ and $\mathbb{T}=\mathbb{T}^n$ (the space of the harmonic numbers). For an introduction to time scale calculus and dynamic equations, see Bohner and Peterson books [7, 8].

Recently, many results have been obtained on the oscillation and nonoscillation of dynamic equations on time scales (see [10, 12, 20, 22] and references cited therein).

Impulsive dynamic equations on time scales have been investigated by Agarwal et al. [1], Belarbi et al. [2], Benchohra *et al.* [3-6] and so forth. Benchohra et al. [6] considered the existence of extremal solutions for a class of second order impulsive dynamic equations on time scales.

The oscillation of impulsive differential equations has been investigated by many authors and many results were obtained (see [13, 15, 17] etc. and the references

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cited therein). But fewer papers are on the oscillation of impulsive dynamic equations on time scales.

A. Zafer [21] considered the second order sublinear impulsive differential equation

$$\begin{cases} x^{''}(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0, t \neq \theta_k, \\ \Delta x^{'}(t)_{|_{t=\theta_k}} + q_k|x(\tau(\theta_k))|^{\alpha-1}x(\tau(\theta_k)) = 0, \end{cases}$$

Kunwen Wen [18] studied the oscillation of second order sublinear delay differential equations with impulses of the form

$$\begin{cases} (r(t)x^{'}(t))^{'} + p(t)|x^{'}(\tau(t))|^{\alpha-1}x^{'}(\tau(t)) = 0, \ t \neq \theta_k, \\ \Delta(r(t)x^{'}(t))_{|_{t=\theta_k}} + q_k|x(\tau(\theta_k))|^{\alpha-1}x(\tau(\theta_k)) = 0, \end{cases}$$

In [19] the authors studied the oscillation criteria for second-order impulsive differential equations of the form

$$\begin{cases} (r(t)|x^{'}(t)|^{\alpha-1}x^{'}(t))^{'} + p(t)f(x(\tau(t))) = 0, \ t \neq \theta_k, \\ \Delta(r(t)|x^{'}(t)|^{\alpha-1}x^{'}(t))_{|_{t=\theta_k}} + b_k h(x(\tau(\theta_k))) = 0, \end{cases}$$

Here, we are concerned with the oscillation of second-order nonlinear dynamic equation with impulses on a time scale $\mathbb{T}$ which is unbounded above

$$\begin{cases} (r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))^{\Delta} + p(t)f(x(\tau(t))) = 0, \ t \in \mathbb{J} := [0,\infty) \cap \mathbb{T}, \ t \neq \theta_k, \ k = 1, 2, ..., \\ \Delta(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))_{|_{t=\theta_k}} + b_k h(x(\tau(\theta_k))) = 0 \end{cases} \tag{1}$$

where
$\Delta(z(t))_{|_{t=\theta}} := z(\theta^+) - z(\theta^-)$, in which $z(\theta^{\mp}) := \lim_{t\to\theta^{\mp}} z(t)$. For Convenience we define $z(\theta) = z(\theta^-)$.

Throughout this paper we assumed the following conditions are satisfied:
(H_1) α, β are quotients of odd positive integers,
(H_2) $r(t)$ and $p(t)$ are positive rd-continuous functions on an arbitrary time scale $\mathbb{T}$ such that

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)} \Delta t = \infty,$$

(H_3) $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing and differentiable function such that $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$,
(H_4) $f \in C(\mathbb{R}, \mathbb{R})$ is continuous and nondecreasing function such that $xf(x) > 0$ and for a positive constant L, we have $\frac{f(x)}{x^{\beta}} \geq L$ for all $x \neq 0$,
(H_5) $h \in C(\mathbb{R}, \mathbb{R})$ is continuous such that $xh(x) > 0$ for all $x \neq 0$ and for a positive constant $c_2 > 0$, we have $|h(x)| \geq c_2|x^{\beta}|$
(H_6) $\{\theta_k\}$ is a fixed strictly increasing unbounded sequence of positive real numbers and $\{b_k\}$ is a sequence of positive real numbers,

The purpose of this paper is to establish some new oscillation criteria for the second-order nonlinear impulsive delay dynamic equations (1) which is not studied before. Our results extend and improve some results established by [11, 18, 19, 21] and can be applied to arbitrary time scales. Some examples are given to show that a dynamic equation is nonoscillatory, it may become oscillatory by adding some impulses to it. In this cases, impulses play a dominating part in oscillations of dynamic equations on time scales.

By a solution of (1), we mean that a nontrivial real valued function x satisfies (1) for $t \in \mathbb{T}$. A solution x of (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Eq. (1) is said to be oscillatory if all of its solutions are oscillatory.

Throughout the remainder of the paper, we assume that, for each $k = 1, 2, ...$, the points of impulses t_k are right-dense (rd for short). In order to define the solutions of (1), we introduce the spaces

$AC^i = \{x : \mathbb{J}_\mathbb{T} \to \mathbb{R}\ is\ i - times\ \Delta - differentiable,\ whose$
$ith\ delta\ derivative\ x^{\Delta^{(i)}}\ is\ absolutely\ continuous\}$.

$PC = \{x : \mathbb{J}_\mathbb{T} \to \mathbb{R}\ is\ right\ dense\ continuous\ except\ at\ t_k, k = 1, 2, ...\ for\ which$
$x(t_k^-), x(t_k^+), x^\Delta(t_k^-)\ and\ x^\Delta(t_k^+)\ exist\ with\ x(t_k^-) = x(t_k),\ x^\Delta(t_k^-) = x^\Delta(t_k)\}$.

2. Main results

Before starting our studies, we begin with the following lemma which will play an important role in the proof of our main results.

Lemma 1 [7] If $x(t)$ is delta differentiable and eventually positive or negative, then

$$((x(t))^\gamma)^\Delta = \gamma \int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{\gamma-1} x^\Delta(t) dh.$$

Lemma 2 (Hardy et al. [14]) If X and Y are nonnegative, then

$$\lambda XY^{\lambda-1} - X^\lambda \le (\lambda - 1)Y^\lambda \qquad when \qquad \lambda > 1,$$

where the equality holds if and only if $X = Y$.

Theorem 1 Assume that (H_1)-(H_6) hold. There exists differentiable positive function $\phi(t)$ such that $\phi^\Delta(t) \ge 0$ and

$$\limsup_{t\to\infty} \left[\int_{t_0}^t \left(Lp(s)\phi(s) - \mu \frac{r(\tau(s))(\phi^\Delta(s))^{\alpha+1}}{(\phi(s)V(s))^\alpha(\tau^\Delta(s))^\alpha} \right) \Delta s + \sum_{t_0 \le \theta_k < t} c_2 b_k \phi(\theta_k) \right] = \infty, \tag{2}$$

where $\mu = \frac{(\alpha/\beta)^\alpha}{(\alpha+1)^{\alpha+1}}$ and

$$V(t) := \begin{cases} k_1 \text{ is any positive constant,} & \text{if } \beta > \alpha, \\ 1, & \text{if } \beta = \alpha, \\ k_2(u^\sigma(\tau(t)))^{\frac{\alpha-\beta}{\alpha}}, k_2 \text{ is any positive constant,} & \text{if } \beta < \alpha. \end{cases}$$

Then the impulsive dynamic equation Eq. (1) is oscillatory.

Proof. Assume that Eq. (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t)$ is eventually positive solution of (1). Since $\tau(t) \to \infty$ as $t \to \infty$, there exists a positive real number T such that $x(\tau(t)) > 0$ for all $t > T$. From Eq. (1), we have

$$(r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t))^\Delta = -p(t)f(x(\tau(t))) \le -Lp(t)x^\beta(\tau(t)) \le 0.$$

Hence the function $r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)$ is nonincreasing on each interval (θ_k, θ_{k+1}) whenever $\theta_k \geq T$.
If $t = \theta_k$, then

$$r(\theta_k^+)|x^{\Delta}(\theta_k^+)|^{\alpha-1}x^{\Delta}(\theta_k^+) - r(\theta_k)|x^{\Delta}(\theta_k)|^{\alpha-1}x^{\Delta}(\theta_k) = -b_k h(x(\tau(\theta_k))) \leq 0.$$

Then, $r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)$ is nonincreasing in (T, ∞). We claim that $x^{\Delta}(t)$ is eventually positive. Assume on the contrary, If $x^{\Delta}(t^*) \leq 0$ for some $t^* \geq T$, then

$$r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t) \leq r(t^*)|x^{\Delta}(t^*)|^{\alpha-1}x^{\Delta}(t^*) \leq 0 \qquad for \qquad t \geq t^*,$$

$$x^{\Delta}(t) \leq \left(\frac{r(t^*)}{r(t)}\right)^{\frac{1}{\alpha}} x^{\Delta}(t^*) \qquad for \qquad t \geq t^*.$$

Integrating the last inequality from t^* to t, we get

$$x(t) - x(t^*) \leq r^{\frac{1}{\alpha}}(t^*)x^{\Delta}(t^*) \int_{t^*}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)} \Delta s.$$

Letting $t \to \infty$ and using (H_2), we conclude $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction. Therefore our claim is true. Define

$$w(t) = \phi(t)\frac{r(t)(x^{\Delta}(t))^{\alpha}}{x^{\beta}(\tau(t))}, \qquad t \neq \theta_k. \tag{3}$$

Then, $w(t) > 0$. Using the delta derivative rules of the product and quotient of two functions and then chain rule (see [[7], Theorem 1.90]), we get

$$\begin{aligned} w^{\Delta}(t) &= \frac{\phi(t)}{x^{\beta}(\tau(t))}(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + \left(\frac{\phi(t)}{x^{\beta}(\tau(t))}\right)^{\Delta}(r(t)(x^{\Delta}(t))^{\alpha})^{\sigma} \\ &= \frac{\phi(t)}{x^{\beta}(\tau(t))}(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + (r(t)(x^{\Delta}(t))^{\alpha})^{\sigma}\left(\frac{\phi^{\Delta}(t)}{x^{\beta}(\tau(\sigma(t)))} - \frac{\phi(t)(x^{\beta}(\tau(t)))^{\Delta}}{x^{\beta}(\tau(t))x^{\beta}(\tau(\sigma(t)))}\right). \end{aligned}$$

From Eq. (1), we have

$$w^{\Delta}(t) = \frac{\phi(t)}{x^{\beta}(\tau(t))}(-p(t)f(x(\tau(t)))) + (r(t)(x^{\Delta}(t))^{\alpha})^{\sigma}\left(\frac{\phi^{\Delta}(t)}{x^{\beta}(\tau(\sigma(t)))} - \frac{\phi(t)(x^{\beta}(\tau(t)))^{\Delta}}{x^{\beta}(\tau(t))x^{\beta}(\tau(\sigma(t)))}\right).$$

Using (H_4) and (3), we get

$$w^{\Delta}(t) \leq -L\phi(t)p(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\phi(t)w^{\sigma}(t)}{\phi^{\sigma}(t)}\frac{(x^{\beta}(\tau(t)))^{\Delta}}{x^{\beta}(\tau(t))}, \tag{4}$$

and

$$\Delta w(t)_{|t=\theta_k} = \frac{\phi(\theta_k)}{x^{\beta}(\tau(\theta_k))}\big(-b_k h(x(\tau(\theta_k)))\big). \tag{5}$$

Also, by Lemma 1, we get

$$\begin{aligned} (x^{\beta}(\tau(t)))^{\Delta} &= \beta\int_0^1 [x(\tau(t)) + h\mu(\tau(t))(x(\tau(t)))^{\Delta}]^{\beta-1} dh\ (x(\tau(t)))^{\Delta} \\ &= \beta\int_0^1 [(1-h)x(\tau(t)) + hx(\tau(\sigma(t)))]^{\beta-1} dh\ (x(\tau(t)))^{\Delta}, \end{aligned}$$

Hence,

$$(x^{\beta}(\tau(t)))^{\Delta} \geq \begin{cases} \beta(x(\tau(\sigma(t))))^{\beta-1}x^{\Delta}(\tau(t))\tau^{\Delta}(t), & \text{if } 0 < \beta \leq 1, \\ \beta x(\tau(t)))^{\beta-1}x^{\Delta}(\tau(t))\tau^{\Delta}(t), & \text{if } \beta > 1. \end{cases} \tag{6}$$

Since $(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))^{\Delta} \leq 0$, we get

$$x^{\Delta}(\tau(t)) \geq \frac{(r^{\sigma}(t))^{\frac{1}{\alpha}} x^{\Delta}(\sigma(t))}{(r(\tau(t)))^{\frac{1}{\alpha}}}. \tag{7}$$

From (6) and (7) in (4), we get

$$w^{\Delta}(t) \leq \begin{cases} -L\phi(t)p(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)} w^{\sigma}(t) - \\ \qquad \beta \frac{\phi(t)(r^{\sigma}(t))^{\frac{1}{\alpha}} x^{\Delta}(\sigma(t))(x(\tau(\sigma(t))))^{\beta-1}\tau^{\Delta}(t)}{\phi^{\sigma}(t)(r(\tau(t)))^{\frac{1}{\alpha}} x^{\beta}(\tau(t))} w^{\sigma}(t), & \text{if } 0 < \beta \leq 1, \\ -L\phi(t)p(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)} w^{\sigma}(t) - \beta \frac{\phi(t)(r^{\sigma}(t))^{\frac{1}{\alpha}} x^{\Delta}(\sigma(t))\tau^{\Delta}(t)}{\phi^{\sigma}(t)(r(\tau(t)))^{\frac{1}{\alpha}} x(\tau(t))} w^{\sigma}(t), & \text{if } \beta > 1. \end{cases}$$

Since $x^{\Delta}(t) > 0$, we get

$$\begin{aligned} w^{\Delta}(t) &\leq -L\phi(t)p(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)} w^{\sigma}(t) - \beta \frac{\phi(t)(r^{\sigma}(t))^{\frac{1}{\alpha}} x^{\Delta}(\sigma(t))\tau^{\Delta}(t)}{\phi^{\sigma}(t)(r(\tau(t)))^{\frac{1}{\alpha}} x(\tau(\sigma(t)))} w^{\sigma}(t) \\ &\leq -L\phi(t)p(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)} w^{\sigma}(t) - \beta \frac{\phi(t)(w^{\sigma}(t))^{\frac{\alpha+1}{\alpha}} \tau^{\Delta}(t)}{(r(\tau(t)))^{\frac{1}{\alpha}} (\phi^{\sigma}(t))^{\frac{\alpha+1}{\alpha}}} x^{\frac{\beta-\alpha}{\alpha}}(\tau^{\sigma}(t)). \end{aligned} \tag{8}$$

Next, we consider the following three cases:
Case (i): Let $\alpha < \beta$. For $t \in [t_1, \infty)$, since $x^{\sigma}(\tau(t)) \geq x(\tau(t)) \geq x(\tau(t_1)) > 0$, we have

$$(x(\tau(\sigma(t))))^{\frac{\beta-\alpha}{\alpha}} \geq (x(\tau(t_1)))^{\frac{\beta-\alpha}{\alpha}} := k_1. \tag{9}$$

Case (ii): Let $\alpha = \beta$. For $t \in [t_1, \infty)$, we have

$$(x(\tau(\sigma(t))))^{\frac{\beta-\alpha}{\alpha}} = 1. \tag{10}$$

Case (iii): Let $\alpha > \beta$. Since $(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))^{\Delta} \leq 0$, for $t \in [t_1, \infty)$ we get

$$r(t)(x^{\Delta}(t))^{\alpha} \leq r(t_1)(x^{\Delta}(t_1))^{\alpha} := b.$$

Hence, we have $x^{\Delta}(t) \leq b^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(t)$. Integrating both sides of the last inequality from t_1 to t, we get

$$x(t) \leq x(t_1) + b^{\frac{1}{\alpha}} \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)} \Delta s$$

Therefore, there exist a constant $b_1 > 0$ and $t_4 > t_1$ such that $x(t) \leq b_1 \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)} \Delta s := b_1 u^{-1}(t)$ for $t \in [t_4, \infty)$. Hence, we get

$$(x(\tau(\sigma(t))))^{\frac{\beta-\alpha}{\alpha}} \geq k_2 (u(\tau(\sigma(t))))^{\frac{\alpha-\beta}{\alpha}}, \tag{11}$$

where $k_2 = (b_1)^{\frac{\beta-\alpha}{\alpha}}$.
Hence, from (9) , (10) and (11), we get

$$w^{\Delta}(t) \leq -L\phi(t)p(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)} w^{\sigma}(t) - \beta \frac{\phi(t)(w^{\sigma}(t))^{\frac{\alpha+1}{\alpha}} \tau^{\Delta}(t)}{(r(\tau(t)))^{\frac{1}{\alpha}} (\phi^{\sigma}(t))^{\frac{\alpha+1}{\alpha}}} V(t). \tag{12}$$

Where

$$V(t) := \begin{cases} k_1 \text{ is any positive constant,} & \text{if } \beta > \alpha, \\ 1, & \text{if } \beta = \alpha, \\ k_2 (u^{\sigma}(\tau(t)))^{\frac{\alpha-\beta}{\alpha}}, k_2 \text{ is any positive constant,} & \text{if } \beta < \alpha. \end{cases}$$

Taking $\lambda = \frac{\alpha+1}{\alpha}$, $X = \frac{(\beta\phi(t)V(t)\tau^{\Delta}(t))^{\frac{1}{\lambda}}}{(r(\tau(t)))^{\frac{1}{\alpha+1}}\phi^{\sigma}(t)} w^{\sigma}(t)$ and $Y = \frac{(\phi^{\Delta}(t))^{\alpha} r^{\frac{1}{\lambda}}(\tau(t))}{\lambda^{\alpha}(\beta\phi(t)\tau^{\Delta}(t)V(t))^{\frac{\alpha}{\lambda}}}$, applying Lemma 2 on (12), we get

$$w^{\Delta}(t) \leq -L\phi(t)p(t) + \mu \frac{(\phi^{\Delta}(t))^{\alpha+1} r(\tau(t))}{(\phi(t)V(t)\tau^{\Delta}(t))^{\alpha}}, \qquad t \neq \theta_k, \tag{13}$$

where $\mu = \frac{\alpha^{\alpha}}{\beta^{\alpha}(\alpha+1)^{\alpha+1}}$.

$$\int_{t_1}^{t} w^{\Delta}(s)\Delta s = w(t) - w(t_1) - \sum_{t_1 \leq \theta_k < t} \Delta w(\theta_k). \tag{14}$$

Integrating (13) from t_1 to t and using (H_6), (5) and (14), we get

$$w(t) \leq w(t_1) - \left[\int_{t_1}^{t} \left(L\phi(s)p(s) - \mu \frac{(\phi^{\Delta}(s))^{\alpha+1} r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}} \right) \Delta s + \sum_{t_1 \leq \theta_k < t} c_2 b_k \phi(\theta_k) \right].$$

Taking lim sup of both sides as $t \to \infty$, we get a contradiction with (2). This completes the proof.

Corollary 1 Assume that (H_1)-(H_6) hold. There exists differentiable positive function $\phi(t)$ such that $\phi^{\Delta}(t) \geq 0$ for all $t \geq t_0$,

$$\int_{t_0}^{\infty} \frac{r(\tau(s))(\phi^{\Delta}(s))^{\alpha+1}}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}} \Delta s + \sum_{t_0 \leq \theta_k < t} c_2 b_k \phi(\theta_k) = \infty, \tag{15}$$

and

$$\liminf_{t\to\infty} \frac{L(\phi(s))^{\alpha+1}(V(s)\tau^{\Delta}(s))^{\alpha} p(s)}{r(\tau(s))(\phi^{\Delta}(s))^{\alpha+1}} > \mu. \tag{16}$$

Then the impulsive dynamic equation Eq. (1) is oscillatory.
Proof. From (16), it follows that there exists $\epsilon > 0$ such that for all large t

$$\frac{L(\phi(s))^{\alpha+1}(V(s)\tau^{\Delta}(s))^{\alpha} p(s)}{r(\tau(s))(\phi^{\Delta}(s))^{\alpha+1}} > \mu + \epsilon.$$

From the proof of Theorem 1 in (5) and (13), we have

$$w^{\Delta}(t) \leq -L\phi(t)p(t) + \mu \frac{(\phi^{\Delta}(t))^{\alpha+1} r(\tau(t))}{(\phi(t)V(t)\tau^{\Delta}(t))^{\alpha}}, \qquad t \neq \theta_k,$$

$$\Delta w(t)_{|t=\theta_k} = \frac{-\phi(\theta_k) b_k h(x(\tau(\theta_k)))}{x^{\beta}(\tau(\theta_k))}. \tag{17}$$

Hence, we get

$$w^{\Delta}(t) \leq -\frac{(\phi^{\Delta}(t))^{\alpha+1} r(\tau(t))}{(\phi(t)V(t)\tau^{\Delta}(t))^{\alpha}} \left[\frac{L(\phi(t))^{\alpha+1}(V(t)\tau^{\Delta}(t))^{\alpha} p(t)}{r(\tau(t))(\phi^{\Delta}(t))^{\alpha+1}} - \mu \right], \qquad t \neq \theta_k,$$

Integrating the above inequality from t_1 to t, we get

$$w(t) \leq w(t_1) - \int_{t_1}^{t} \frac{(\phi^{\Delta}(s))^{\alpha+1} r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}} \qquad \left[L \frac{(\phi(s))^{\alpha+1}(V(s)\tau^{\Delta}(s))^{\alpha} p(s)}{r(\tau(s))(\phi^{\Delta}(s))^{\alpha+1}} - \mu \right] \Delta s$$
$$- \frac{\phi(\theta_k) b_k h(x(\tau(\theta_k)))}{x^{\beta}(\tau(\theta_k))}.$$

Therefore, we get

$$w(t) \leq w(t_1) - \left[\epsilon \int_{t_1}^{t} \frac{(\phi^{\Delta}(s))^{\alpha+1} r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}} \Delta s + \sum_{t_1 \leq \theta_k < t} c_2 b_k \phi(\theta_k)\right].$$

Taking limit of both sides as $t \to \infty$, we get a contradiction with $w(t) > 0$. This completes the proof.

Theorem 2 Assume that (H_1)-(H_6) hold. Let H be an rd-continuous function defined as follows:

$$H : D_{\mathbb{T}} = \{(t, s) \in \mathbb{T} \times \mathbb{T} : t \geq s \geq t_0, t, s \in [t_0, s)_{\mathbb{T}}\} \to \mathbb{R},$$

$$H(t, t) = 0,\ t \geq t_0, \qquad H(t, s) > 0\ for\ t > s \geq t_0,\ t, s \in [t_0, s)_{\mathbb{T}}, \tag{18}$$

and H has a non positive continuous Δ-partial derivative $H^{\Delta_s}(t, s)$ with respect to the second variable. If there exists differentiable positive function $\phi(t)$ such that $\phi^{\Delta}(t) \geq 0$ for all $t \geq t_0$ such that

$$\limsup_{t\to\infty} \left[\frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s) \left(Lp(s)\phi(s) - \mu \frac{r(\tau(s))(\phi^{\Delta}(s))^{\alpha+1}}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}}\right)\Delta s + \frac{1}{H(t, t_0)} \sum_{t_0 \leq \theta_k < t} H(t, \theta_k) c_2 b_k \phi(\theta_k)\right] = \infty. \tag{19}$$

Then the impulsive dynamic equation Eq. (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). Suppose that $x(t) > 0$ is eventually positive for $t \geq t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Proceeding as in the proof of Theorem 1, we get

$$w^{\Delta}(t) \leq -\left[L\phi(t)p(t) - \mu \frac{(\phi^{\Delta}(t))^{\alpha+1} r(\tau(t))}{(\phi(t)V(t)\tau^{\Delta}(t))^{\alpha}}\right], \qquad t \neq \theta_k, \tag{20}$$

$$\Delta w(t)_{|t=\theta_k} = \frac{-\phi(\theta_k) b_k h(x(\tau(\theta_k)))}{x^{\beta}(\tau(\theta_k))}. \tag{21}$$

Multiplying Eq. (20) by $H(t, s)$ and integrating from t_1 to t, we get

$$\int_{t_1}^{t} H(t, s) w^{\Delta}(s)\Delta s \leq -\int_{t_1}^{t} H(t, s) \left[L\phi(s)p(s) - \mu \frac{(\phi^{\Delta}(s))^{\alpha+1} r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}}\right]\Delta s - \sum_{t_1 \leq \theta_k < t} H(t, \theta_k) \frac{\phi(\theta_k) b_k h(x(\tau(\theta_k)))}{x^{\beta}(\tau(\theta_k))},\ t \neq \theta_k,$$

therefore, we get

$$\int_{t_1}^{t} H(t, s)\left[L\phi(s)p(s) - \mu \frac{(\phi^{\Delta}(s))^{\alpha+1} r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}}\right]\Delta s + \sum_{t_1 \leq \theta_k < t} H(t, \theta_k) \frac{\phi(\theta_k) b_k h(x(\tau(\theta_k)))}{x^{\beta}(\tau(\theta_k))} \leq -\int_{t_1}^{t} H(t, s) w^{\Delta}(s)\Delta s, \tag{22}$$

where

$$\int_{t_1}^{t} H(t, s) w^{\Delta}(s)\Delta s = -H(t, t_1)w(t_1) - \int_{t_1}^{t} H^{\Delta_s}(t, s) w^{\sigma}(s)\Delta s. \tag{23}$$

From (23) in (22), we get

$$\int_{t_1}^{t} H(t,s)\Big[L\phi(s)p(s) - \mu\frac{(\phi^{\Delta}(s))^{\alpha+1}r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}}\Big]\Delta s + \sum_{t_1\le\theta_k<t} H(t,\theta_k)\frac{\phi(\theta_k)b_k h(x(\tau(\theta_k)))}{x^{\beta}(\tau(\theta_k))}$$
$$\le H(t,t_1)w(t_1) + \int_{t_1}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s \le H(t,t_1 w(t_1).$$

Hence,

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t} H(t,s)\Big[L\phi(s)p(s) - \mu\frac{(\phi^{\Delta}(s))^{\alpha+1}r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}}\Big]\Delta s + \frac{1}{H(t,t_1)}\sum_{t_1\le\theta_k<t} H(t,\theta_k)\frac{\phi(\theta_k)b_k h(x(\tau(\theta_k)))}{x^{\beta}(\tau(\theta_k))} \le w(t_1). \quad (24)$$

Taking lim sup of (24) as $t\to\infty$, we get a contradiction with (19). This completes the proof.

Theorem 3 Assume that (H_1)-(H_6) hold. Let H be an rd-continuous function defined as follows:

$$H: D_{\mathbb{T}} = \{(t,s)\in\mathbb{T}\times\mathbb{T}: t\ge s\ge t_0, t,s\in[t_0,s)_{\mathbb{T}}\}\to\mathbb{R},$$
$$H(t,t)=0,\ t\ge t_0, \qquad H(t,s)>0\ for\ t>s\ge t_0,\ t,s\in[t_0,s)_{\mathbb{T}}, \quad (25)$$

and H has a non positive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable. Let $h: D_{\mathbb{T}}\to\mathbb{R}$ be an rd-continuous function satisfying

$$H^{\Delta_s}(t,s) + H(t,s)\frac{\phi^{\Delta}(s)}{\phi^{\sigma}(s)} = \frac{h(t,s)}{\phi^{\sigma}(s)}(H(t,s))^{\frac{\alpha}{\alpha+1}}, \quad (26)$$

if there exists differentiable positive function $\phi(t)$ such that $\phi^{\Delta}(t)\ge 0$ for all $t\ge t_0$ such that

$$\limsup_{t\to\infty}\Big[\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big(LH(t,s)\phi(s)p(s) - \mu\int_{t_1}^{t}\frac{(h(t,s))^{\alpha+1}r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}}\Big)\Delta s + \frac{1}{H(t,t_0)}\sum_{t_0\le\theta_k<t} H(t,\theta_k)c_2 b_k\phi(\theta_k)\Big] = \infty. \quad (27)$$

Then the impulsive dynamic equation Eq. (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). Suppose that $x(t)>0$ is eventually positive for $t\ge t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Proceeding as in the proof of Theorem 1, we get

$$w^{\Delta}(t) \le -L\phi(t)p(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)}w^{\sigma}(t) - \beta\frac{\phi(t)(w^{\sigma}(t))^{\frac{1+\alpha}{\alpha}}\tau^{\Delta}(t)}{(r(\tau(t)))^{\frac{1}{\alpha}}(\phi^{\sigma}(t))^{\frac{1+\alpha}{\alpha}}}V(t), \quad (28)$$

$$\Delta w(t)_{|t=\theta_k} = \frac{-\phi(\theta_k)b_k h(x(\tau(\theta_k)))}{x^{\beta}(\tau(\theta_k))}. \quad (29)$$

Multiplying Eq. (28) by $H(t,s)$ and integrating from t_1 to t, we get

$$\int_{t_1}^{t} H(t,s)w^{\Delta}(s)\Delta s \le -\int_{t_1}^{t} H(t,s)L\phi(s)p(s)\Delta s + \int_{t_1}^{t} H(t,s)\frac{\phi^{\Delta}(s)}{\phi^{\sigma}(s)}w^{\sigma}(s)\Delta s - \int_{t_1}^{t} H(t,s)\beta\frac{\phi(s)(w^{\sigma}(s))^{\frac{1+\alpha}{\alpha}}\tau^{\Delta}(s)}{(r(\tau(s)))^{\frac{1}{\alpha}}(\phi^{\sigma}(s))^{\frac{1+\alpha}{\alpha}}}V(s)\Delta s + \sum_{t_1\le\theta_k<t} H(t,\theta_k)\Delta w(\theta_k). \quad (30)$$

Integrating by parts and using (25) and (26), we get

$$\int_{t_1}^{t} LH(t,s)\phi(s)p(s)\Delta s \le H(t,t_1)w(t_1) + \int_{t_1}^{t}\big[H^{\Delta_s}(t,s) + H(t,s)\frac{\phi^{\Delta}(s)}{\phi^{\sigma}(s)}\big]w^{\sigma}(s)\Delta s$$
$$-\int_{t_1}^{t} H(t,s)\beta\frac{\phi(s)(w^{\sigma}(s))^{\frac{1+\alpha}{\alpha}}\tau^{\Delta}(s)}{(r(\tau(s)))^{\frac{1}{\alpha}}(\phi^{\sigma}(s))^{\frac{1+\alpha}{\alpha}}}V(s)\Delta s - \sum_{t_1\le\theta_k<t} H(t,\theta_k)c_2b_k\phi(\theta_k),$$

$$\int_{t_1}^{t} LH(t,s)\phi(s)p(s)\Delta s \le H(t,t_1)w(t_1) + \int_{t_1}^{t}\big[\frac{h(t,s)}{\phi^{\sigma}(s)}(H(t,s))^{\frac{\alpha}{\alpha+1}}\big]w^{\sigma}(s)\Delta s -$$
$$\int_{t_1}^{t} H(t,s)\beta\frac{\phi(s)(w^{\sigma}(s))^{\frac{1+\alpha}{\alpha}}\tau^{\Delta}(s)}{(r(\tau(s)))^{\frac{1}{\alpha}}(\phi^{\sigma}(s))^{\frac{1+\alpha}{\alpha}}}V(s)\Delta s - \sum_{t_1\le\theta_k<t} H(t,\theta_k)c_2b_k\phi(\theta_k). \quad (31)$$

Taking $\lambda = \frac{\alpha+1}{\alpha}$, $X = \frac{(\beta\phi(t)V(t)\tau^{\Delta}(t))^{\frac{1}{\lambda}}}{r^{\frac{1}{\alpha+1}}(\tau(t))\phi^{\sigma}(t)}(H(t,s))^{\frac{1}{\lambda}}w^{\sigma}(t)$ and $Y = \frac{(h(t,s))^{\alpha}r^{\frac{1}{\lambda}}(\tau(t))}{\lambda^{\alpha}(\beta\phi(t)\tau^{\Delta}(t)V(t))^{\frac{\alpha}{\lambda}}}$, applying Lemma 2 on (31), we get

$$\int_{t_1}^{t} LH(t,s)\phi(s)p(s)\Delta s \le \quad H(t,t_1)w(t_1) \quad +\mu\int_{t_1}^{t}\frac{(h(t,s))^{\alpha+1}r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}}\Delta s$$
$$-\sum_{t_1\le\theta_k<t} H(t,\theta_k)c_2b_k\phi(\theta_k),$$

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\Big[\quad LH(t,s)\phi(s)p(s) - \mu\frac{(h(t,s))^{\alpha+1}r(\tau(s))}{(\phi(s)V(s)\tau^{\Delta}(s))^{\alpha}}\Big]\Delta s +$$
$$\frac{1}{H(t,t_1)}\sum_{t_1\le\theta_k<t} H(t,\theta_k)c_2b_k\phi(\theta_k) \le w(t_1). \quad (32)$$

Taking $\limsup$ of (32) as $t\to\infty$, we get a contradiction with (27). This completes the proof.

3. Examples

Example 1 Consider the equation ($\mathbb{T}=\mathbb{R}$)

$$\begin{cases}(t^3(x^{'}(t))^3)^{'} + \frac{1}{t^4}x^3(\tau(t)) = 0,\ t\ge t_0,\ \ t\ne k\\ \Delta(t^3(x^{'}(t))^3)_{|t=k} + b_kh(x(\tau(k))) = 0,\end{cases} \quad (33)$$

Here, $\alpha=\beta=3$, $r(t)=t^3$, $p(t)=\frac{1}{t^4}$, $b_k = k^{\frac{-1}{3}}$, $\tau(t)=\frac{t}{2}$ and $\theta_k = k$. To apply Theorem 1, take $\phi(t)=1$, $c_2=1$ and $L=1$. Note that

$$\limsup_{t\to\infty}\Big[\int_{t_0}^{t}\Big(Lp(s)\phi(s) \quad - \quad \mu\frac{r(\tau(s))(\phi^{\Delta}(s))^{\alpha+1}}{(\phi(s)V(s))^{\alpha}(\tau^{\Delta}(s))^{\alpha}}\Big)\Delta s + \sum_{t_0\le\theta_k<t} c_2b_k\phi(\theta_k)\Big]$$
$$= \limsup_{t\to\infty}\Big(\int_{t_0}^{t}\frac{1}{s^4}ds + \sum_{t_0\le\theta_k<t} k^{\frac{-1}{3}}\Big) = \infty,$$

Hence, every solution of Eq. (33) is oscillatory.

Remark 1
The results of [19] can not be applied to equation (33). This is because condition (3) in [19] is not satisfied. But, according to Theorem 1, when $\mathbb{T}=\mathbb{R}$, this equation is oscillatory.

Example 2 Consider the equation ($\mathbb{T}=\mathbb{R}$)

$$\begin{cases} (\frac{1}{t}x^{'}(t))^{'}+\frac{1}{t^3}x(\tau(t))(x^2(\tau(t))+1)=0,\ t\geq t_0,\ \ t\neq k \\ \Delta(\frac{1}{t}x^{'}(t))_{|t=k}+b_k h(x(\tau(k)))=0, \end{cases} \tag{34}$$

Here, $\alpha=\beta=1$, $r(t)=\frac{1}{t}$, $p(t)=\frac{1}{t^3}$, $b_k=k^{\frac{-3}{2}}$, $\tau(t)=t$ and $\theta_k=k$. To apply Theorem 1, take $\phi(t)=t$, $c_2=1$ and $L=1$. Note that

$$\limsup_{t\to\infty}\left[\int_{t_0}^{t}\left(Lp(s)\phi(s) \quad - \quad \mu\frac{r(\tau(s))(\phi^{\Delta}(s))^{\alpha+1}}{(\phi(s)V(s))^{\alpha}(\tau^{\Delta}(s))^{\alpha}}\right)\Delta s+\sum_{t_0\leq\theta_k<t}c_2 b_k\phi(\theta_k)\right]$$
$$= \limsup_{t\to\infty}\left(\frac{3}{4}\int_{t_0}^{t}\frac{1}{s^2}ds+\sum_{t_0\leq\theta_k<t}k^{\frac{-1}{2}}\right)=\infty,$$

Hence, every solution of Eq. (34) is oscillatory.

Remark 2
(1) The results of [18] can not be applied to equation (34) for $r(t)=\frac{1}{t}$. But, according to Theorem 1, when $\mathbb{T}=\mathbb{R}$, this equation is oscillatory.
(2) The results of [11] can not be applied to equation (34) for $r(t)=\frac{1}{t}$ and condition (2.1) in [11] is not satisfied. But, according to Theorem 1, when $\mathbb{T}=\mathbb{R}$, this equation is oscillatory.
(3) In the above example, we note that the dynamic equation without impulses is nonoscillatory, but when some impulses are added to it, it becomes oscillatory. Therefore, this example shows that impulses play an important part in the oscillations of dynamic equations on time scales.

Example 3 Consider the second order impulsive dynamic equation

$$\begin{cases} (t|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))^{\Delta}+\frac{1}{t\sigma(t)}x^{\beta}(\tau(t))=0, \qquad t\neq k \\ \Delta(t|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))_{|t=k}+b_k h(x(\tau(k)))=0, \end{cases} \tag{35}$$

where $t\geq 2$ and $k\geq 2$.
Here, α, β are quotients of odd positive integers such that $\beta\geq\alpha$, $r(t)=t$, $p(t)=\frac{1}{t\sigma(t)}$, $b_k=(k-1)^{\frac{-3}{4}}$, $\tau(t)=t-1$ and $\theta_k=k$. To apply Theorem 1, take $\phi(t)=1$, $c_2=1$ and $L=1$. Note that

$$\limsup_{t\to\infty}\quad\left[\quad\int_{t_0}^{t}\left(Lp(s)\phi(s)-\mu\frac{r(\tau(s))(\phi^{\Delta}(s))^{\alpha+1}}{(\phi(s)V(s))^{\alpha}(\tau^{\Delta}(s))^{\alpha}}\right)\Delta s+\sum_{t_0\leq\theta_k<t}c_2 b_k\phi(\theta_k)\right]$$
$$= \limsup_{t\to\infty}\left(\int_{t_0}^{t}\frac{1}{s\sigma(s)}\Delta s+\sum_{t_0\leq\theta_k<t}(k-1)^{\frac{-3}{4}}\right)=\infty,$$

Hence, every solution of Eq. (35) is oscillatory.

Example 4 Consider the equation ($\mathbb{T} = \mathbb{N}$)

$$\begin{cases} \Delta(\frac{t^\alpha}{(\sigma(t))^\alpha}\Delta(x(t))^\alpha) + \frac{1}{t(\sigma(t))^\alpha}x^\alpha(\tau(t)) = 0, & t \geq t_0, \ \ t \neq k+1 \\ \Delta(\frac{t^\alpha}{(\sigma(t))^\alpha}\Delta(x(t))^\alpha)_{|_{t=k+1}} + b_k h(x(\tau(k+1))) = 0, & \end{cases} \tag{36}$$

where α, β are quotients of odd positive integers such that $\beta = \alpha$, $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$ and $\sigma(t) = t+1$. Here $r(t) = \frac{t^\alpha}{(\sigma(t))^\alpha}$, $p(t) = \frac{1}{t(\sigma(t))^\alpha}$, $b_k = \frac{1}{\sqrt{k}(k+1)}$, $\tau(t) = t$ and $\theta_k = k+1$.
To apply Theorem 1, take $\phi(t) = t$, $c_2 = 1$ and $L = 1$. Note that

$$\begin{aligned} \limsup_{t\to\infty} & \left[\int_{t_0}^t \left(Lp(s)\phi(s) - \mu \frac{r(\tau(s))(\phi^\Delta(s))^{\alpha+1}}{(\phi(s)V(s))^\alpha(\tau^\Delta(s))^\alpha} \right)\Delta s + \sum_{t_0 \leq \theta_k < t} c_2 b_k \phi(\theta_k) \right] \\ &= \limsup_{t\to\infty} \left((1-\mu)\int_{t_0}^t \frac{1}{(\sigma(s))^\alpha}\Delta s + \sum_{t_0\leq\theta_k<t}\frac{1}{\sqrt{k}} \right) \\ &= \limsup_{t\to\infty} \left((1-\mu)\int_{t_0}^t \frac{1}{(t+1)^\alpha}\Delta s + \sum_{t_0\leq\theta_k<t} k^{\frac{-1}{2}} \right) = \infty, \end{aligned}$$

Hence, every solution of Eq. (36) is oscillatory.

Example 5 Consider the second order impulsive dynamic equation

$$\begin{cases} \left(\frac{1}{(t+\sigma(t))^\alpha}|x^\Delta(t)|^{\alpha-1}x^\Delta(t)\right)^\Delta + \frac{1}{t\sigma(t)}x^\beta(\tau(t)) = 0, & t \neq k \\ \Delta(\frac{1}{(t+\sigma(t))^\alpha}|x^\Delta(t)|^{\alpha-1}x^\Delta(t))_{|_{t=k}} + b_k h(x(\tau(k))) = 0, & \end{cases} \tag{37}$$

where $t \geq 2$ and $k \geq 2$.
Here, α, β are quotients of odd positive integers such that $\beta \geq \alpha$, $r(t) = \frac{1}{(t+\sigma(t))^\alpha}$, $p(t) = \frac{1}{t\sigma(t)}$, $b_k = (k-1)^{\frac{-3}{4}}$, $\tau(t) = t$ and $\theta_k = k$. To apply Theorem 1, take $\phi(t) = 1$, $c_2 = 1$ and $L = 1$. Note that

$$\begin{aligned} \limsup_{t\to\infty} & \left[\int_{t_0}^t \left(Lp(s)\phi(s) - \mu \frac{r(\tau(s))(\phi^\Delta(s))^{\alpha+1}}{(\phi(s)V(s))^\alpha(\tau^\Delta(s))^\alpha} \right)\Delta s + \sum_{t_0 \leq \theta_k < t} c_2 b_k \phi(\theta_k) \right] \\ &= \limsup_{t\to\infty} \left(\int_{t_0}^t \frac{1}{s\sigma(s)}\Delta s + \sum_{t_0\leq\theta_k<t}(k-1)^{\frac{-3}{4}} \right) = \infty, \end{aligned}$$

Hence, every solution of Eq. (37) is oscillatory.

Remark 3
(1) The results of [9] can not be applied to equation (37). But, according to Theorem 1, this equation is oscillatory.
(2) In the above example, we note that the dynamic equation without impulses is nonoscillatory, but when some impulses are added to it, it becomes oscillatory. Therefore, this example shows that impulses play an important part in the oscillations of dynamic equations on time scales.

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