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ANALYZING THE RELIABILITY OF QUANTUM AMPLITUDE AMPLIFICATION TECHNIQUES IN THE PRESENCE OF NOISE

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Abstract. This paper studies three different amplitude amplification techniques; solution marking via phase shift, solution marking via entanglement and solution marking via conditional global phase shift; when the extra qubit is in different states. The ability of the three techniques to amplify the probability of solution items is analyzed against the bit-flip error problem on the auxiliary qubit which is used for oracle evaluation. It is found that the solution marking via conditional global phase shift is more robust against the bit-flip error than the other two techniques.

1. Introduction

Quantum computers exploit quantum mechanics to do different calculations with speed surpass that of classical computers [1]. This computing discipline has gained intensive attention by research community and many quantum algorithms were presented for different applications. One of these algorithms is the quantum search algorithm which utilizes amplitude amplification techniques to amplify the probability of finding solution items and to de-amplify the probability of non-solution items in an unstructured database. The quantum search algorithm has been presented by Grover [2] which comes as a subroutine in different algorithms [3, 4, 5]. This algorithm locates the solution item in a given list of size N with complexity $O(\sqrt{N})$ better than the classical unstructured search algorithms. The quantum parallelism mechanism is used to express the superposition of all states simultaneously. Then, the algorithm iterates two operators; the oracle operator and the diffusion operator. The oracle operator marks the solution item by applying a phase shift of $\pi(e^{i\pi} = -1)$ and does nothing on the non-solution items. The diffusion operator achieves inversion about the mean of the whole space to magnify the amplitude of the solution item.

Many researchers generalized Grover's search operator in several ways for studying the effect of any selective phase shift other than π phase shift on the success probability of the search algorithm [6, 7, 8, 9, 10]. These ways repeat the operation $HD(\phi)HU_f(\varphi)$ on the initial states of the given system where H is the Hadamard

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operator, $D(\phi) = I - (1 - e^{i\phi})|s\rangle\langle s|$ is the diffusion operator, $|s\rangle$ is the initial state of the system, $U_f(\varphi) = I - (1 - e^{i\varphi})|t\rangle\langle t|$ is the oracle operator, $|t\rangle$ is the target states and I is the identity operator. The researchers used the same phase shift for the oracle and the diffusion operator. The generalization of the search algorithm included the study of the arbitrary superposition effect on the algorithm performance [6, 11, 12, 13, 14, 15]. All of the above search algorithms used the phase shift technique for marking the solutions with phase shift φ and doing nothing on other non-solution items.

Younes et al. in [16] proposed quantum search algorithm using local diffusion operator that resists the de-amplification behavior in Grover algorithm with quadratic speedup. They used entanglement technique for marking the solutions where solutions are entangled with $|1\rangle$ and non-solutions are entangled with $|0\rangle$. Yoder et al. in [17] presented fixed-point amplitude amplification technique that avoids the overcooking problem by always magnifying the amplitude of solution items with quadratic speedup. They used conditional global phase shift as a marking solution technique. Furthermore, many researchers studied Grover search algorithm in the presence of different errors with noisy environment [18, 19, 20, 21, 22, 23, 24].

This paper studies three different solution marking techniques; solution marking via phase shift, solution marking via entanglement and solution marking via conditional global phase shift; by assuming the extra qubit is in different states. The three techniques are analyzed and evaluated in the presence of bit-flip error on the extra qubit that used for oracle evaluation.

The remainder of the paper is organized as follows. Section 2 introduces unstructured database search problem. Section 3 presents solution marking techniques in noiseless environment while section 4 analyzes the three techniques with noisy environment. Section 5 discusses the results in details. Finally, this paper is concluded in section 6.

2. Unstructured Database Search Problem

Suppose an unstructured database of $N = 2^n$ items where n is a positive integer and these items are labelled by an index j where $j\epsilon[0, 1, ..., N - 1]$. Consider an oracle U_f that maps the database items to either 0 or 1 based on some properties those items should satisfy,

$$U_f(x) = \begin{cases} |x\rangle & f(x) = 0 \\ e^{i\varphi}|x\rangle & f(x) = 1. \end{cases} \tag{1}$$

The search problem is to find any x in the database where $U_f(x) = 1$ assuming that x exists in the database and $e^{i\varphi}$ is the phase shift for marking the solution items i.e. search for M solutions in the database that satisfy the boolean oracle where $1 \leq M \leq N$. Classical computers solve this problem in $O(N/M)$ calls to the oracle(query) while the quantum computers solve in $O(\sqrt{N/M})$. Solution marking using boolean oracle needs an extra qubit for oracle evaluation as in classical search. The following section evaluates three different solution marking techniques with noisy environment.

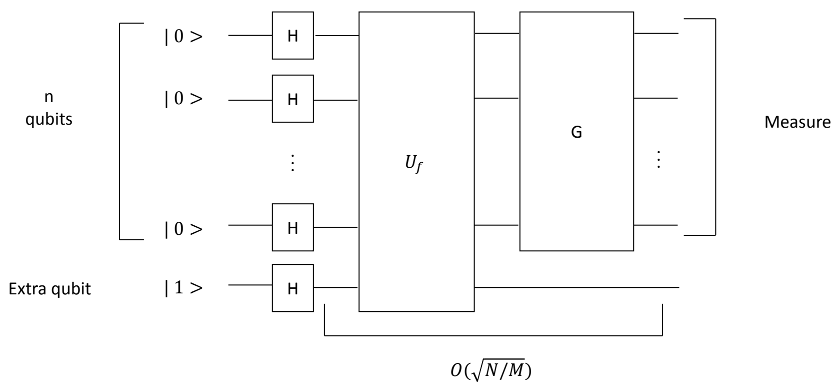

FIGURE 1. Quantum circuit for Grover search algorithm [6].

3. Solution Marking Techniques

The amplitude amplification techniques are subjected to different noises. One of these noises is the bit-flip error on the extra qubit. Here are the solution marking techniques in the absence of noise on the extra qubit.

3.1. **Solution Marking Via Phase Shift.** In amplitude amplification corresponds to Grover's algorithm [6], the oracle operator marks the solution items by applying a phase shift of $\pi (e^{i\pi} = -1)$ on that items and does nothing on the other non-solution items. Then, the diffusion operator does inversion operation about the mean of the whole space to amplify the amplitude of the solution items and de-amplify the amplitude of the non-solution items. The diagonal representation that describes the diffusion operator is $D = H^{\otimes n}[2|0\rangle\langle 0| - I]H^{\otimes n}$. This operator is applied on n qubits where the vector $|0\rangle$ is of length 2^n and I is the identity matrix of size $2^n \times 2^n$. Furthermore, the diffusion operator can be written as a $2^n \times 2^n$ matrix in the computational basis as follows,

$$D = H^{\otimes n} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} H^{\otimes n} = \begin{bmatrix} \frac{2}{N} - 1 & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N} - 1 & \cdots & \frac{2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} - 1 \end{bmatrix}. \quad (2)$$

Figure 1. Shows the quantum circuit for Grover search algorithm. The algorithm with first iteration goes through the following steps.

(1) Register Preparation: The quantum register is prepared with $n + 1$ qubits. The first n qubits are in the state $|0\rangle$ to represent the initial states and the last qubit is the ancilla qubit with the state $|1\rangle$ to evaluate the oracle U_f.

$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |1\rangle. \quad (3)$$

(2) Register Initialization: Hadamard gate is applied on the $n+1$ qubits simultaneously to represent the initial 2^n states with uniform superposition.

$$|\psi_1\rangle = H^{\otimes n+1}|\psi_0\rangle = (\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}|j>) \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}. \quad (4)$$

(3) Applying Oracle: The oracle U_f is applied on $|\psi_1\rangle$ that change the amplitudes of the solution items by phase shift $-1 = e^{i\pi}$ *i.e.* $U_f|j\rangle \rightarrow (-1)^{f(j)}|j\rangle$, so the system is

$$\begin{aligned}|\psi_2\rangle &= U_f|\psi_1\rangle \\ &= \frac{1}{\sqrt{N}}\sum_{j=0}^{M-N-1}{}''|j\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} - \frac{1}{\sqrt{N}}\sum_{j=0}^{M-1}{}'|j\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}},\end{aligned} \quad (5)$$

where $\sum_j'$ points to the sum over j for solution items and $\sum_j''$ points to the sum over j for non-solution items. The oracle in this algorithm uses phase shift technique for solution marking where the solutions are marked by phase shift $-1 = e^{i\pi}$ while the non-solutions are not. The system $|\psi_2\rangle$ can be written as

$$|\psi_2\rangle = a_0\sum_{j=0}^{N-M-1}{}''|j\rangle + b_0\sum_{j=0}^{M-1}{}'|j\rangle, \quad (6)$$

where $a_0 = \frac{1}{\sqrt{N}}$, $b_0 = \frac{-1}{\sqrt{N}}$. Notice that, the state of the ancilla qubit will not change so we can remove it from the system for simplicity.

(4) Applying Diffusion Operator: The inversion about the mean is done on the whole space of the system using the diffusion operator. This operator is applied on a general system $\sum_{x=0}^{N-1}\gamma_x|x\rangle$ as follows,

$$D(\sum_{x=0}^{N-1}\gamma_x|x\rangle) = \sum_{x=0}^{N-1}(2 < \mu > -\gamma_x)|j\rangle, \quad (7)$$

where $< \mu >= \frac{1}{N}\sum_{x=0}^{N-1}\gamma_x$ is the mean of the amplitudes of the whole space. The system after applying the D on $|\psi_2\rangle$ becomes

$$\begin{aligned}|\psi_3\rangle &= D_p|\psi_2\rangle \\ &= a_1\sum_{j=0}^{N-M-1}{}''|j\rangle + b_1\sum_{j=0}^{M-1}{}'|j\rangle,\end{aligned} \quad (8)$$

$$< \mu >= \frac{1}{N}(\frac{N-M}{\sqrt{N}} - \frac{M}{\sqrt{N}}) = \frac{1}{\sqrt{N}}(1 - \frac{2M}{N}), \quad (9)$$

$$a_1 = 2 < \mu > -a_0 = 2 < \mu_1 > -\frac{1}{N} = \frac{1}{\sqrt{N}}(1 - \frac{4M}{N}), \quad (10)$$

$$b_1 = 2 < \mu > -b_0 = 2 < \mu_1 > +\frac{1}{N} = \frac{1}{\sqrt{N}}(3 - \frac{4M}{N}). \quad (11)$$

The amplitude of the solution items b_1 are amplified and becomes larger than the amplitudes of the non-solution items a_1.

(5) Measurement: After the first iteration, the first n qubits are measured to calculate the system probability.
   - Solutions Probability P_s: The probability of the solutions is computed as
   $$P_s = M(b_1^2). \tag{12}$$
   - Non-Solutions Probability P_{ns}: The probability of the non-solutions is computed as
   $$P_{ns} = (N-M)a_1^2, \tag{13}$$

   such that $(N-M)a_1^2 + Mb_1^2 = 1$.

It is found that the success probability (P_s) is amplified and becomes larger than the failure probability (P_{ns}) in the absence of noise on the extra qubit.

3.2. **Solution Marking Via Entanglement.** Younes et al in [16] proposed a quantum search algorithm based on the entanglement technique and the local diffusion operator which do inversion operation about the mean on a local space. The local diffusion operator is applied on $n+1$ qubits where the last qubit is used for entanglement purpose. The diagonal representation that describes this local operator is $Y = (H^{\otimes n} \otimes I_1)(2|0\rangle\langle 0| - I_{2N})(H^{\otimes n} \otimes I_1)$ where the vector $|0\rangle$ is of length $2N = 2^{n+1}$. Additionally, this operator can be written as a $2^{n+1} \times 2^{n+1}$ matrix in the computational basis as follows.

$$Y = H^{\otimes n} \otimes I_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} H^{\otimes n} \otimes I_1$$
$$= \begin{bmatrix} \frac{2}{N}-1 & 0 & \frac{2}{N} & 0 & \cdots & \frac{2}{N} & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{2}{N} & 0 & \frac{2}{N}-1 & 0 & \cdots & \frac{2}{N} & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{2}{N} & 0 & \frac{2}{N} & 0 & \cdots & \frac{2}{N}-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix} \tag{14}$$

It is found that applying the operator Y on a given system will perform the inversion about the mean only on the subspace entangled with the extra qubit in state $|0\rangle$ (even elements) and will only change the sign of the amplitudes for the rest of the system entangled with the extra qubit in state $|1\rangle$ (odd elements). Figure 2. Shows the circuit of the quantum search algorithm using entanglement in [16].

Their algorithm goes through the following steps:

(1) Register Preparation: The quantum register is prepared with $n+1$ qubits all in the state $|0\rangle$ where the ancilla qubit is used to evaluate the oracle U_f.
$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |0\rangle. \tag{15}$$

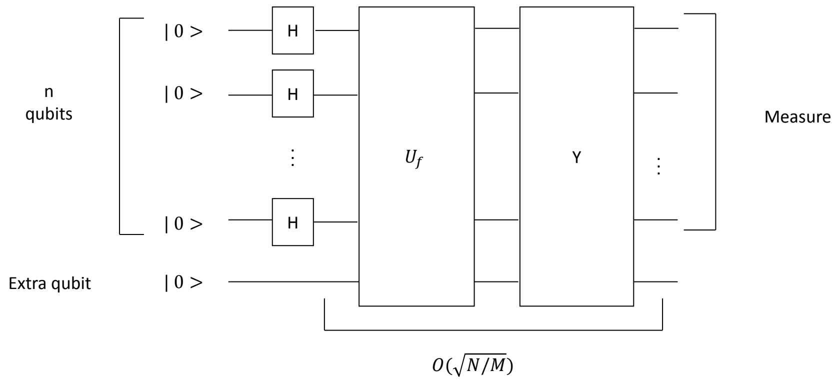

FIGURE 2. Quantum circuit for the proposed algorithm in [16].

(2) Register Initialization: Hadamard gate is applied on the first n qubits simultaneously to represent the initial 2^n states with uniform superposition.

$$\begin{aligned} |\psi_1\rangle &= (H^{\otimes n} \otimes I)|\psi_0\rangle = (\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j >) \otimes |0\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |0\rangle), \end{aligned} \tag{16}$$

where $\sum_j'$ refers to the sum of solutions over j and $\sum_j''$ refers to the sum of non-solutions.

(3) Applying Oracle: The oracle U_f maps the items to either 0 or 1 simultaneously and stores the results in the extra qubit.

$$|\psi_2\rangle = U_f|\psi_1\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} (|j\rangle \otimes |0 \oplus f(j)\rangle) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} (|j\rangle \otimes |f(j)\rangle). \tag{17}$$

So the system will be

$$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |1\rangle). \tag{18}$$

where $\sum_j'$ refers to the sum of M states of solutions over j and $\sum_j''$ refers to the sum of $N - M$ states of non-solutions.

The oracle in this algorithm uses entanglement for solution marking where the solutions are entangled with the extra qubit in state $|1\rangle$ while the non-solutions are entangled with the extra qubit in state $|0\rangle$.

(4) Applying Local Diffusion Operator: The local diffusion operator does the inversion about the mean only on local space of the system that entangled with extra qubit in state $|0\rangle$ (even elements) and changes the sign of the rest system entangled with extra qubit in state $|1\rangle$ (odd elements). If the

general system can be written as

$$\sum_{x=0}^{2N-1} \gamma_x |x\rangle = \sum_{j=0}^{N-1} \alpha_j (|j\rangle \otimes |0\rangle) + \sum_{j=0}^{N-1} \beta_j (|j\rangle \otimes |1\rangle), \tag{19}$$

where α_j is the amplitude of the even items of the system and β_j is the amplitude of the odd items, then the local diffusion operator is applied on the system as follows,

$$Y(\sum_{x=0}^{2N-1} \gamma_x |x\rangle) = \sum_{j=0}^{N-1} (2 < \mu > -\alpha_j)(|j\rangle \otimes |0\rangle) - \sum_{j=0}^{N-1} \beta_j (|j\rangle \otimes |1\rangle), \tag{20}$$

where $< \mu >= \frac{1}{N} \sum_{j=0}^{N-1} \alpha_j$ is the mean of the amplitudes of the local space which entangled with the extra qubit in state $|0\rangle$. If the system before applying Y operator on $|\psi_2\rangle$ can be written as

$$|\psi_2\rangle = a_0 \sum_{j=0}^{N-M-1} {}''(|j\rangle \otimes |0\rangle) + b_0 \sum_{j=0}^{M-1} {}'(|j\rangle \otimes |0\rangle) + c_0 \sum_{j=0}^{M-1} {}'(|j\rangle \otimes |1\rangle), \tag{21}$$

where

$$a_0 = \frac{1}{\sqrt{N}}, \quad b_0 = 0 \quad \text{and} \quad c_0 = \frac{1}{\sqrt{N}}. \tag{22}$$

Then the system after applying Y operator becomes

$$|\psi_3\rangle = a_1 \sum_{j=0}^{N-M-1} {}''(|j\rangle \otimes |0\rangle) + b_1 \sum_{j=0}^{M-1} {}'(|j\rangle \otimes |0\rangle) + c_1 \sum_{j=0}^{M-1} {}'(|j\rangle \otimes |1\rangle), \tag{23}$$

$$< \mu >= \frac{1}{N}(\frac{N-M}{\sqrt{N}} + M.0) = \frac{N-M}{N\sqrt{N}}, \tag{24}$$

$$a_1 = 2 < \mu > -a_0 = 2 < \mu > -\frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}}(1 - \frac{2M}{N}), \tag{25}$$

$$b_1 = 2 < \mu > -b_0 = 2 < \mu >= \frac{1}{\sqrt{N}}(2 - \frac{2M}{N}), \tag{26}$$

$$c_1 = -c_0 = \frac{-1}{\sqrt{N}}. \tag{27}$$

The amplitude of the solution items $(b_1 + c_1)$ are amplified and becomes larger than the amplitudes of the non-solution items a_1.

(5) Measurement: After the first iteration, the first n qubits are measured to get the system probability.
- Solutions Probability P_s: The probability of the solutions is computed as

$$P_s = M(b_1^2 + c_1^2) = \frac{M}{N}\{1 + (2 - \frac{2M}{N})^2\}. \tag{28}$$

- Non-Solutions Probability P_{ns}: The probability of the non-solutions is computed as

$$P_{ns} = (N-M)a_1^2 = \frac{N-M}{N}(1 - \frac{2M}{N})^2, \tag{29}$$

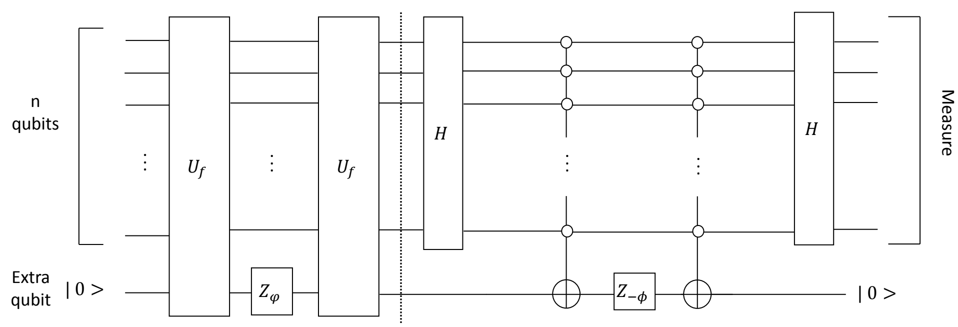

FIGURE 3. Quantum circuit for the proposed algorithm in [17].

Such that $(N-M)a_1^2 + M(b_1^2 + c_1^2) = 1$. It is noted that the probability of the solution items is amplified and becomes larger than the probability of the non-solution items in the absence of noise on the extra qubit.

3.3. **Solution Marking Via Conditional Global Phase Shift.** Thodor et al. in [17, 25] proposed quantum search algorithm based on the conditional global phase shift. They used two boolean oracles where the oracle flips the ancilla qubit with solution items. The conditional phase shift for all items in the list is done using Z_φ gate between the two oracles where Z_φ gate represents a rotation about Z axis by phase φ and is represented using the following matrix,

$$Z_\varphi = \begin{bmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{bmatrix}. \tag{30}$$

Figure 3. shows the circuit of the proposed algorithm in [25]. The first part (before the dotted line) implements the oracle with global phase shift as follows,

$$e^{-i\varphi/2}U_f(\varphi) = e^{-i\varphi/2}[I - (1 - e^{i\varphi})|t\rangle\langle t|]. \tag{31}$$

While the second part (after the dotted line) in Figure 3. implements the diffusion operator that can be performed by the following matrices,

$$T = H^{\otimes n} \otimes I_1 \begin{bmatrix} e^{-i\phi/2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & e^{i\phi/2} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & e^{i\phi/2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & e^{-i\phi/2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & e^{i\phi/2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-i\phi/2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & e^{-i\phi/2} \end{bmatrix} H^{\otimes n} \otimes I_1. \tag{32}$$

The behaviour of the diffusion operator with the even elements (entangled with extra qubit in state $|0\rangle$) is expressed using the following equation,

$$T_{even} = (H^{\otimes n})e^{i\phi/2}[I_n - (1 - e^{-i\phi})|0\rangle\langle 0|](H^{\otimes n}), \tag{33}$$

$$T_{even} = H^{\otimes n} \begin{bmatrix} e^{-i\phi/2} & 0 & 0 & 0 & \cdots \\ 0 & e^{i\phi/2} & 0 & 0 & \cdots \\ 0 & 0 & e^{i\phi/2} & 0 & \cdots \\ 0 & 0 & 0 & e^{i\phi/2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & e^{i\phi/2} \end{bmatrix} H^{\otimes n}. \tag{34}$$

while, the behaviour of the diffusion operator with the odd elements (entangled with extra qubit in state $|1\rangle$) is shown in the following equation,

$$T_{odd} = H^{\otimes n} e^{-i\phi/2}[I_n - (1 - e^{i\phi})|0\rangle\langle 0|]H^{\otimes n}, \tag{35}$$

$$T_{odd} = H^{\otimes n} \begin{bmatrix} e^{i\phi/2} & 0 & 0 & 0 & \cdots \\ 0 & e^{-i\phi/2} & 0 & 0 & \cdots \\ 0 & 0 & e^{-i\phi/2} & 0 & \cdots \\ 0 & 0 & 0 & e^{-i\phi/2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & e^{-i\phi/2} \end{bmatrix} H^{\otimes n}. \tag{36}$$

The algorithm without noise goes through the following steps:

(1) Register Preparation: The quantum register is prepared with $n + 1$ qubits all in the state $|0\rangle$ where the ancilla qubit is used to evaluate the oracle U_f but can be reused.

$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |0\rangle. \tag{37}$$

(2) Register Initialization: Hadamard gate is applied on the first n qubits simultaneously to point the initial 2^n states with uniform superposition.

$$\begin{aligned} |\psi_1\rangle &= (H^{\otimes n} \otimes I)|\psi_0\rangle = (\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j >) \otimes |0\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |0\rangle). \end{aligned} \tag{38}$$

(3) Applying the First Oracle: The oracle U_f will flip the ancilla qubit for solution items,

$$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |1\rangle). \tag{39}$$

(4) Applying Z_φ Gate: The Z_φ gate is applied on the ancilla qubit as follows,

$$\begin{aligned} |\psi_3\rangle &= (I_n \otimes Z_\varphi)|\psi_2\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes Z_\varphi|0\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes Z_\varphi|1\rangle) \\ &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes e^{-i\varphi/2}|0\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes e^{i\varphi/2}|1\rangle). \end{aligned} \tag{40}$$

(5) Applying the Second Oracle: The oracle U_f will flip again the ancilla qubit for solution items,

$$|\psi_4\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes e^{-i\varphi/2}|0\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes e^{i\varphi/2}|0\rangle), \quad (41)$$

$$|\psi_4\rangle = a_0 \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + b_0 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |0\rangle), \quad (42)$$

where $a_0 = \frac{e^{-i\varphi/2}}{\sqrt{N}}$ and $b_0 = \frac{e^{i\varphi/2}}{\sqrt{N}}$.

(6) Applying Diffusion operator: The diffusion operator that is applyied on the system when extra qubit has $|0\rangle$ is

$$T_{even} = (H^{\otimes n})e^{i\phi/2}[I_n - (1 - e^{-i\phi})|0\rangle\langle 0|](H^{\otimes n}), \quad (43)$$

$$T_{even}(\sum_{x=0}^{N-1} (\gamma_x|x\rangle) = e^{i\phi/2}[\gamma_x - (1 - e^{-i\phi}) < \mu >] \sum_{x=0}^{N-1} |x\rangle, \quad (44)$$

where $< \mu >= \frac{1}{N}\sum_{x=0}^{N-1} \gamma_x$ is the mean of the amplitudes of the subspace space entangled with $|0\rangle$.

The system after applying the diffusion operator T_{even} on $|\psi_4\rangle$ becomes

$$|\psi_5\rangle = T_{even}|\psi_4\rangle = a_1 \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + b_1 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |0\rangle), \quad (45)$$

where

$$\langle\mu\rangle = \frac{1}{N}(\frac{(N-M)e^{-i\varphi/2}}{\sqrt{N}} + \frac{(M)e^{i\varphi/2}}{\sqrt{N}}), \quad (46)$$

$$\begin{aligned} a_1 &= e^{i\phi/2}[a_0 - (1 - e^{-i\phi})\langle\mu\rangle] \\ &= \frac{1}{\sqrt{N}}[e^{-i(\frac{\phi+\varphi}{2})} - 2i\frac{M}{N}(e^{i\phi/2} - e^{-i\phi/2})sin(\varphi/2)] \\ &= \frac{1}{\sqrt{N}}[e^{-i(\frac{\phi+\varphi}{2})} + 4\frac{M}{N}sin(\phi/2)sin(\varphi/2)], \end{aligned} \quad (47)$$

$$\begin{aligned} b_1 &= e^{i\phi/2}[b_0 - (1 - e^{-i\phi})\langle\mu\rangle] \\ &= \frac{1}{\sqrt{N}}[e^{-i(\frac{\phi+\varphi}{2})} - 2i\frac{M}{N}(e^{i\phi/2} - e^{-i\phi/2})sin(\varphi/2) + 2ie^{i\phi/2}sin(\varphi/2)] \\ &= \frac{1}{\sqrt{N}}[e^{-i(\frac{\phi+\varphi}{2})} + 4\frac{M}{N}sin(\phi/2)sin(\varphi/2) + 2ie^{i\phi/2}sin(\varphi/2)]. \end{aligned} \quad (48)$$

There is a condition on the phase shift of the oracle and diffusion operator for first iteration as shown bellow.

$$\phi = -\varphi = 2cot^{-1}(tan(2pi/3)\sqrt{(}1 - \gamma^2)), \quad (49)$$

where $\gamma^{-1} = T_{1/3}(1/\delta)$, $T_L(x) = cos[Lcos^{-1}(x)]$ is the L^{th} Chebyshev polynomial of the first kind and δ is a tunable parameter for bounding the error success probability [17].

(7) Measurement: After the first iteration, the first n qubits are measured in order to calculate the system probability.

- Solutions Probability P_s: The probability of the solutions is computed as
$$P_s = M|b_1|^2. \tag{50}$$
- Non-Solutions Probability P_{ns}: The probability of the non-solutions is computed as
$$P_{ns} = (N-M)|a_1|^2, \tag{51}$$

such that $(N-M)a_1^2 + Mb_1^2 = 1$. The success probability P_s is amplified and becomes larger than the failure probability P_{ns} in the absence of noise on the extra qubit.

4. Solution Marking Technique in The Presence of Noise

This section analyzes the three techniques with noisy environment and studies the effects of the bit-flip error on the success probability of these techniques.

4.1. **Solution Marking Via Phase Shift.** The following steps describe the algorithm behaviour when the extra qubit is flipped from $|1>$ to $|0>$.

(1) Register Preparation: The quantum register is prepared with $n+1$ qubits all in the state $|0\rangle$ to represent the initial states where the ancilla qubit is used to evaluate the oracle U_f,
$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |0\rangle. \tag{52}$$
(2) Register Initialization: Hadamard gate is applied on the $n+1$ qubits simultaneously to represent the initial 2^n states with uniform superposition,
$$|\psi_1\rangle = H^{\otimes n+1}|\psi_0\rangle = (\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}|j>) \otimes \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}. \tag{53}$$
(3) Applying Oracle: The oracle U_f is applied on $|\psi_1\rangle$ as follows,
$$\begin{aligned}|\psi_2\rangle = U_f|\psi_1\rangle &= \frac{1}{\sqrt{2N}}\sum_{j=0}^{N-1}|j\rangle(|0 \oplus U_f(j)\rangle + |1 \oplus U_f(j)\rangle) \\ &= \frac{1}{\sqrt{N}}\sum_{j=0}^{M-N-1}{}''|j\rangle \otimes \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} + \frac{1}{\sqrt{N}}\sum_{j=0}^{M-1}{}'|j\rangle \otimes \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}.\end{aligned} \tag{54}$$
The state of the extra qubit is not subjected to change, so we can remove it from the system for simplicity. The oracle in this state does nothing and the amplitudes of the solutions and non-solutions are $a_0 = \frac{1}{\sqrt{N}} \quad and \quad b_0 = \frac{1}{\sqrt{N}}$.

(4) Applying Diffusion Operator: The system after applying the D on $|\psi_2\rangle$ becomes
$$|\psi_3\rangle = D|\psi_2\rangle = a_1\sum_{j=0}^{N-M-1}{}''|j\rangle + b_1\sum_{j=0}^{M-1}{}'|j\rangle, \tag{55}$$
$$<\mu>= \frac{1}{N}[(N-M)\frac{1}{\sqrt{N}} + \frac{M}{\sqrt{N}}] = \frac{1}{\sqrt{N}}, \tag{56}$$
$$a_1 = 2<\mu> - a_0 = \frac{2}{\sqrt{N}} - \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}}, \tag{57}$$

$$b_1 = 2 < \mu > -b_0 = \frac{2}{\sqrt{N}} - \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}}. \tag{58}$$

The amplitudes of the non-solutions (a_1) equal to the amplitudes of the solutions (b_1), so there is no amplification on the solutions probability.

(5) Measurement: The first n qubits are measured to get the system probability.
- Solutions Probability P_s: The probability of the solutions is computed as
$$P_s = M(b_1^2) = \frac{M}{N}. \tag{59}$$
- Non-Solutions Probability P_{ns}: The probability of the non-solutions is computed as
$$P_{ns} = (N - M)a_1^2 = \frac{N - M}{N}. \tag{60}$$

In reality, the success probability (P_s) equal to the failure probability (P_{ns}) in the presence of noise. Therefore, the algorithm performance is affected by the value of the extra qubit due to the bit-flip error and no amplification to the solutions state occurs.

4.2. **Solution Marking Via Entanglement.** The following steps describe the algorithm behaviour when the extra qubit is flipped from $|0\rangle$ to $|1\rangle$ due to the bit-flip error.

(1) Register Preparation: The quantum register is prepared with n qubits all in the state $|0\rangle$ and the ancilla qubit is prepared with $|1\rangle$ to be used for evaluating the oracle U_f,
$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |1\rangle. \tag{61}$$

(2) Register Initialization: Hadamard gate is applied on the first n qubits simultaneously to represent the initial 2^n states with uniform superposition,
$$|\psi_1\rangle = (H^{\otimes n} \otimes I)|\psi_0\rangle = (\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j >) \otimes |1\rangle, \tag{62}$$

$$\begin{aligned} |\psi_1\rangle = a_0 & \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + b_0 \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |1\rangle) \\ & + c_0 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |0\rangle) + d_0 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |1\rangle), \end{aligned} \tag{63}$$

where
$$a_0 = 0, \quad b_0 = \frac{1}{\sqrt{N}}, \quad c_0 = 0 \quad and \quad d_0 = \frac{1}{\sqrt{N}}.$$

(3) Applying Oracle: The oracle U_f will swap the amplitudes of the solution items, so the system after applying the oracle will be
$$\begin{aligned} |\psi_2\rangle = U_f|\psi_1\rangle = a_0 & \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + b_0 \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |1\rangle) \\ & + c_0 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |0\rangle) + d_0 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |1\rangle), \end{aligned} \tag{64}$$

where

$$a_0 = 0, \quad b_0 = \frac{1}{\sqrt{N}}, \quad c_0 = \frac{1}{\sqrt{N}} \quad and \quad d_0 = 0$$

(4) Applying Local Diffusion Operator: The system after applying the Y operator on $|\psi_2\rangle$ becomes

$$\begin{aligned} |\psi_3\rangle = Y|\psi_2\rangle = a_1 \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + b_1 \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |1\rangle) \\ + c_1 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |0\rangle) + d_1 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |1\rangle), \end{aligned} \tag{65}$$

$$< \mu >= \frac{1}{N}(\frac{N-M}{0} + M(\frac{1}{\sqrt{N}})) = \frac{M}{N\sqrt{N}}, \tag{66}$$

$$a_1 = 2 < \mu > -a_0, \quad b_1 = -b_0, c_1 = 2 < \mu > -c_0 \quad \text{and} \quad d_1 = -d_0$$

$$a_1 = \frac{2M}{N\sqrt{N}}, \quad b_1 = \frac{-1}{\sqrt{N}}, \quad c_1 = \frac{2M}{N\sqrt{N}} - \frac{1}{\sqrt{N}} \quad \text{and} \quad d_1 = 0. \tag{67}$$

The amplitude of the non-solutions $(a_1 + b_1)$ is larger than the amplitudes of the solutions $(c_1 + d_1)$, so there is de-amplification on solutions probability.

(5) Measurement: The first n qubits are measured in order to calculate the system probability.
- Solutions Probability P_s: The probability of the solutions is computed as

$$P_s = M(c_1^2 + d_1^2) = \frac{M}{N}(-1 + \frac{2M}{N})^2. \tag{68}$$

- Non-Solutions Probability P_{ns}: The probability of the non-solutions is computed as

$$P_{ns} = (N-M)(a_1^2 + b_1^2) = \frac{N-M}{N}(1 + 4(\frac{M}{N})^2). \tag{69}$$

It is found that the success probability (P_s) is de-amplified while the failure probability (P_{ns}) is amplified in the presence of noise. Therefore, the algorithm performance is affected by the value of the extra qubit due to the bit-flip error.

4.3. **Solution Marking Via Conditional Global Phase Shift.** The following steps describe the algorithm behaviour when the extra qubit is flipped from $|0\rangle$ to $|1\rangle$ due to the bit-flip error.

(1) Register Preparation: The quantum register is prepared with n qubits all in the state $|0\rangle$ and the ancilla qubit is in the state $|1\rangle$.

$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |1\rangle. \tag{70}$$

(2) Register Initialization: Hadamard gate is applied on the first n qubits simultaneously to acts the initial 2^n states with uniform superposition.

$$|\psi_1\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |1\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |1\rangle). \tag{71}$$

(3) Applying the First Oracle: The oracle U_f will flip the ancilla qubit for solution items

$$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |1\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |0\rangle). \tag{72}$$

(4) Applying Z_φ Gate: The Z_φ gate is applied on the ancilla qubit as follows.

$$|\psi_3\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes e^{i\varphi/2}|1\rangle) + \frac{1}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes e^{-i\varphi/2}|0\rangle). \tag{73}$$

(5) Applying the Second Oracle: The oracle U_f will flip again the ancilla qubit for solution items

$$|\psi_4\rangle = \frac{e^{i\varphi/2}}{\sqrt{N}} \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |1\rangle) + \frac{e^{-i\varphi/2}}{\sqrt{N}} \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |1\rangle). \tag{74}$$

The system can be written as

$$|\psi_4\rangle = a_0 \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |1\rangle) + b_0 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |1\rangle). \tag{75}$$

where $a_0 = \frac{e^{i\varphi/2}}{\sqrt{N}}$ and $b_0 = \frac{e^{-i\varphi/2}}{\sqrt{N}}$.

(6) Applying Diffusion Operator: The diffusion operator that is applied on the system when extra qubit has $|1\rangle$ is

$$T_{odd} = (H^{\otimes n}) e^{-i\phi/2} [I_n - (1 - e^{i\phi})|0\rangle\langle 0|](H^{\otimes n})$$

$$T_{odd}\left(\sum_{x=0}^{N-1} (\gamma_x |x\rangle)\right) = e^{-i\phi/2} [\gamma_x - (1 - e^{i\phi}) < \mu >] \sum_{x=0}^{N-1} |x\rangle \tag{76}$$

where $< \mu > = \frac{1}{N} \sum_{x=0}^{N-1} \gamma_x$ is the mean of the amplitudes of the subspace entangled with $|1\rangle$.

The system after applying the diffusion operator T_{odd} on $|\psi_4\rangle$ becomes

$$|\psi_5\rangle = T_{odd}|\psi_4\rangle = a_1 \sum_{j=0}^{N-M-1}{}'' (|j\rangle \otimes |0\rangle) + b_1 \sum_{j=0}^{M-1}{}' (|j\rangle \otimes |0\rangle) \tag{77}$$

where

$$\langle \mu \rangle = \frac{1}{N} \left(\frac{(N-M)e^{i\varphi/2}}{\sqrt{N}} + \frac{(M)e^{-i\varphi/2}}{\sqrt{N}} \right), \tag{78}$$

$$\begin{aligned} a_1 &= e^{-i\phi/2}[a_0 - (1 - e^{i\phi})\langle\mu\rangle] \\ &= \frac{1}{\sqrt{N}}[e^{i(\frac{\phi+\varphi}{2})} - 2i\frac{M}{N}(e^{i\phi/2} - e^{-i\phi/2})sin(\varphi/2)] \\ &= \frac{1}{\sqrt{N}}[e^{i(\frac{\phi+\varphi}{2})} + 4\frac{M}{N}sin(\phi/2)sin(\varphi/2)], \end{aligned} \quad (79)$$

$$\begin{aligned} b_1 &= e^{-i\phi/2}[b_0 - (1 - e^{i\phi})\langle\mu\rangle] \\ &= \frac{1}{\sqrt{N}}[e^{i(\frac{\phi+\varphi}{2})} - 2i\frac{M}{N}(e^{i\phi/2} - e^{-i\phi/2})sin(\varphi/2) - 2ie^{-i\phi/2}sin(\varphi/2)] \\ &= \frac{1}{\sqrt{N}}[e^{i(\frac{\phi+\varphi}{2})} + 4\frac{M}{N}sin(\phi/2)sin(\varphi/2) - 2ie^{-i\phi/2}sin(\varphi/2)] \end{aligned} \quad (80)$$

(7) Measurement: The first n qubits are measured for getting the system probability.
   - Solutions Probability P_s: The probability of the solutions is computed as
$$P_s = M|b_1|^2 \quad (81)$$
   - Non-Solutions Probability P_{ns}: The probability of the non-solutions is computed as
$$P_{ns} = (N - M)|a_1|^2 \quad (82)$$

   The success probability P_s is amplified and becomes larger than the failure probability P_{ns} in the presence of noise on the extra qubit.

5. Results and Discussion

This section evaluates the performance of the three amplitude amplification techniques for quantum search algorithms in the absence and presence of noise on the extra qubit.

5.1. **Experiment in the Absence of Noise.**

The performance of the three amplitude amplification techniques is evaluated and compared with three different values of M/N in case of no error that occurs on the extra qubit.

(1) Solution Marking via Phase Shift: When the value of $M/N = 25/100$, then the amplitude of solution items is $b_1 = 0.2$ and the amplitude of non-solution items is $a_1 = 0$. Therefore, the probability of solution items is amplified and becomes one while the probability of non-solution items is de-amplified and becomes zero. When the value of $M/N = 50/100$, then the amplitude of solution items is $b_1 = 0.1$ and the amplitude of non-solution items is $a_1 = -0.1$. Therefore, the probability of solution items equals to the probability of non-solution items which is 0.5. When the value of $M/N = 75/100$, then the amplitude of solution items is $b_1 = 0$ and the amplitude of non-solution items is $a_1 = -0.2$. Therefore, the probability of solution items is de-amplified and becomes zero while the probability of non-solution items is amplified and becomes one.

(2) Solution Marking via Entanglement: When the value of $M/N = 25/100$, then the amplitudes of solution items are $b_1 = 0.15, c_1 = -0.1$ and the

Table 1. Comparing the success probability of the three amplitude amplification techniques with three different values of M/N in the absence of noise.

M/N	Success Probability% of		
	Phase Shift	Entanglement	Conditional Global Phase Shift
0.25	100.00%	81.25%	78.44%
0.50	50.00%	100.00%	99.70%
0.75	00.00%	93.75%	96.17%

amplitude of non-solution items is $a_1 = 0.05$. Therefore, the probability of solution items is amplified and becomes 0.8125 while the probability of non-solution items is de-amplified and becomes 0.1875. When the value of $M/N = 50/100$, then the amplitudes of solution items are $b_1 = 0.1, c_1 = -0.1$ and the amplitude of non-solution items is $a_1 = 0$. Therefore, the probability of solution items is amplified and becomes one while the probability of non-solution items is de-amplified and becomes zero. When the value of $M/N = 75/100$, then the amplitudes of solution items are $b_1 = 0.05, c_1 = -0.1$ and the amplitude of non-solution items is $a_1 = -0.05$. Therefore, the probability of solution items is amplified and becomes 0.9375 while the probability of non-solution items is de-amplified and becomes 0.0625.

(3) Solution Marking via conditional phase shift: When the value of $M/N = 25/100$, then the amplitude of solution items is $b_1 = 0.1464$ and the amplitude of non-solution items is $a_1 = 0.0536$. Therefore, the probability of solution items is amplified and becomes 0.7844 while the probability of non-solution items is de-amplified and becomes 0.2156. When the value of $M/N = 50/100$, then the amplitude of solution items is $b_1 = 0.1$ and the amplitude of non-solution items is $a_1 = 0.0072$. Therefore, the probability of solution items is amplified and becomes 0.9974 while the probability of non-solution items is de-amplified and becomes 0.0026. When the value of $M/N = 75/100$, then the amplitude of solution items is $b_1 = 0.0536$ and the amplitude of non-solution items is $a_1 = -0.0392$. Therefore, the probability of solution items is amplified and becomes 0.9617 while the probability of non-solution items is de-amplified and becomes 0.0383.

Table 1. summarises the success probability of the three amplitude amplification techniques with three different values of M/N in the absence of noise. Figure 4. shows the behaviour of the success probability for the three techniques in the absence of noise. Solution marking via phase shift solves the case where $M = N/4$ with certainty. Then the success probability is below one-half for $M > N/2$ and fails with certainty for $M = 3N/4$. Solution marking via entanglement solves the case where $M = N/2$ with certainty. Then the success probability stays more reliable with probability at least 92.6%. Additionally, Solution marking via conditional global phase shift solves the case where $M = N/2$ with certainty. Then the success probability stays more reliable with probability at least 95.00% with the phase shift 1.4985 for oracle operator and -1.4985 for diffusion operator.

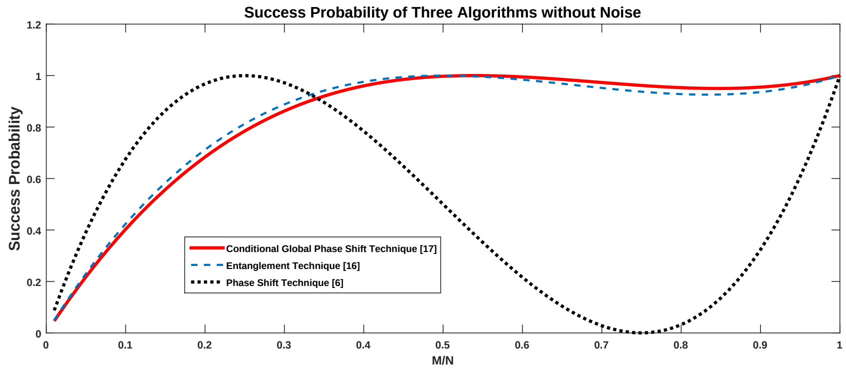

FIGURE 4. Success probability of the three algorithms in the absence of noise.

5.2. **Experiment in the Presence of Noise.** The performance of the three amplitude amplification techniques is evaluated and compared with three different values of M/N in case of bit-flip error that occurs on the extra qubit.

(1) Solution Marking via Phase Shift: The amplitude of solution items equals to the amplitude of non-solution items which is 0.1 in three cases of $M/N = 25/100$, $50/100$ and $75/100$. Therefore, there is no amplification on the amplitude of solution items so the probability of solution items equals to the value of M/N.
(2) Solution Marking via Entanglement: When the value of $M/N = 25/100$, then the amplitudes of solution items are $c_1 = -0.05, d_1 = 0$ and the amplitudes of non-solution items are $a_1 = 0.05, b_1 = -0.1$. Therefore, the probability of solution items is de-amplified and becomes 0.0625 while the probability of non-solution items is amplified and becomes 0.9375. When the value of $M/N = 50/100$, then the amplitudes of solution items are $c_1 = 0, d_1 = 0$ and the amplitudes of non-solution items are $a_1 = 0.1, b_1 = -0.1$. Therefore, the probability of solution items is de-amplified and becomes zero while the probability of non-solution items is amplified and becomes one. When the value of $M/N = 75/100$, then the amplitudes of solution items are $c_1 = 0.05, d_1 = 0$ and the amplitudes of non-solution items are $a_1 = 0.15, b_1 = -0.1$. Therefore, the probability of solution items is de-amplified and becomes 0.1875 while the probability of non-solution items is amplified and becomes 0.8125.
(3) Solution Marking via conditional phase shift: The same results are obtained when the extra qubit is in different states ($|0\rangle$ or $|1\rangle$). When the value of $M/N = 25/100$, $50/100$ and $75/100$, then the success probability of solution items is amplified and become 0.7844, 0.9974 and 0.9617 respectively.

Table 2. summarises the success probability of the three amplitude amplification techniques with three different values of M/N in the presence of noise. Figure 5. describes the success probability of the three techniques in the presence of noise. In Solution marking technique via phase shift, there is no amplification for the solutions probability and the amplitudes of the solutions and non-solutions are equal. Therefore, the algorithm has the same behaviour of the classical search algorithm.

Table 2. Comparing the success probability of the three amplitude amplification techniques with three different values of M/N in the presence of noise.

M/N	Success Probability% of		
	Phase Shift	Entanglement	Conditional Global Phase Shift
0.25	25.00%	6.25%	78.44%
0.50	50.00%	0.00%	99.70%
0.75	75.00%	18.75%	96.17%

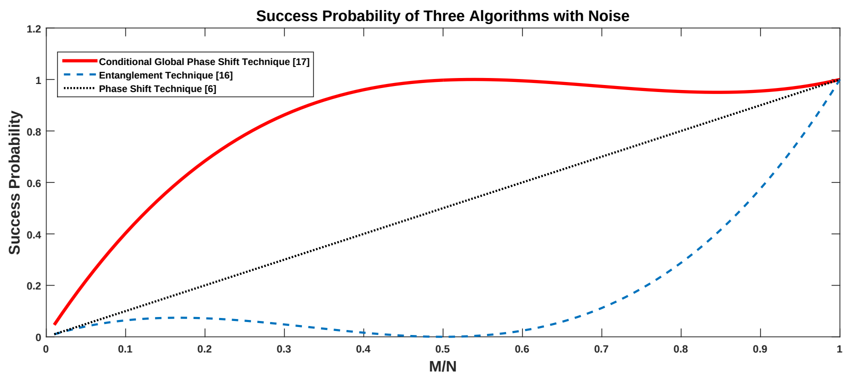

Figure 5. Success probability of the three algorithms in the presence of noise.

In solution marking technique via entanglement, there is de-amplification on the solutions probability with bit-flip error problem on the extra qubit contrary to the behaviour of the same algorithm without bit-flip error. While solution marking technique via conditional global phase shift maintains the same behaviour which is the amplification on the solutions probability in both cases.

Table 3. summarises the probability of the three techniques with/without noise on the extra qubit. It is noted that the success probability (P_s) of the three techniques is amplified and becomes larger than the failure probability (P_{ns}) in the absence of noise on the extra qubit. On the other hand, the behaviour of the success probability becomes different in the presence of noise with solution marking via phase shift and solution marking via entanglement. In solution marking via phase shift, the success probability (P_s) equal to the failure probability (P_{ns}) in the presence of noise where the algorithm performance is affected by the bit-flip error of the extra qubit and no amplification to the solutions state occurs. In solution marking via entanglement, the success probability (P_s) is de-amplified and becomes smaller than the failure probability (P_{ns}) that is amplified in the presence of noise. On the other hand, the success probability P_s is amplified and becomes larger than the failure probability P_{ns} in the absence/presence of noise on the extra qubit with solution marking technique via conditional global phase shift.

Table 3. Comparing the probability (P) of the three algorithms in both cases.

Solution Marking Technique	P without noise	P with noise
Via Phase Shift	$P_s > P_{ns}$	$P_s = P_{ns}$
Via Entanglement	$P_s > P_{ns}$	$P_s < P_{ns}$
Via Conditional Global Phase Shift	$P_s > P_{ns}$	$P_s > P_{ns}$

6. Conclusion

Quantum search algorithms are very important due to their superiority over the classical search algorithms. Those algorithms depend on amplitudes amplification of the solutions in the database. This paper reviewed three different marking solution techniques for quantum search algorithms and described the system in terms of N-dimensional Hilbert space. The three techniques are solutions marking via entanglement, solutions marking via phase shift and solutions marking via conditional phase shift. The three quantum search algorithms are analyzed and evaluated with bit-flip error problem on the extra qubit that used for oracle evaluation. In the case of non-noise on the extra qubit, the success probability of solution items is larger than the failure probability ($P_s > P_{ns}$) of non-solution items. Whereas in case of noise on the extra qubit, the success probability equals to the failure probability ($P_s = P_{ns}$) with solution marking via phase shift. Additionally, the success probability is smaller than the failure probability ($P_s < P_{ns}$) with solution marking via entanglement. The solution marking via conditional global phase shift is robust against bit-flip error and preserves its behaviour which is the amplification of solutions probability ($P_s > P_{ns}$).

In the future, we will compare the performance of the three techniques against the bit-flip error on the initial states of the system (arbitrary initial states). Additionally, we will study the performance of the three techniques against of phase shift errors.

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