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ANALYSIS OF PERIODICITY FOR A NEW CLASS OF NON-LINEAR DIFFERENCE EQUATIONS BY USING A NEW METHOD

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Abstract. This paper aims to investigate the periodicity of solutions of the following delay nonlinear difference equation

$$x_{n+1} = ax_{n-k} + bx_{n-l} + \frac{\sum\limits_{i=0}^{k} a_i x_{n-i}}{\sum\limits_{j=0}^{l} b_j x_{n-j}},\ n = 0, 1, ...$$

where the parameters $a, b, a_0, \ldots, a_k, b_0, b_1, \ldots, b_l$ are non-zero real numbers, $k, l \in \mathbb{Z}^+$ and the initial values $x_{-\max\{k,l\}}, ..., x_{-1}, x_0 \in \mathbb{R} - \{0\}$. Moreover, several numerical simulations are provided to support obtained results.

1. Introduction

Rational recursive sequences are also called rational recursive difference equations. These types seem very simple and some of their properties can also be observed and conjectured by computers' simulations, however, it is extremely difficult to prove completely the properties observed and conjectured by computers' simulations. Therefore, the qualitative analysis of recursive difference equations has been the object of recent study. For example, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references cited therein.

In [7], Elsayed has obtained the period two and three solution using a new method of the following rational difference equation

$$x_{n+1} = \alpha + \frac{\beta x_n}{x_{n-1}} + \frac{\gamma x_{n-1}}{x_n},\ n = 0, 1, ... \tag{1}$$

with positive parameters and positive initial conditions.

Also, in [16] Moaaz has investigated the results of [7] and he has also revealed some important results.

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In [15], Moaaz et al. have studied the behavior of solutions of the following rational difference equation

$$x_{n+1} = \alpha + \frac{\beta x_{n-r}}{x_{n-s}} + \frac{\gamma x_{n-r}}{x_{n-t}},\ n = 0, 1, ... \tag{2}$$

where the initial conditions are arbitrary positive real numbers and α, β, γ are positive constans.

The purpose of this paper is investigate the periodic nature via Elsayed's new method [7] of solutions of the following higher-order difference equation

$$x_{n+1} = ax_{n-k} + bx_{n-l} + \frac{\sum\limits_{i=0}^{k} a_i x_{n-i}}{\sum\limits_{j=0}^{l} b_j x_{n-j}},\ n = 0, 1, ... \tag{3}$$

where the parameters $a, b, a_0, \ldots, a_k, b_0, b_1, \ldots, b_l$ are non-zero real numbers, $k, l \in \mathbb{Z}^+$ and the initial values $x_{-\max\{k,l\}}, ..., x_{-1}, x_0 \in \mathbb{R} - \{0\}$.

As far as we examine, there is no paper dealing with Eq.(3). Therefore, it is meaningful to study their elagance results.

2. Main Result

In this section, we will study the existence of two periodic solutions using the new method which was introduced by E. M. Elsayed in [7]. Thanks to this method, demonstration of the existence of two periodic solutions is quite easier than the method commonly used in the literature. It also provides short and easy proof for periodic solutions of Eq.(3).

Theorem 1. *Assume that $n \in \mathbb{R} - \{0, \pm 1\}$, the parameters $a, b, a_0, \ldots, a_k, b_0, b_1, \ldots, b_l$ are non-zero real numbers, $k, l \in \mathbb{Z}^+$ and the initial values $x_{-\max\{k,l\}}, ..., x_{-1}, x_0 \in \mathbb{R} - \{0\}$.*
(i) Let k be odd and l be odd, then Eq.(3) possesses eventual prime period two solutions if and only if

$$y_1 z_2 = n y_2 z_1 \text{ and } a + b \neq 1 \tag{4}$$

where

$$\begin{aligned} y_1 &= \left(\frac{a_0}{n} + a_1 + \frac{a_2}{n} + \ldots + \frac{a_{k-1}}{n} + a_k\right), \\ z_1 &= \left(\frac{b_0}{n} + b_1 + \frac{b_2}{n} + \ldots + \frac{b_{l-1}}{n} + b_l\right), \\ y_2 &= (a_0 n + a_1 + a_2 n + \ldots + a_{k-1} n + a_k), \\ z_2 &= (b_0 n + b_1 + b_2 n + \ldots + b_{l-1} n + b_l). \end{aligned}$$

(ii) Let k be even and l be even, then Eq.(3) possesses eventual prime period two solutions if and only if

$$n y_1 z_2 = y_2 z_1 \text{ and } a + b \neq 1$$

where

$$
\begin{aligned}
y_1 &= (\frac{a_0}{n} + a_1 + \frac{a_2}{n} + \ldots + \frac{a_{k-1}}{n} + a_k),\\
z_1 &= (\frac{b_0}{n} + b_1 + \frac{b_2}{n} + \ldots + \frac{b_{l-1}}{n} + b_l),\\
y_2 &= (a_0 n + a_1 + a_2 n + \ldots + a_{k-1} n + a_k),\\
z_2 &= (b_0 n + b_1 + b_2 n + \ldots + b_{l-1} n + b_l).
\end{aligned}
$$

(iii) Let k be odd and l be even, then Eq.(3) possesses eventual prime period two solutions if and only if

$$y_1(1-x_2)z_2 = n y_2(1-x_1)z_1 \tag{5}$$

where

$$
\begin{aligned}
x_1 &= a + \frac{b}{n},\\
y_1 &= (\frac{a_0}{n} + a_1 + \frac{a_2}{n} + \ldots + \frac{a_{k-1}}{n} + a_k),\\
z_1 &= (\frac{b_0}{n} + b_1 + \frac{b_2}{n} + \ldots + b_{l-1} + \frac{b_l}{n}),\\
x_2 &= a + bn,\\
y_2 &= (a_0 n + a_1 + a_2 n + \ldots + a_{k-1} n + a_k),\\
z_2 &= (b_0 n + b_1 + b_2 n + \ldots + b_{l-1} + b_l n).
\end{aligned}
$$

(iv) Let k be even and l be odd, then Eq.(3) possesses eventual prime period two solutions if and only if

$$\frac{y_2(1-x_1)z_1}{y_1(1-x_2)z_2} = n.$$

Proof. (i) First, we assume that Eq.(3) has eventual prime period two solutions in the following form

$$..., x, y, x, y, ...$$

We shall show that Condition (4) holds. By using Elsayed's new method, from (3) we get

$$x = ax + bx + \frac{a_0 y + a_1 x + a_2 y + \ldots + a_{k-1} y + a_k x}{b_0 y + b_1 x + b_2 y + \ldots + b_{l-1} y + b_l x}$$

and

$$y = ay + by + \frac{a_0 x + a_1 y + a_2 x + \ldots + a_{k-1} x + a_k y}{b_0 x + b_1 y + b_2 x + \ldots + b_{l-1} x + b_l y}.$$

We assume that $n = \frac{x}{y}$. So, we rewrite the equilities above

$$x = x(a+b) + \frac{x(\frac{a_0}{n} + a_1 + \frac{a_2}{n} + \ldots + \frac{a_{k-1}}{n} + a_k)}{x(\frac{b_0}{n} + b_1 + \frac{b_2}{n} + \ldots + \frac{b_{l-1}}{n} + b_l)}$$

and

$$y = y(a+b) + \frac{y(a_0 n + a_1 + a_2 n + \ldots + a_{k-1} n + a_k)}{y(b_0 n + b_1 + b_2 n + \ldots + b_{l-1} n + b_l)}.$$

From the second equility, we have

$$ny = yn(a+b) + \frac{n(a_0 n + a_1 + a_2 n + \ldots + a_{k-1} n + a_k)}{(b_0 n + b_1 + b_2 n + \ldots + b_{l-1} n + b_l)}.$$

We set

$$\begin{aligned} y_1 &= (\frac{a_0}{n} + a_1 + \frac{a_2}{n} + \ldots + \frac{a_{k-1}}{n} + a_k), \\ z_1 &= (\frac{b_0}{n} + b_1 + \frac{b_2}{n} + \ldots + b_{l-1} + \frac{b_l}{n}), \\ y_2 &= (a_0 n + a_1 + a_2 n + \ldots + a_{k-1} n + a_k), \\ z_2 &= (b_0 n + b_1 + b_2 n + \ldots + b_{l-1} + b_l n). \end{aligned}$$

Then we obtain

$$x = x(a+b) + \frac{y_1}{z_1} \text{ and } y = y(a+b) + \frac{y_2}{z_2}.$$

So, it can derived that

$$x = \frac{y_1}{(1-(a+b))z_1} \tag{6}$$

and

$$ny = \frac{n y_2}{(1-(a+b))z_2} \tag{7}$$

Subtracting (7) from (6) gives

$$x - ny = 0 = \frac{y_1 z_2 - n y_2 z_1}{(1-(a+b))z_1 z_2}$$

and so

$$y_1 z_2 = n y_2 z_1.$$

Therefore, we obtain

$$\begin{aligned} & (\frac{a_0}{n} + a_1 + \frac{a_2}{n} + \ldots + \frac{a_{k-1}}{n} + a_k)(b_0 n + b_1 + b_2 n + \ldots + b_{l-1} + b_l n) \\ = \; & n(a_0 n + a_1 + a_2 n + \ldots + a_{k-1} n + a_k)(\frac{b_0}{n} + b_1 + \frac{b_2}{n} + \ldots + b_{l-1} + \frac{b_l}{n}). \end{aligned}$$

Thus, Condition (4) holds.

Secondly, assume that Condition (4) holds. We shall show that Eq.(3) has eventual prime period two solutions. Let

$$x = \frac{y_1}{(1-(a+b))z_1}$$

and

$$y = \frac{y_2}{(1-(a+b))z_2}.$$

where x and y are distinct real numbers with $n \in \mathbb{R} \backslash \{0, \pm 1\}$. We assume in Eq.(3) that $k > l$. We choose the initial conditions as $x_{-k} = y, x_{-k+1} = x, ..., x_{-l} = y$, $x_{-l+1} = x, \ldots, x_{-1} = y$, $x_0 = x$. We shall show that $x_1 = y$, $x_2 = x$. From Eq.(3)

$$x_1 = ax + bx + \frac{a_0 x + a_1 y + \ldots + a_{k-1} x + a_k y}{b_0 x + b_1 y + \ldots + b_{l-1} x + b_l y}$$

$$\begin{aligned} x_1 &= (a+b)\frac{y_1}{(1-(a+b))z_1} + \frac{a_0 \frac{y_1}{(1-x_1)z_1} + a_1 \frac{y_2}{(1-x_2)z_2} + \ldots + a_{k-1} \frac{y_1}{(1-x_1)z_1} + a_k \frac{y_2}{(1-x_2)z_2}}{b_0 \frac{y_1}{(1-x_1)z_1} + b_1 \frac{y_2}{(1-x_2)z_2} + \ldots + b_{l-1} \frac{y_1}{(1-x_1)z_1} + b_l \frac{y_2}{(1-x_2)z_2}} \\ &= y \end{aligned}$$

$$\begin{aligned} x_2 &= (a+b)\frac{y_2}{(1-x_2)z_2} + \frac{a_0\frac{y_2}{(1-x_2)z_2} + a_1\frac{y_1}{(1-x_1)z_1} + \ldots + a_{k-1}\frac{y_2}{(1-x_2)z_2} + a_k\frac{y_1}{(1-x_1)z_1}}{b_0\frac{y_2}{(1-x_2)z_2} + b_1\frac{y_1}{(1-x_1)z_1} + \ldots + b_{l-1}\frac{y_2}{(1-x_2)z_2} + b_l\frac{y_1}{(1-x_1)z_1}} \\ &= x. \end{aligned}$$

By induction, we can obtain $x_{2n} = x$ and $x_{2n+1} = y$ for all $n \geq -k$. Therefore, Eq.(3) has a prime period two solution of the following form

$$y, x, y, x, \ldots,$$

where $x \neq y$. This completes the proof.

(ii) The proof of this case is proven the same way of the proof (i).

(iii) First, we assume that Eq.(3) has eventual prime period two solutions in the following form

$$..., x, y, x, y, ...$$

We shall show that Condition (5) holds. By using Elsayed's new method, from (3) we get

$$x = ax + by + \frac{a_0y + a_1x + a_2y + \ldots + a_{k-1}y + a_kx}{b_0y + b_1x + b_2y + \ldots + b_{l-1}x + b_ly}$$

and

$$y = ay + bx + \frac{a_0x + a_1y + a_2x + \ldots + a_{k-1}x + a_ky}{b_0x + b_1y + b_2x + \ldots + b_{l-1}y + b_lx}.$$

We assume that $n = \frac{x}{y}$. So, we rewrite the equilities above

$$x = ax + \frac{bx}{n} + \frac{x(\frac{a_0}{n} + a_1 + \frac{a_2}{n} + \ldots + \frac{a_{k-1}}{n} + a_k)}{x(\frac{b_0}{n} + b_1 + \frac{b_2}{n} + \ldots + b_{l-1} + \frac{b_l}{n})}$$

and

$$y = ay + bny + \frac{y(a_0n + a_1 + a_2n + \ldots + a_{k-1}n + a_k)}{y(b_0n + b_1 + b_2n + \ldots + b_{l-1} + b_ln)}.$$

From the second equility, we have

$$ny = any + bn^2y + \frac{n(a_0n + a_1 + a_2n + \ldots + a_{k-1}n + a_k)}{(b_0n + b_1 + b_2n + \ldots + b_{l-1} + b_ln)}.$$

We set

$$\begin{aligned} x_1 &= a + \frac{b}{n}, \\ y_1 &= (\frac{a_0}{n} + a_1 + \frac{a_2}{n} + \ldots + \frac{a_{k-1}}{n} + a_k), \\ z_1 &= (\frac{b_0}{n} + b_1 + \frac{b_2}{n} + \ldots + b_{l-1} + \frac{b_l}{n}), \\ x_2 &= a + bn, \\ y_2 &= (a_0n + a_1 + a_2n + \ldots + a_{k-1}n + a_k), \\ z_2 &= (b_0n + b_1 + b_2n + \ldots + b_{l-1} + b_ln). \end{aligned}$$

Then we obtain

$$x = xx_1 + \frac{y_1}{z_1} \text{ and } y = yx_2 + \frac{y_2}{z_2}.$$

So, it can derived that

$$x = \frac{y_1}{(1-x_1)z_1} \tag{8}$$

and

$$ny = \frac{ny_2}{(1-x_2)z_2} \tag{9}$$

Subtracting (9) from (8) gives

$$x-ny=0=\frac{y_1(1-x_2)z_2-ny_2(1-x_1)z_1}{(1-x_1)(1-x_2)z_1z_2}$$

and so

$$y_1(1-x_2)z_2=ny_2(1-x_1)z_1.$$

Therefore, we obtain

$$\begin{aligned}&(\frac{a_0}{n}+a_1+\frac{a_2}{n}+\ldots+\frac{a_{k-1}}{n}+a_k)(1-(a+bn))(b_0n+b_1+b_2n+\ldots+b_{l-1}+b_ln)\\ =\ & n(a_0n+a_1+a_2n+\ldots+a_{k-1}n+a_k)(1-(a+\frac{b}{n}))(\frac{b_0}{n}+b_1+\frac{b_2}{n}+\ldots+b_{l-1}+\frac{b_l}{n})\end{aligned}$$

Thus, Condition (5) holds.

Secondly, assume that Condition (5) holds. We shall show that Eq.(3) has eventual prime period two solutions. Let

$$x=\frac{y_1}{(1-x_1)z_1}$$

and

$$y=\frac{y_2}{(1-x_2)z_2}.$$

where x and y are distinct real numbers with $n\in\mathbb{R}\backslash\{0,\pm1\}$. We assume in Eq.(3) that $k>l$. We choose the initial conditions as $x_{-k}=y, x_{-k+1}=x,...,x_{-l}=x$, $x_{-l+1}=y,\ldots,x_{-1}=y$, $x_0=x$. We shall show that $x_1=y$, $x_2=x$. From Eq.(3)

$$x_1=ay+bx+\frac{a_0x+a_1y+\ldots+a_{k-1}x+a_ky}{b_0x+b_1y+\ldots+b_{l-1}y+b_lx}$$

$$\begin{aligned}x_1 &= a\frac{y_2}{(1-x_2)z_2}+b\frac{y_1}{(1-x_1)z_1}+\frac{a_0\frac{y_1}{(1-x_1)z_1}+a_1\frac{y_2}{(1-x_2)z_2}+\ldots+a_{k-1}\frac{y_1}{(1-x_1)z_1}+a_k\frac{y_2}{(1-x_2)z_2}}{b_0\frac{y_1}{(1-x_1)z_1}+b_1\frac{y_2}{(1-x_2)z_2}+\ldots+b_{l-1}\frac{y_2}{(1-x_2)z_2}+b_l\frac{y_1}{(1-x_1)z_1}}\\ &= y\end{aligned}$$

$$\begin{aligned}x_2 &= a\frac{y_1}{(1-x_1)z_1}+b\frac{y_2}{(1-x_2)z_2}+\frac{a_0\frac{y_2}{(1-x_2)z_2}+a_1\frac{y_1}{(1-x_1)z_1}+\ldots+a_{k-1}\frac{y_2}{(1-x_2)z_2}+a_k\frac{y_1}{(1-x_1)z_1}}{b_0\frac{y_2}{(1-x_2)z_2}+b_1\frac{y_1}{(1-x_1)z_1}+\ldots+b_{l-1}\frac{y_1}{(1-x_1)z_1}+b_l\frac{y_2}{(1-x_2)z_2}}\\ &= x.\end{aligned}$$

By induction, we can obtain $x_{2n}=x$ and $x_{2n+1}=y$ for all $n\geq -k$. Therefore, Eq.(3) has a prime period two solution of the following form

$$y,x,y,x,\ldots,$$

where $x\neq y$. This completes the proof.

(iv) The proof of this case is proven the same way of the proof (iii). □

3. Numerical Simulations

In order to verify our theoretical results we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions of Eq.(3). All plots in this section are drawn with Mathematica.

Example (1) Consider the following non-linear difference equations

$$x_{n+1}=ax_{n-1}+bx_{n-3}+\frac{a_0x_n+a_1x_{n-1}}{b_0x_n+b_1x_{n-1}+b_2x_{n-2}+b_3x_{n-3}},\ n=0,1,.... \quad (10)$$

with $a = 0.5, b = 0.4, a_0 = \frac{4}{29}, a_1 = 2, b_0 = 1, b_1 = 2, b_2 = 3, b_3 = 1$ (if we take $n = 2$, then $y_1 z_2 = n y_2 z_1$) and the initial values $x_{-3} = 0.1$, $x_{-2} = 0.8$, $x_{-1} = 0.6$, $x_0 = 0.8$. Then Eq.(10) possesses eventual prime period two solutions (see Figure 1).

Example (2) Consider the following non-linear difference equations

$$x_{n+1} = a x_{n-3} + b x_{n-2} + \frac{a_0 x_n + a_1 x_{n-1} + a_2 x_{n-2} + a_3 x_{n-3}}{b_0 x_n + b_1 x_{n-1} + b_2 x_{n-2}}, \ n = 0, 1, ... \quad (11)$$

with $a = 0.5, b = 0.1, a_0 = 0.5, a_1 = 12.5, a_2 = 0.5, a_3 = 0.5, b_0 = 8, b_1 = 2, b_2 = 6$ and the initial values $x_{-3} = 0.1$, $x_{-2} = 0.8$, $x_{-1} = 0.6$, $x_0 = 0.8$. Eq.(11) has possesses eventual prime period two solutions (see Figure 2).

FIGURE 1. The plot of Eq.(10)

FIGURE 2. The plot of Eq.(11)

4. CONCLUSIONS

It is well known that using difference equations in problems involving time-dependent fluid flows, neutron diffusion and convection, radiation flow, thermonuclear reactions and the solution of several partial differential equations at the same time provides great convenience. Differently the utilization of difference equations as approximations to ODEs and PDEs, they also avail a powerful method for the analysis of electrical, mechanical, thermal, and other systems in which there is a recurrence of identic parts. By usage the difference equations, the investigation of the behavior of electric-wave filters, multistage booster, magnetic amplifiers, insulator strings, continuous beams of equal span, crankshafts of multicylinder engines, acoustical filters, etc., is hugely simplified. The standard methods for solving such systems are in general very long when the number of elements related is grand.

In the paper, we completed the picture as regards the periodicity of positive solutions of Eq.(3). The main aim of dynamical systems theory is to approach the global behavior of solutions. So, we here give the asymptotic behavior of solutions for a class of non-linear difference equations. The results obtained here improve and generalize [15, 16, 7]. Also, we present some results about the general behavior of solutions of Eq.(3) and some numerical effective examples are provided to support our theoretical results.

5. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this manuscript.

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