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NEW SHARP BOUNDS FOR THE LOGARITHMIC FUNCTION

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Abstract. In this article we establish new bounds for the logarithmic function. These bounds have the feature to involve the arctan function and to be sharper than well-known sharp bounds.

1. Introduction

Inequalities for the logarithmic function are useful in all the areas of mathematics. The most famous logarithmic inequality is without doubt the scholar one:

$$\log(1+x) \le x, \quad x \ge 0.$$

It has a place of choice in terms of simplicity and enormous amount of applications. More complex but sharpest logarithmic inequalities are available in the literature. Those list below are often considered in various situations:

$$\log(1+x) \le \frac{x}{\sqrt{x+1}}, \quad x \ge 0, \tag{1.1}$$

$$\log(1+x) \le \frac{x(2+x)}{2(1+x)}, \quad x \ge 0, \tag{1.2}$$

$$\log(1+x) \le \frac{x(6+x)}{2(3+2x)}, \quad x \ge 0 \tag{1.3}$$

and

$$\log(1+x) \le \frac{(x+2)[(x+1)^3-1]}{3(x+1)[(x+1)^2+1]}, \quad x \ge 0. \tag{1.4}$$

Details can be found in [3] for (1.1), (1.2) and (1.3). See [1, page 1, only for $x > 1$] for (1.4). Other sharp upper bounds defined as ratio of two polynomial terms are presented in [3, Table 1]. Further developments can also be found in [2].

In this paper, we present new sharp bounds for $\log(1+x)$. We prove that our upper bound is sharper than all the upper bounds presented above. Moreover, it has the surprising feature to involve the arctan function, with a relatively tractable expression. A graphical study supports the theoretical findings. A lower bound is also proved for $x \in (-1, 0]$, with discussion.

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The rest of the paper is as follows. Section 2 is devoted to the main results of the paper, the proofs are postponed in Section 3.

2. Results

The result below presents a new upper bound for $\log(1+x)$ for $x \geq 0$.

Proposition 1. *For any $x \geq 0$, we have*

$$\log(1+x) \leq \frac{f(x)}{\sqrt{x+1}}, \tag{2.1}$$

where

$$f(x) = \pi + \frac{1}{2}(4+\pi)x - 2(x+2)\arctan\left(\sqrt{x+1}\right). \tag{2.2}$$

The proof of Proposition 1 is based on an analytical study of the function $g(x) = f(x) - \sqrt{x+1}\log(x+1)$, with the use of sharp lower bound of $\log(x+1)$ i. e. $\log(1+x) \geq 2x/(2+x)$ for $x \geq 0$.

Now we claim that the obtained upper bound $f(x)/\sqrt{x+1}$ is shaper to those in (1.1), (1.2), (1.3) and (1.4). For a first approach, we illustrate this claim graphically in Figure 1, where $\log(1+x)$ and all the presented upper bounds are depicted. At least for x large, we clearly see that the new upper bound is the closest to $\log(1+x)$.

We now prove analytically that $f(x)/\sqrt{x+1}$ is the best in Lemmas 1, 2, 3 and 4 below. To facilitate the comparison in the proofs, we only use the numerator $f(x)$ of the bound.

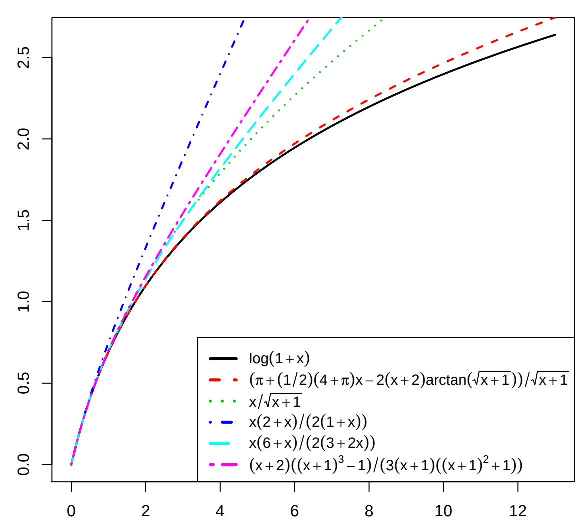

Figure 1. Graphs of the functions of the upper bounds (2.1), (1.1), (1.2) and (1.3) for $x \in (0, 13)$.

Lemma 1. *Let $f(x)$ be the function given by* (2.2). *Then, for any $x \geq 0$, we have*

$$f(x) \leq x.$$

It follows from Lemma 1 that $f(x)/\sqrt{x+1}$ is shaper than the one in (1.2).

Lemma 2. *Let $f(x)$ be the function given by* (2.2). *Then, for any $x \geq 0$, we have*

$$f(x) \leq \frac{x(2+x)}{2\sqrt{1+x}}.$$

An immediate consequence of Lemma 2 is that $f(x)/\sqrt{x+1}$ is sharper than the one in (1.1).

Lemma 3. *Let $f(x)$ be the function given by* (2.2). *Then, for any $x \geq 0$, we have*

$$f(x) \leq \frac{x(6+x)\sqrt{x+1}}{2(3+2x)}.$$

Thus, Lemma 3 shows that $f(x)/\sqrt{x+1}$ is better in comparison to (1.3).

Lemma 4. *Let $f(x)$ be the function given by* (2.2). *Then, for any $x \geq 0$, we have*

$$f(x) \leq \frac{(x+2)[(x+1)^3-1]}{3\sqrt{x+1}[(x+1)^2+1]}.$$

Lemma 3 shows that $f(x)/\sqrt{x+1}$ is shaper than the one in (1.3).

Let us now present results on lower bound for $\log(1+x)$. We first present a reverse version of the inequality (2.1) for $x \in (-1, 0]$.

Proposition 2. *For any $x \in (-1, 0]$, we have*

$$\log(1+x) \geq \frac{f(x)}{\sqrt{x+1}},$$

where $f(x)$ is defined by (2.2).

The proof of Proposition 2 is an adaptation of the proof of Proposition 1.

Again, we claim that the obtained lower bound is sharp. The following result shows that the obtained lower bound, i.e. $f(x)/\sqrt{x+1}$, is shaper than those in the following inequality (see, for instance, [3, Equation (4)]): for any $x \in (-1, 0]$,

$$\log(1+x) \geq \frac{x(2+x)}{2(1+x)}.$$

Lemma 5. *Let $f(x)$ be the function given by* (2.2). *Then, for any $x \in (-1, 0]$, we have*

$$f(x) \geq \frac{x(2+x)}{2\sqrt{1+x}}.$$

The proof of Lemma 5 is an adaptation of the proof of Lemma 2.

3. Proofs

Proof of Proposition 1. For $x \geq 0$, we consider the function $g(x)$ given by

$$\begin{aligned} g(x) &= f(x) - \sqrt{x+1}\, \log(x+1) \\ &= \pi + \frac{1}{2}(4+\pi)x - 2(x+2)\ \arctan\left(\sqrt{x+1}\right) - \sqrt{x+1}\, \log(x+1). \end{aligned}$$

Then, by differentiation, we have

$$g'(x) = -\frac{2}{\sqrt{x+1}} - \frac{\log(x+1)}{2\sqrt{x+1}} - 2\arctan\left(\sqrt{x+1}\right) + 2 + \frac{\pi}{2}$$

and

$$g''(x) = \frac{(x+2)\log(x+1) - 2x}{4(x+1)^{3/2}(x+2)}.$$

Using the (nontrivial sharp) inequality $\log(1+x) \geq 2x/(2+x)$ for $x \geq 0$ (see [3, Equation (3)]), we arrive at $g''(x) \geq 0$. So $g'(x)$ is increasing for $x \geq 0$ and $g'(x) \geq g'(0) = 0$. Therefore $g(x)$ is increasing for $x \geq 0$ and $g(x) \geq g(0) = 0$, ending the proof of Proposition 1. □

Proof of Lemma 1. For $x \geq 0$, let us denote by $h(x)$ the function given by

$$h(x) = f(x) - x = \pi + \frac{1}{2}(4+\pi)x - 2(x+2)\ \arctan\left(\sqrt{x+1}\right) - x.$$

Then, by differentiation, we have

$$h'(x) = -\frac{1}{\sqrt{x+1}} - 2\arctan\left(\sqrt{x+1}\right) + 1 + \frac{\pi}{2}$$

and, by another differentiation, we get

$$h''(x) = -\frac{x}{2(x+1)^{3/2}(x+2)}.$$

Thus $h''(x) \leq 0$ for $x \geq 0$. So $h'(x)$ is decreasing for $x \geq 0$ and $h'(x) \leq h'(0) = 0$. Therefore $h(x)$ is decreasing for $x \geq 0$ and $h(x) \leq h(0) = 0$. The proof of Lemma 1 is completed. □

Proof of Lemma 2. For $x \geq 0$, let us consider the function $k(x)$ given by

$$\begin{aligned} k(x) &= f(x) - \frac{x(2+x)}{2\sqrt{1+x}} \\ &= \pi + \frac{1}{2}(4+\pi)x - 2(x+2)\ \arctan\left(\sqrt{x+1}\right) - \frac{x(2+x)}{2\sqrt{1+x}}. \end{aligned}$$

Then, by differentiation, we have

$$k'(x) = -\frac{(x+2)(3x+4)}{4(x+1)^{3/2}} - 2\ \arctan\left(\sqrt{x+1}\right) + \frac{\pi}{2} + 2$$

and, by another differentiation, we get

$$k''(x) = -\frac{x(x+4)(3x+4)}{8(x+1)^{5/2}(x+2)}.$$

Thus $k''(x) \leq 0$ for $x \geq 0$. So $k'(x)$ is decreasing for $x \geq 0$ and $k'(x) \leq k'(0) = 0$. Therefore $k(x)$ is decreasing for $x \geq 0$ and $k(x) \leq k(0) = 0$. This ends the proof of

Lemma 2. □

Proof of Lemma 3. For $x \geq 0$, let us consider the function $\ell(x)$ given by

$$\begin{aligned}\ell(x) &= f(x) - \frac{x(6+x)\sqrt{x+1}}{2(3+2x)}\\ &= \pi + \frac{1}{2}(4+\pi)x - 2(x+2)\ \arctan\left(\sqrt{x+1}\right) - \frac{x(6+x)\sqrt{x+1}}{2(3+2x)}.\end{aligned}$$

Then, by differentiation, we have

$$\ell'(x) = -\frac{6x^3+47x^2+114x+72}{4\sqrt{x+1}(2x+3)^2} - 2\arctan\left(\sqrt{x+1}\right) + \frac{\pi}{2} + 2$$

and

$$\ell''(x) = -\frac{x^2(12x^3+108x^2+239x+144)}{8(x+1)^{3/2}(x+2)(2x+3)^3}.$$

Thus $\ell''(x) \leq 0$ for $x \geq 0$. So $\ell'(x)$ is decreasing for $x \geq 0$ and $\ell'(x) \leq \ell'(0) = 0$. Therefore $\ell(x)$ is decreasing for $x \geq 0$ and $\ell(x) \leq \ell(0) = 0$. This ends the proof of Lemma 3. □

Proof of Lemma 4. For $x \geq 0$, let us consider the function $m(x)$ given by

$$\begin{aligned}m(x) &= f(x) - \frac{(x+2)[(x+1)^3-1]}{3\sqrt{x+1}[(x+1)^2+1]}\\ &= \pi + \frac{1}{2}(4+\pi)x - 2(x+2)\ \arctan\left(\sqrt{x+1}\right) - \frac{(x+2)[(x+1)^3-1]}{3\sqrt{x+1}[(x+1)^2+1]}.\end{aligned}$$

Then, by differentiation, we have

$$\begin{aligned}m'(x) = &-\frac{3x^6+25x^5+87x^4+178x^3+222x^2+156x+48}{6(x+1)^{3/2}(x^2+2x+2)^2}\\ &- 2\ \arctan\left(\sqrt{x+1}\right) + \frac{\pi}{2} + 2\end{aligned}$$

and, by another differentiation, we get

$$m''(x) = -\frac{x^3(3x^6+35x^5+171x^4+460x^3+700x^2+564x+188)}{12(x+1)^{5/2}(x+2)(x^2+2x+2)^3}$$

Thus $m''(x) \leq 0$ for $x \geq 0$. So $m'(x)$ is decreasing for $x \geq 0$ and $m'(x) \leq m'(0) = 0$. Therefore $m(x)$ is decreasing for $x \geq 0$ and $m(x) \leq m(0) = 0$. This ends the proof of Lemma 4. □

Proof of Proposition 2. The first part of the proof is identical to the one of Proposition 1. For $x \in (-1, 0]$, we consider again the function $g(x) = f(x) - \sqrt{x+1}\log(x+1)$. Then recall that

$$g'(x) = -\frac{2}{\sqrt{x+1}} - \frac{\log(x+1)}{2\sqrt{x+1}} - 2\ \arctan\left(\sqrt{x+1}\right) + 2 + \frac{\pi}{2}$$

and

$$g''(x) = \frac{(x+2)\log(x+1) - 2x}{4(x+1)^{3/2}(x+2)}.$$

Using the upper bound $\log(1+x) \leq 2x/(2+x)$ for $x \in (-1, 0]$ (see [3, Equation (4)]), we arrive at $g''(x) \leq 0$. So $g'(x)$ is decreasing for $x \in (-1, 0]$ and $g'(x) \geq g'(0) = 0$. Therefore $g(x)$ is increasing for $x \in (-1, 0]$ and $g(x) \leq g(0) = 0$. The proof of Proposition 2 is completed. □

Proof of Lemma 5. The proof follows the lines of Lemma 2. For $x \geq 0$, let us consider the function $k(x) = f(x) - x(2+x)/(2\sqrt{1+x})$. Then recall that

$$k'(x) = -\frac{(x+2)(3x+4)}{4(x+1)^{3/2}} - 2\arctan\left(\sqrt{x+1}\right) + \frac{\pi}{2} + 2$$

and

$$k''(x) = -\frac{x(x+4)(3x+4)}{8(x+1)^{5/2}(x+2)}.$$

Thus $k''(x) \geq 0$ for $x \in (-1, 0]$. So $k'(x)$ is increasing for $x \in (-1, 0]$ and $k'(x) \leq k'(0) = 0$. Therefore $k(x)$ is decreasing for $x \in (-1, 0]$ and $k(x) \geq k(0) = 0$. This ends the proof of Lemma 5. □

Concluding Remarks: In this paper, we proposed the function $f(x)/\sqrt{x+1}$ as upper and lower bound for $\log(1+x)$ according as $x \geq 0$ and $x \in (-1, 0]$. We proved analytically that it is better to some existing sharp bounds in the literature. A graphical study of the upper bound supports the theory.

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