Electronic Journal of Mathematical Analysis and Applications
Vol. 8(1) Jan. 2020, pp. 117-129.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR SOME THIRD ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

HOCINE GABSI, ABDELOUAHEB ARDJOUNI AND AHCENE DJOUDI

ABSTRACT. We establish the existence of positive periodic solutions for a third order differential equations with periodic delay. For that purpose, we use the fixed point technique. By choosing available operators and applying Schauder's fixed-point theorem we obtain sufficient conditions for the existence of positive and periodic solutions. We end by giving an example to illustrate our claim.

1. INTRODUCTION

In this article, we investigate the existence of positive periodic solutions for the third order functional differential equation with variable delay

$$\dddot{x} = a(t)\ddot{x} + b(t)\dot{x} + \lambda c(t) g(x(t-\tau(t))), \tag{1}$$

where $\dddot{x} = \dfrac{d^3x}{dt^3}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$, $\dot{x} = \dfrac{dx}{dt}$ and $g \in C(\mathbb{R},\mathbb{R})$, $c \in C(\mathbb{R},(0,\infty))$, $\tau \in C(\mathbb{R},\mathbb{R}^+)$ are continuous positive ω-periodic functions in t with ω is a positive constant. To reach our desired end we have to transform (1) into an integral equation and then use Schauder's fixed point theorem to show the existence of positive periodic solution. The obtained equation writes as a compact functional differential equation mapping with periodic delays. This kind of equations appear in a number of ecological models. In particular, our equation can be interpreted as an extension equation of the standard Malthus population model $\dddot{x} = a(t)\ddot{x} + b(t)\dot{x}$ subject to a perturbation with periodical delay. One important question is whether these equations can support positive periodic solutions. Particular question has been studied extensively by a number of authors; see for example [1]-[10], [12]-[16] and the references therein. In this paper, we will obtain existence criteria for ω-periodic solutions of (1) by means of the well known fixed point theorem due to Schauder's.

Our subject is to establish some sufficient condition ensuring that (1) has at least one positive ω-periodic solution. To describe the main result we need some preparations and use the following notations. For $\omega > 0$, let C_ω be the set of all

---

2010 *Mathematics Subject Classification.* 34K13, 35B09, 47H10.
Key words and phrases. Fixed points, delay differential equations, positive periodic solutions.
Submitted Oct. 18, 2018. Revised May 6, 2019.

continuous scalar functions x, periodic in t of period ω. Then $(C_\omega, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,\omega]} |x(t)|. \tag{2}$$

Define

$$C_\omega^+ = \{x \in C_\omega \mid x > 0\} \text{ and } C_\omega^- = \{x \in C_\omega \mid x < 0\}. \tag{3}$$

Denote

$$M = \max\{b(t) \mid t \in [0,\omega]\},\ m = \min\{b(t) \mid t \in [0,\omega]\},\ \beta = \sqrt{M}. \tag{4}$$

Throughout this paper, we will denote by

$$\mu = e^{-\int_0^\omega a(v)dv}, \tag{5}$$

and let

$$\zeta = \frac{\exp\left(-\frac{\beta\omega}{2}\right)}{\beta(1-\exp(-\beta\omega))},\ \eta = \frac{1+\exp(-\beta\omega)}{2\beta(1-\exp(-\beta\omega))}.$$

In order to simplify notations, we define the functions F and ϕ by

$$\phi(t) = \left(\dot{b} - ab\right)(t),$$
$$F(t,x) = \lambda c(t) g(x(t-\tau(t))) - \phi(t) x(t).$$

Throughout this section we assume that $F(t,x) > 0$ for all $t \in [0,\omega]$, $x \in C_\omega$. λ is a positive parameter and a, b, c, τ and g are ω-periodic in t where ω is a positive constant. For convenience, the conditions needed for our criteria are listed as follows

H1) $a, b, \dot{b}, \phi, c \in C(\mathbb{R}, (0,\infty))$.

L1) $\lim\limits_{x\to 0} \frac{g(x)}{x} = \infty$.

L2) $\lim\limits_{x\to \infty} \frac{g(x)}{x} = \infty$.

L3) $\lim\limits_{x\to 0} \frac{g(x)}{x} = 0$.

L4) $\lim\limits_{x\to \infty} \frac{g(x)}{x} = 0$.

2. Preliminaries

Our investigation needs the following helpful lemmas.

Lemma 2.1 ([7]). *The equation*

$$\frac{d^2}{dt^2} y(t) - My(t) = h(t),\ h \in C_\omega^-,$$

has a unique ω-periodic solution

$$y(t) = \int_t^{t+\omega} K(t,s)(-h(s))\, ds,$$

where

$$K(t,s) = \frac{\exp(-\beta(s-t)) + \exp(\beta(s-t-\omega))}{2\beta(1-\exp(-\beta\omega))},\ s \in [t, t+\omega].$$

Lemma 2.2 ([7]). *$\zeta \le K(t,s) \le \eta$ and $\int_t^{t+\omega} K(t,s)\, ds = \frac{1}{M}$ for all $t \in [0,\omega]$ and $s \in [t, t+\omega]$.*

Lemma 2.3 ([7]). *The equation*

$$\frac{d^2}{dt^2}y(t) - a(t)y(t) = h(t), h \in C_\omega^-$$

has a unique ω-periodic solution

$$y(t) = (Dh)(t) = (I - TB)^{-1} Th(t),$$

where

$$(Th)(t) = \int_t^{t+\omega} K(t,s)(-h(s))\, ds \text{ and } (By)(t) = [a(t) - M]\, y(t).$$

Next, we define operators $A, D, P : C_\omega \longrightarrow C_\omega$ by

$$(Dh)(t) = ((I - TB)^{-1} Th)(t),$$

$$(A\varphi)(t) := -\int_t^{t+\omega} G(t,s)\, F(s, \varphi(s))\, ds,$$

and

$$(P\varphi)(t) = (DA\varphi)(t) = ((I - TB)^{-1} TA\varphi)(t), \tag{6}$$

where the function G is given by

$$G(t,s) = \frac{e^{-\int_t^s a(v)dv}}{1 - e^{-\int_0^\omega a(v)dv}},\ t \le s \le t+\omega,\ t \in \mathbb{R}. \tag{7}$$

Remark 2.4. From the fact that $t \le s \le u \le s+\omega \le t+2\omega$ we have

$$\int_t^{t+\omega}\int_s^{s+\omega} \left(e^{-\beta(s-t)} + e^{-\beta(t+\omega-s)}\right) e^{-\int_s^u a dv} y(u)\, duds$$
$$= \int_t^{t+\omega}\int_s^{t+\omega} \left(e^{-\beta(s-t)} + e^{-\beta(t+\omega-s)}\right) e^{-\int_s^u a dv} y(u)\, duds$$
$$+ \int_t^{t+\omega}\int_{t+\omega}^{s+\omega} \left(e^{-\beta(s-t)} + e^{-\beta(t+\omega-s)}\right) e^{-\int_s^u a dv} y(u)\, duds.$$

By change variables in second summand $z = u - \omega$ and use the ω–periodicity of y we have

$$\int_t^{t+\omega}\int_s^{t+\omega} \left(e^{-\beta(s-t)} + e^{-\beta(t+\omega-s)}\right) e^{-\int_s^u a dv} y(u)\, duds$$
$$= \int_t^{t+\omega}\int_t^{s} \left(e^{-\beta(s-t)} + e^{-\beta(t+\omega-s)}\right) e^{-\int_s^{z+\omega} a dv} y(z)\, dzds.$$

By interchanging the integrating order, one gets

$$\int_t^{t+\omega}\int_s^{s+\omega} \left(e^{-\beta(s-t)} + e^{-\beta(t+\omega-s)}\right) e^{-\int_s^u a dv} y(u)\, duds$$
$$= \int_t^{t+\omega}\left(\int_t^{u} \left(e^{-\beta(s-t)} + e^{-\beta(t+\omega-s)}\right) e^{-\int_s^u a dv} ds\right) y(u)\, du$$
$$+ \int_t^{t+\omega}\left(\int_z^{t+\omega} \left(e^{-\beta(s-t)} + e^{-\beta(t+\omega-s)}\right) e^{-\int_s^{z+\omega} a dv} ds\right) y(z)\, dz.$$

Then

$$(TA\varphi)(t) = \int_t^{t+\omega} H(t,s)\, \varphi(s)\, ds,$$

for $s \in [t, t+\omega]$ with

$$\begin{aligned} H(t,s) &= \frac{\int_t^s \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) e^{-\int_u^s a dv} du}{2\beta\left(1-e^{-\beta\omega}\right)(1-\mu)} \\ &+ \frac{\int_s^{t+\omega} \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) e^{-\int_u^{s+\omega} a dv} du}{2\beta\left(1-e^{-\beta\omega}\right)(1-\mu)}. \end{aligned}$$

On the other hand, it is easy to see that

$$\frac{\mu}{\beta^2(1-\mu)} \le H(t,s) \le \frac{1}{\beta^2(1-\mu)}. \tag{8}$$

Because

$$\begin{aligned} &\int_t^s \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) e^{-\int_u^s a dv} du \\ &+ \int_s^{t+\omega} \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) e^{-\int_u^{s+\omega} a dv} du \\ &\ge \mu \int_t^s \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) du + \int_s^{t+\omega} \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) du \\ &= \mu \int_t^{t+\omega} \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) du = \frac{2\mu}{\beta}\left(1-e^{-\beta\omega}\right), \end{aligned}$$

and

$$\begin{aligned} &\int_t^s \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) e^{-\int_u^s a dv} du \\ &+ \int_s^{t+\omega} \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) e^{-\int_u^{s+\omega} a dv} du \\ &\le \int_t^s \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) du + \int_s^{t+\omega} \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) du \\ &\le \int_t^{t+\omega} \left(e^{-\beta(u-t)} + e^{-\beta(t+\omega-u)}\right) du = \frac{2}{\beta}\left(1-e^{-\beta\omega}\right). \end{aligned}$$

Then,

$$\frac{\omega\mu}{M(1-\mu)} = \frac{\omega\mu}{\beta^2(1-\mu)} \le \int_t^{t+\omega} H(t,s)\, ds \le \frac{\omega}{\beta^2(1-\mu)} = \frac{\omega}{M(1-\mu)}. \tag{9}$$

Then we get the following.

Lemma 2.5. *The mappings D and P are completely continuous. Further, D satisfies*

$$0 < (Th)(t) \le (Dh)(t) \le \frac{M}{m}\|Th\|,\ h \in C_\omega^-.$$

Proof. The proof is very similar to the proof of Lemma 2 in [14]. □

Theorem 2.6 (**Schauder** [11])**.** *Let S be a closed convex bounded subset of a Banach space X. Assume that $A : S \to S$ is compact operator. Then, A has at least one fixed point in S.*

3. Main Results

To apply Theorem 2.6, we need to define a Banach space X, a closed convex subset Γ of X and construct a fixed point mapping that is a completely continuous. So, we let $(X, \|\cdot\|) = (C_\omega, \|\cdot\|)$ and $\Gamma = \{\varphi \in B : l \leq \varphi \leq L\}$, where l is non-negative constant and L is positive constant. In this section we obtain the existence of a positive periodic solution of (1) by considering the two cases primary $\lambda = 1$ and secondary $\lambda > 0$.

Lemma 3.1. *The function $x \in C_\omega$ is a solution of equation (1) if and only if*

$$x(t) = (Px)(t), \tag{10}$$

where the function P is given by (6).

Proof. Let $x \in C_\omega$ be a solution of (1). Taking

$$u(t) = \frac{d^2}{dt^2}x(t) - b(t)x(t),$$

and

$$F(t,x) = \lambda c(t) g(x(t-\tau(t))) - \phi(t)x(t),$$

then (1) can be rewritten as

$$\frac{d}{dt}u(t) - a(t)u(t) = F(t,x). \tag{11}$$

We proceed formally from (11) to obtain

$$\frac{d}{dt}\left(u(t)e^{\int_t^\infty a(v)dv}\right) = e^{\int_t^\infty a(v)dv}F(t,x(t)).$$

After integration from t to $t+\omega$, we obtain

$$\frac{d^2}{dt^2}x(t) - b(t)x(t) = -\int_t^{t+\omega} G(t,s)F(s,x(s))ds. \tag{12}$$

Clearly, the right hand side of (12) is negative and $\omega-$periodic. Then from Lemma 2.3, we have

$$x(t) = \int_t^{t+\omega} K(t,s)[M - b(s)]x(s)ds + \int_t^{t+\omega} K(t,s)\int_s^{s+\omega} G(s,u)F(u,x(u))duds.$$

This yields

$$x(t) = (TBx)(t) + (TAx)(t).$$

Therefore, since $\|TB\| \leq 1 - \frac{m}{M} < 1$, then, the solution of (1) can be written in the form

$$x(t) = \left((I - TB)^{-1}TAx\right)(t).$$

It is clear that the existence of periodic solutions for (1) is equivalent to the existence of solutions for the operator equation $x = Px$ in C_ω. $\square$

First we consider a special case when $\lambda = 1$. So we have the following theorem.

Theorem 3.2. *Let (H1) holds. In addition, suppose that F satisfies*

$$\frac{\mu\left(1-\mu\right)}{\omega}\leq F\left(t,x\right)\leq\frac{M\left(1-\mu\right)}{\omega}\ \text{for } x\in\left[\frac{\mu^2}{M},\frac{M}{m}\right]\ \text{and } t\in\left[0,\omega\right].$$

Then, (1) has at least one positive ω-periodic solution x with $0<\frac{\mu^2}{M}\leq x\left(t\right)\leq\frac{M}{m}$ for $t\in[0,\omega]$.

Proof. Let $\Gamma=\left\{x\in X\mid x\in\left[\frac{\mu^2}{M},\frac{M}{m}\right]\right\}$. It is obvious that Γ is a bounded closed convex set in X. Moreover, for any $x\in\Gamma$, it is easy to verify that P is continuous and $\left(Px\right)\left(t+\omega\right)=\left(Px\right)\left(t\right)$, that is, $P\left(\Gamma\right)\subset X$.

Next, we claim that $P\varphi\in\Gamma$ for all $\varphi\in\Gamma$. That is, P maps Γ into itself. To see this, note that since $0<\mu\left(1-\mu\right)/\omega\leq F\left(t,x\right)\leq M\left(1-\mu\right)/\omega$, then for any $\varphi\in\Gamma$, by Lemmas 2.2 and 2.5, we have

$$\begin{aligned}\left(P\varphi\right)\left(t\right)&=\left(I-TB\right)^{-1}\int_t^{t+\omega}K\left(t,s\right)\int_s^{s+\omega}G\left(s,u\right)F\left(u,\varphi\left(u\right)\right)duds\\&=\left(I-TB\right)^{-1}\int_t^{t+\omega}H\left(t,s\right)F\left(s,\varphi\left(s\right)\right)ds\\&\leq\left\|\left(I-TB\right)^{-1}\right\|\int_t^{t+\omega}\left|H\left(t,s\right)F\left(s,\varphi\left(s\right)\right)\right|ds\\&\leq\frac{M}{m}\int_t^{t+\omega}\left|H\left(t,s\right)F\left(s,\varphi\left(s\right)\right)\right|ds\\&\leq\frac{M}{m}\frac{M\left(1-\mu\right)}{\omega}\frac{\omega}{M\left(1-\mu\right)}\\&=\frac{M}{m}.\end{aligned}\tag{13}$$

On the other hand, by Lemmas 2.2 and 2.5,

$$\begin{aligned}\left(P\varphi\right)\left(t\right)&=\left(I-TB\right)^{-1}\int_t^{t+\omega}H\left(t,s\right)F\left(s,\varphi\left(s\right)\right)ds\\&\geq\int_t^{t+\omega}H\left(t,s\right)F\left(s,\varphi\left(s\right)\right)ds\\&\geq\frac{\mu\left(1-\mu\right)}{\omega}\int_t^{t+\omega}H\left(t,s\right)ds\\&\geq\frac{\mu\left(1-\mu\right)}{\omega}\frac{\omega\mu}{M\left(1-\mu\right)}\\&=\frac{\mu^2}{M}>0.\end{aligned}\tag{14}$$

Combining (13) and (14), we get $P\varphi\in\Gamma$ for all $\varphi\in\Gamma$. Moreover, from Lemma 2.5, P is completely continuous in X. Hence by Theorem 2.6, P has a fixed point $x\in\Gamma$, that is to say, (1) has a positive ω-periodic solution $x\left(t\right)$ with $0<\frac{\mu^2}{M}\leq x\left(t\right)\leq\frac{M}{m}$. □

Case $\lambda>0$.

In this case, we let

$$L\left(r\right)=\max_{0\leq x\leq r}g\left(x\right),\tag{15}$$

and

$$l(r) = \min_{0 \leq x \leq r} g(x). \tag{16}$$

Here, we assume further that for all $t \in [0, \omega]$, $x \in C_\omega$ and

$$F(t, x) = \lambda c(t) g(x(t - \tau(t))) - \phi(t) x(t) \geq 0. \tag{17}$$

Theorem 3.3. *Suppose that (H1) and the conditions (17), (L3) hold. Then (1) has a positive ω-periodic solution for*

$$\lambda > \frac{M(1-\mu) r_1}{l(r_1) \mu \int_0^\omega c(u) du}. \tag{18}$$

Proof. In view of (L3) there is an $0 < r_1$ so that $g(x) \leq \varepsilon x$ for $x \in [0, r_1]$ where $\varepsilon > 0$ satisfies

$$\frac{\varepsilon \lambda}{m(1-\mu)} \int_0^\omega c(u) du \leq 1. \tag{19}$$

Let $\Gamma := \{x \in X \mid x \in [0, r_1]\}$, it is easy to see that Γ is a bounded closed convex set of X. Also, we will show that $P\varphi \in \Gamma$ for all $\varphi \in \Gamma$. In fact, for any $\varphi \in \Gamma$, by Lemmas 2.2, 2.5 and from (19) we can get

$$\begin{aligned}
&(P\varphi)(t) \\
&= (I - TB)^{-1} \int_t^{t+\omega} K(t, s) \int_s^{s+\omega} G(s, u) F(u, \varphi(u)) duds \\
&\leq \frac{M}{m} \int_t^{t+\omega} K(t, s) \int_s^{s+\omega} G(s, u) [\lambda \varepsilon c(u) \varphi(u - \tau(u)) - \phi(u) \varphi(u)] duds \\
&\leq \varepsilon \frac{M}{m} \lambda \int_t^{t+\omega} K(t, s) \int_s^{s+\omega} G(s, u) c(u) \varphi(u - \tau(u)) duds \\
&\leq \varepsilon \frac{M}{m} \lambda r_1 \int_t^{t+\omega} K(t, s) \int_s^{s+\omega} G(s, u) c(u) duds \\
&\leq \varepsilon \frac{M}{m} \lambda r_1 \frac{1}{1-\mu} \int_t^{t+\omega} K(t, s) ds \int_0^\omega c(u) du \\
&\leq \varepsilon \frac{M}{m} \lambda r_1 \frac{1}{1-\mu} \frac{1}{M} \int_0^\omega c(u) du \leq \left(\frac{\varepsilon \lambda}{m(1-\mu)} \int_0^\omega c(u) du \right) r_1 \\
&\leq r_1.
\end{aligned}$$

Similarly, by Lemmas 2.2, 2.5 and $g(\varphi) \geq l(r_1)$ for $\varphi \in \Gamma$, for any φ we get

$$\begin{aligned}
&(P\varphi)(t) \\
&\geq \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\left[\lambda c(u) g(\varphi(u-\tau(u))) - \phi(u)\varphi(u)\right] duds \\
&\geq \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\lambda c(u) g(\varphi(u-\tau(u))) duds \\
&- \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\phi(u)\varphi(u) duds \\
&\geq \lambda l(r_1)\frac{\mu}{\mu-1}\int_t^{t+\omega} K(t,s)\int_s^{s+\omega} c(u) duds \\
&- r_1 \int_t^{t+\omega} K(t,s)\int_s^{s+\omega} G(s,u)\phi(u) duds.
\end{aligned}$$

in fact that $G(t,s) = e^{-\int_t^s a(v)dv}/(1-\mu)$ and $\mu = e^{-\int_0^\omega adv}$ so

$$\int_s^{s+\omega} G(s,u)\phi(u) du = b(s).$$

Thus, from Lemma 2.2

$$\begin{aligned}
(P\varphi)(t) &\geq \lambda l(r_1)\frac{\mu}{1-\mu}\int_t^{t+\omega} K(t,s) ds \int_0^\omega c(u) du - r_1 M \int_t^{t+\omega} K(t,s) ds. \\
&\geq \frac{\lambda l(r_1)\mu\int_0^\omega c(u) du}{M(1-\mu)} - r_1 > 0.
\end{aligned}$$

Clearly if $\lambda > M(1-\mu) r_1/\mu l(r_1)\int_0^\omega c(u) du$ so $P\varphi > 0$. Then $P\varphi \in \Gamma$ for all $\varphi \in \Gamma$. Hence, $P(\Gamma) \subset \Gamma$. Furthermore, from Lemma 2.5, the operator P is completely continuous. Clearly, all the hypotheses of the Schauder's theorem are satisfied. Thus there exists a fixed point $x \in \Gamma$ such that $Px = x$. By Lemma 3.1 this fixed point is a solution of (1) and the proof is complete. □

Theorem 3.4. *Suppose that (H1) and the conditions (17), (L4) hold. Then, (1) has a positive ω-periodic solution for*

$$\lambda > \frac{r_2 M(1-\mu)}{l(r_2)\mu\int_0^\omega c(u) du}. \tag{20}$$

Proof. In view of (L4) we can choose $1 < R$ so that $g(x) \leq \varepsilon x$ for $x \geq R$. Let $r_2 = \frac{M}{m}R$ where $\varepsilon > 0$ satisfies

$$\frac{\varepsilon\lambda}{m(1-\mu)}\int_0^\omega c(s) ds - \eta\int_0^\omega b(s) ds \leq 1. \tag{21}$$

Let $\Gamma := \{x \in X \mid R \leq x \leq r_2\}$, it is easy to see that Γ is a bounded closed convex set of X. Next, we will show that $P\varphi \in \Gamma$ for all $\varphi \in \Gamma$. That is, $R \leq P\varphi \leq r_2$. In fact, for any $\varphi\in\Gamma$, by Lemmas 2.2, 2.5 and from (21) we can get

$$\begin{aligned}
&(P\varphi)(t)\\
&=(I-TB)^{-1}\int_t^{t+\omega}K(t,s)\int_s^{s+\omega}G(s,u)\,F(u,\varphi(u))\,duds\\
&\le\frac{M}{m}\int_t^{t+\omega}K(t,s)\,ds\int_s^{s+\omega}G(s,u)\left[\lambda\varepsilon c(u)\,\varphi(u-\tau(u))-\phi(u)\,\varphi(u)\right]du\\
&\le\frac{M}{m}r_2\lambda\varepsilon\int_t^{t+\omega}K(t,s)\,ds\int_s^{s+\omega}G(s,u)\,c(u)\,du\\
&-\frac{M}{m}R\int_t^{t+\omega}K(t,s)\,ds\int_s^{s+\omega}G(s,u)\,\phi(u)\,du\\
&\le\frac{M}{m}r_2\lambda\varepsilon\int_t^{t+\omega}K(t,s)\,ds\int_s^{s+\omega}G(s,u)\,c(u)\,du-\frac{M}{m}R\int_t^{t+\omega}K(t,s)\,b(s)\,ds\\
&\le r_2\left(\frac{1}{m}\frac{\lambda\varepsilon}{1-\mu}\int_0^{\omega}c(s)\,ds\right)-\frac{M}{m}R\left(\eta\int_0^{\omega}b(s)\,ds\right)\\
&\le r_2\left[\frac{\varepsilon\lambda}{m(1-\mu)}\int_0^{\omega}c(s)\,ds-\eta\int_0^{\omega}b(s)\,ds\right]\le r_2.
\end{aligned}\tag{22}$$

On the other hand, for any $\varphi\in\Gamma$ by Lemmas 2.2, 2.5 and $g(\varphi)\ge l(r_2)$ on Γ, we get

$$\begin{aligned}
(P\varphi)(t)&\ge\int_t^{t+\omega}K(t,s)\int_s^{s+\omega}G(s,u)\left[\lambda c(u)\,g(\varphi(u-\tau(u)))-\phi(u)\,\varphi(u)\right]duds\\
&\ge\int_t^{t+\omega}K(t,s)\int_s^{s+\omega}G(s,u)\,\lambda c(u)\,g(\varphi(u-\tau(u)))\,duds\\
&-\int_t^{t+\omega}K(t,s)\int_s^{s+\omega}G(s,u)\,\phi(u)\,\varphi(u)\,duds\\
&\ge\lambda l(r_2)\frac{\mu}{1-\mu}\int_t^{t+\omega}K(t,s)\,ds\int_0^{\omega}c(u)\,du\\
&-r_2\int_t^{t+\omega}K(t,s)\int_s^{s+\omega}G(s,u)\,\phi(u)\,duds.\\
&\ge\lambda l(r_2)\frac{\mu}{1-\mu}\frac{1}{M}\int_0^{\omega}c(u)\,du-r_2\int_t^{t+\omega}K(t,s)\,b(s)\,ds.
\end{aligned}$$

In fact that

$$\int_t^{t+\omega}K(t,s)\,b(s)\,ds\le M\int_t^{t+\omega}K(t,s)=1,$$

$$\int_t^{t+\omega}K(t,s)\int_s^{s+\omega}G(s,u)\,c(u)\,duds\ge\frac{\mu}{M(1-\mu)}\int_0^{\omega}c(u)\,du,$$

by condition (20) we deduce

$$(P\varphi)(t)\ge\lambda\tfrac{l(r_2)\mu}{M(1-\mu)}\int_0^{\omega}c(u)\,du-r_2\ge(R+r_2)-r_2\ge R.\tag{23}$$

So from (22) and (23), we get $P:\Gamma\to\Gamma$, it is clear that from Lemmas 2.5, 3.1, and Theorem 2.6 there exists $x\in\Gamma$ such that $Px=x$. Hence $R\le Px\le\frac{M}{m}R$, that is to say (1) has at least a positive $\omega-$periodic solution x with $R\le x\le\frac{M}{m}R$. $\square$

Theorem 3.5. *Suppose that (H1) and the conditions (17), (L1) hold. Then (1) has a positive ω-periodic solution for*

$$0<\lambda\leq\frac{m\left(1-\mu\right)r_{1}}{L\left(r_{1}\right)\int_{0}^{\omega}c\left(s\right)ds}. \tag{24}$$

Proof. Let $Q_0>0$. So in view of (L1) there is an $0<r_1<1$ so that $g(x)\geq Q_0x$ for $x\in[0,r_1]$ for a choice of Q_0 that satisfies

$$\frac{\lambda\mu Q_{0}}{\left(1-\mu\right)M}\int_{0}^{\omega}c\left(u\right)du-\frac{M}{m}\geq1. \tag{25}$$

Let $\Gamma:=\{x\in X\mid\hat{r}_1\leq x\leq r_1\}$. It is easy to see that Γ is a bounded closed convex set of X, and $P(\Gamma)\subset X$. We will show that, indeed, $P\varphi\in\Gamma$ for all $\varphi\in\Gamma$. In fact, for any $\varphi\in\Gamma$, from Lemmas 2.2 and 2.5 we can get

$$\begin{aligned}
\left(P\varphi\right)\left(t\right)&=\left(I-TB\right)^{-1}\int_{t}^{t+\omega}K\left(t,s\right)\int_{s}^{s+\omega}G\left(s,u\right)\left(-F\left(u,\varphi\left(u\right)\right)\right)duds\\
&\geq\int_{t}^{t+\omega}K\left(t,s\right)\int_{s}^{s+\omega}G\left(s,u\right)\left[\lambda Q_{0}c\left(u\right)\varphi\left(u-\tau\left(u\right)\right)-\phi\left(u\right)\varphi\left(u\right)\right]duds\\
&\geq\int_{t}^{t+\omega}K\left(t,s\right)\int_{s}^{s+\omega}G\left(s,u\right)\lambda Q_{0}c\left(u\right)\varphi\left(u-\tau\left(u\right)\right)duds\\
&-\int_{t}^{t+\omega}K\left(t,s\right)\int_{s}^{s+\omega}G\left(s,u\right)\phi\left(u\right)\varphi\left(u\right)duds\\
&\geq\lambda Q_{0}\int_{t}^{t+\omega}K\left(t,s\right)\int_{s}^{s+\omega}G\left(s,u\right)c\left(u\right)\varphi\left(u-\tau\left(u\right)\right)duds\\
&-r_{1}\int_{t}^{t+\omega}K\left(t,s\right)\int_{s}^{s+\omega}G\left(s,u\right)\phi\left(u\right)duds.
\end{aligned}$$

Let $\hat{Q}_0>Q_0$ so in view of (L1) there is an $0<\hat{r}_1<r_1$ so that $g(x)\geq\hat{Q}_0x$ for $x\in[0,\hat{r}_1]$, so $g(x)\geq Q_0\hat{r}_1$ for $x\in[\hat{r}_1,r_1]$ thus

$$\left(P\varphi\right)\left(t\right)\geq\frac{Q_{0}\lambda\mu\hat{r}_{1}}{\left(1-\mu\right)}\frac{1}{M}\int_{0}^{\omega}c\left(u\right)du-r_{1}.$$

Choose $\hat{r}_1=\frac{m}{M}r_1$ and from (25) we deduce that

$$\left(P\varphi\right)\left(t\right)\geq\hat{r}_{1}. \tag{26}$$

Similarly, by Lemma 2.2, 2.5 and $g(\varphi)\leq L(r_1)$ for $\varphi\in[0,r_1]$, for any φ we get

$$\begin{aligned}
\left(P\varphi\right)\left(t\right)&\leq\frac{M}{m}\int_{t}^{t+\omega}K\left(t,s\right)\int_{s}^{s+\omega}G\left(s,u\right)\lambda c\left(u\right)g\left(\varphi\left(u-\tau\left(u\right)\right)\right)duds\\
&\leq\lambda L\left(r_{1}\right)\frac{M}{m}\int_{t}^{t+\omega}K\left(t,s\right)\int_{s}^{s+\omega}G\left(s,u\right)c\left(u\right)duds\\
&\leq\lambda L\left(r_{1}\right)\frac{M}{m}\frac{1}{1-\mu}\int_{t}^{t+\omega}K\left(t,s\right)ds\int_{0}^{\omega}c\left(s\right)ds\\
&\leq\frac{\lambda L\left(r_{1}\right)}{m\left(1-\mu\right)}\int_{0}^{\omega}c\left(s\right)ds\leq r_{1}.
\end{aligned} \tag{27}$$

Clearly from (26) and (27) we get $P\varphi\in\Gamma$ for all $\varphi\in\Gamma$ so $P(\Gamma)\subset\Gamma$. Also, from Lemma 2.5, the operator P is completely continuous. Thus by Theorem 2.6 there exists a fixed point $x \in \Gamma$ such that $Px = x$. By Lemma 3.1 this fixed point is a solution of (1). □

Theorem 3.6. *Suppose that (H1) and the conditions (17), (L2) hold. Then (1) has a positive ω-periodic solution for*

$$0 < \lambda \leq \frac{(\hat{r}_2 + r_2)\, m\,(1-\mu)}{L(\hat{r}_2)\int_0^{\omega} c(s)\,ds}. \tag{28}$$

Proof. Let $Q_1 > 0$. If (L2) holds so we can choose $r_2 > 1$ so that $g(x) \geq Q_1 x$ for $x \geq r_2$. Let $\hat{r}_2 = \frac{M}{m} r_2$ where Q_1 satisfies

$$\frac{Q_1 \lambda \mu}{M(1-\mu)} \int_0^{\omega} c(s)\,ds - \frac{M}{m} \geq 1. \tag{29}$$

Next, consider the subset $\Gamma := \{x \in X \mid r_2 \leq x \leq \hat{r}_2\}$. It is easy to see that Γ is a bounded closed convex of X. We will show that $P : \Gamma \to \Gamma$. Since for any $\varphi \in \Gamma$

$$\begin{aligned}
(P\varphi)(t) &= (I - TB)^{-1} \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\,(-F(u,\varphi(u)))\,duds \\
&\geq \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\,[\lambda Q_1 c(u)\,\varphi(u-\tau(u)) - \phi(u)\,\varphi(u)]\,duds \\
&\geq \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\,\lambda Q_1 c(u)\,\varphi(u-\tau(u))\,duds \\
&- \hat{r}_2 \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\,\phi(u)\,\varphi(u)\,duds \\
&\geq \frac{r_2 \lambda \mu Q_1}{M(1-\mu)} \int_0^{\omega} c(s)\,ds - \hat{r}_2 = \left(\frac{\lambda \mu Q_1}{M(1-\mu)} \int_0^{\omega} c(s)\,ds - \frac{M}{m}\right) r_2 \\
&\geq r_2.
\end{aligned} \tag{30}$$

Similarly, by Lemma 2.2, 2.5 and $g(\varphi) \leq L(\hat{r}_2)$ for $\varphi \in [r_2, \hat{r}_2]$, for any φ we get

$$\begin{aligned}
&(P\varphi)(t) \\
&\leq \frac{M}{m} \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\,[\lambda Q_1 c(u)\,\varphi(u-\tau(u)) - \phi(u)\,\varphi(u)]\,duds \\
&\leq \frac{M}{m} \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\,\lambda Q_1 c(u)\,\varphi(u-\tau(u))\,duds \\
&- \frac{M}{m} \int_t^{t+\omega} K(t,s) \int_s^{s+\omega} G(s,u)\,\phi(u)\,\varphi(u)\,duds \\
&\leq \frac{\lambda L(\hat{r}_2)}{m(1-\mu)} \int_0^{\omega} c(s)\,ds - r_2 \\
&\leq \hat{r}_2.
\end{aligned} \tag{31}$$

By (30) and (31) we get $P\varphi \in \Gamma$ for all $\varphi \in \Gamma$ so $P(\Gamma) \subset \Gamma$. Again by Lemma 2.5, the operator P is completely continuous. Thus by Theorem 2.6 we know P has fixed point $x \in \Gamma$ that is $Px = x$ and by Lemma 3.1 this fixed point is nothing but a solution of (1). □

In view of the hypothesis (L1)–(L2), (L3)–(L4) and if (H1) holds, then from Theorems 3.3 and 3.4, we can easily get the following theorems.

Theorem 3.7. *Suppose that the conditions (17), (L3) and (L4) hold, then (1) has two positive ω-periodic for*

$$\lambda \geq \max\left\{\frac{M(1-\mu)r_1}{l(r_1)\mu\int_0^\omega c(u)\,du}, \frac{r_2 M(1-\mu)}{l(r_2)\mu\int_0^\omega c(u)\,du}\right\}. \tag{32}$$

From Theorems 3.5 and 3.6, we can easily get the following Theorem.

Theorem 3.8. *Suppose that the conditions (17), (L1) and (L2) hold, then (1) has two positive ω-periodic for*

$$0 < \lambda \leq \min\left\{\frac{m(1-\mu)r_1}{L(r_1)\int_0^\omega c(s)\,ds}, \frac{(\hat{r}_2 + r_2)m(1-\mu)}{L(\hat{r}_2)\int_0^\omega c(s)\,ds}\right\}. \tag{33}$$

Example 3.9. Consider the following equation

$$\begin{aligned} \dddot{x}(t) &= \sin^2(t)\,\ddot{x}(t) + \frac{1}{2}\left(e^{\cos 2t}\right)\dot{x}(t) \\ &\quad + e^{\cos 2t}\left(0.3\frac{\arctan x}{x}e^{-\cos 2t} - \left(\sin 2t + \frac{1}{2}\sin^2 t\right)x(s)\right). \end{aligned} \tag{34}$$

So, $\sin^2(t)$ $\cos 2t$ are continuous positive $\pi-$periodic functions in t. A simple calculation yields

$$\begin{aligned} \phi(t) &= -\left(\sin 2t + \frac{1}{2}\sin^2 t\right)e^{\cos 2t}, \\ F(t,x) &= 0.3\frac{\arctan x}{x}, \end{aligned}$$

and

$$\mu = e^{-\frac{\pi}{2}},\ M = \frac{e}{2} \text{ and } m = \frac{1}{2e}.$$

Moreover, for $x \in \left[\frac{\mu^2}{M}, \frac{M}{m}\right] = \left[2e^{-\pi-1}, e^2\right]$ we have

$$\frac{\mu(1-\mu)}{\pi} \leq 0.3\frac{\arctan e^2}{e^2} \leq F(t,x) \leq 0.3\frac{\arctan 2e^{-\pi-1}}{2e^{-\pi-1}} \leq \frac{M(1-\mu)}{\pi},$$

because the function F is a strictly decreasing on $(0,+\infty)$. By Theorem 3.2, we see that (34) has at least one positive π-periodic solution.

Acknowledgment. The authors would like to thank the anonymous referee for his valuable comments.

References

[1] A. Ardjouni, A. Rezaiguia, A. Djoudi, Existence of positive periodic solutions for two types of second-order nonlinear neutral differential equations with infinite distributed delay, J. Appl. Math. Comput. 47 (2015), 291–314.

[2] S. Cheng, G. Zhang, Existence of positive periodic solutions for non-autonomous functional differential equations, Electronic Journal of Differential Equations 2001(59) (2001), 1–8.

[3] W. S. Cheung. J. Ren and W. Han, Positive periodic solutions of second order neutral functional differential equations, Nonlinear Analysis 71 (2009), 3948–3955.

[4] M. Fan and K. Wang, Optimal harvesting policy for single population with periodic coefficients, Math. Biosci. 152 (1998), 165–177.

[5] M. Fan and K. Wang, Uniform ultimate boundedness and periodic solutions of functional differential equations with infinite delay, J. Sys. Sci. Math. Sci., 17(3) (1999), 323–327 (in Chinese).

[6] M. Gregus, Third Order Linear Differential Equations, Reidel, Dordrecht, 1987.

[7] W. Han, J. Ren, Some results on second-order neutral functional differential equations with infinite distributed delay, Nonlinear Anal. 70 (2009), 1393–1406.
[8] D. Q. Jiang and J. J. Wei, Existence of positive periodic solutions of nonautonomous functional differential equations, Chinese Ann. Math. A20(6) (1999), 715–720 (in Chinese).
[9] Y. K. Li, Existence and global attractivity of positive periodic solutions for a class of delay differential equations, Chinese Science, A28(2) (1998), 108–118 (in Chinese).
[10] S. Padhi, S. Pati, Theory of Third-Order Differential Equations, Springer, 2014.
[11] D. R. Smart, Fixed Point Theorems, Cambridge Univ. Press, Cambridge, 1980.
[12] C. Tunç, On the existence of periodic solutions of functional differential equations of the third order, Appl. Comput. Math. 15(2) (2016), 189–199.
[13] C. Tunç, S. Erdur, On the existence of periodic solutions to certain nonlinear differential equations of third order, Proc. Pak. Acad. Sci. A 54(2) (2017), 207–218.
[14] Y. Wang, H. Lian, W. Ge, Periodic solutions for a second order nonlinear functional differential equation, Applied Mathematics Letters 20 (2007), 110–115.
[15] J. Wu, Z. C. Wang, Two periodic solutions of second-order neutral functional differential equations, Journal of Mathematical Analysis and Applications 329 (2007), 677–689.
[16] S. N. Zhang, Periodicity in functional differential equations, Ann. Diff. Eqns. 12(2) (1996), 252–257.

Hocine Gabsi
Department of Mathematics, University of El-Oued, El-Oued, Algeria.
E-mail address: hocinegabsi@gmail.com

Abdelouaheb Ardjouni
Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria
E-mail address: abd_ardjouni@yahoo.fr

Ahcene Djoudi
Applied Mathematics Lab, Faculty of Sciences, Department of Mathematics, University of Annaba, P.O. Box 12, Annaba 23000, Algeria
E-mail address: adjoudi@yahoo.com