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FOUR NEW TENSOR PRODUCTS OF GRAPHS AND THEIR ZAGREB INDICES AND COINDICES

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Abstract. For a (molecular) graph, the first Zagreb index is equal to the sum of squares of the degrees of vertices, and the second Zagreb index is equal to the sum of the products of the degrees of pairs of adjacent vertices. In this paper, we introduce four new tensor products of graphs and study the first and second Zagreb indices and coindices of the resulting graphs and their complements. Also some new relations between these indices are obtained.

1. Introduction

Let $G = (V, E)$ be a simple graph. The number of vertices and edges of G are denoted by n and m, respectively. As usual, n is said to be the order and m is the size of G. A graph of order n and size m will briefly be referred to as an (n, m)-graph. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The degree of a vertex $w \in V(G)$ is the number of vertices adjacent to w and is denoted by $d_G(w)$. The degree of an edge $e = xy$ in G, denoted by $d_G(e)$, is defined by $d_G(e) = d_G(x) + d_G(y) - 2$. The complement of G, denoted by $\overline{G}$, is a graph which has the same vertex set as G, in which two vertices are adjacent if and only if they are not adjacent in G. For graph theoretic terminology, we refer to [15, 16].

A graphical invariant is a number related to a graph. In other words, it is a fixed number under graph automorphisms. In chemical graph theory, these invariants are also called the topological indices. The first and second Zagreb indices are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

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respectively. The first Zagreb index can also be expressed as [10]

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

Došlić [9], has recently conceived the first and second Zagreb coindices as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)]$$

and

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v),$$

respectively.

The vertex-degree-based graph invariant

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2]$$

was encountered in [13]. Recently there has been some interest to F, called "forgotten topological index"[12].

Shirdel et al., [21], introduced a new Zagreb index of a graph G named hyper-Zagreb index and defined as

$$HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2.$$

Milićević et al. [19] in 2004 reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees. The first and second reformulated Zagreb indices are defined respectively as

$$EM_1(G) = \sum_{e \in E(G)} d_G(e)^2 = \sum_{e \sim f} [d_G(e) + d_G(f)]$$

and

$$EM_2(G) = \sum_{e \sim f} d_G(e) d_G(f),$$

where $e \sim f$ means that the edges e and f are adjacent.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $L(G)$ be the line graph of G. There are four related graphs as follows (see Figure 1):

• *Subdivision graph* $S = S(G)$, [15], with $V(S) = V(G) \cup E(G)$ and the added new vertex of S corresponding to the edge uv of G is inserted into the middle of the edge uv of G;

• *Semitotal-point graph* $T_2 = T_2(G)$, [20], with $V(T_2) = V(G) \cup E(G)$ and $E(T_2) = E(S) \cup E(G)$;

• *Semitotal-line graph* $T_1 = T_1(G)$, [20], with $V(T_1) = V(G) \cup E(G)$ and $E(T_1) = E(S) \cup E(L)$;

• *Total graph* $T = T(G)$, [6], with $V(T) = V(G) \cup E(G)$ and $E(T) = E(S) \cup E(G) \cup E(L)$.

In Figure 1, the vertices of transformation graphs $S(G)$, $T_2(G)$, $T_1(G)$ and $T(G)$ corresponding to the vertices of the parent graph G, are indicated by circles. The vertices of these graphs corresponding to the edges of the parent graph G are indicated by squares.

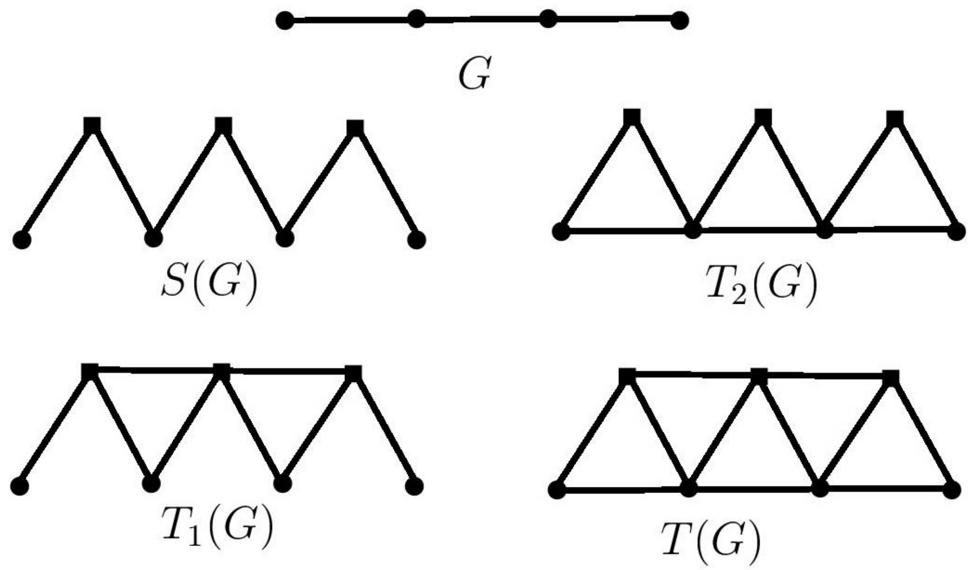

FIGURE 1. Graph G and $S(G)$, $T_2(G)$, $T_1(G)$ and $T(G)$.

2. New tensor products of graphs

Let $i = 1, 2$. For a given graph G_i, its vertex and edge sets will be denoted by $V(G_i)$ and $E(G_i)$, and their cardinalities by n_i and m_i, respectively.

The cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_1, v_1)(u_2, v_2)$ is an edge of $G_1 \times G_2$ if and only if $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$. Based on the cartesian product of graphs, Eliasi and Taeri, [11], introduced the concept of four new sums of graphs as follows:

Let $F \in X$, where $X = \{S, T_2, T_1, T\}$. The F-sum of G_1 and G_2, denoted by $G_1 +_F G_2$, is another graph with the set of vertices $V(G_1 +_F G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(F(G_1))$.

Thus, they obtained four new operations as $G_1 +_S G_2$, $G_1 +_{T_2} G_2$, $G_1 +_{T_1} G_2$ and $G_1 +_T G_2$ and studied the Wiener indices of these graphs. In [5, 8], authors gave the expressions for first Zagreb, second Zagreb and hyper Zagreb indices of these new graphs.

The tensor product $G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u_1, u_2) is adjacent with (v_1, v_2) whenever u_1 is adjacent with v_1 in G_1 and u_2 is adjacent with v_2 in G_2.

Motivated from [11], we introduce four new tensor products of graphs by extending F-sums of graphs on Cartesian product to tensor product as follows:

Definition 2.1. Let F be the one of the symbols S, T_2, T_1, or T. The F-tensor product $G_1 \otimes_F G_2$ is a graph with the set of vertices $V(G_1 \otimes_F G_2)$=$(V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \otimes_F G_2$ are adjacent if and only if u_1 is adjacent to v_1 in $E(F(G_1))$ and u_2 is adjacent to v_2 in G_2.

We illustrate this definition in Fig. 2.

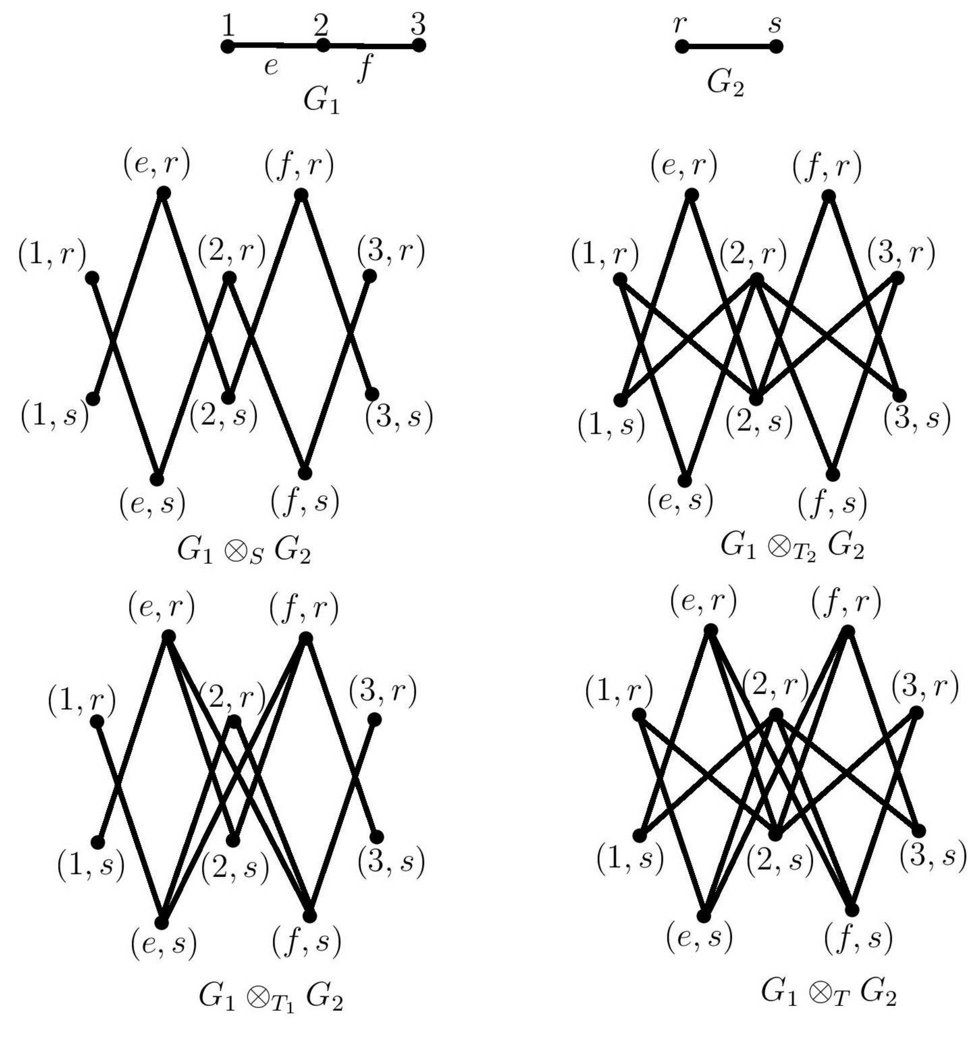

FIGURE 2. Graphs G_1 and G_2 and $G_1 \otimes_F G_2$.

In this paper, we study the first and second Zagreb indices and coindices of $G_1 \otimes_S G_2$, $G_1 \otimes_{T_2} G_2$, $G_1 \otimes_{T_1} G_2$ and $G_1 \otimes_T G_2$. Readers interested in more information on computing topological indices of graph operations can be referred to

[1, 3, 4, 5, 7, 8, 17, 18, 21].

The following earlier established results will be needed for the present considerations:

2.1. **Theorem.** [14, 22] For any (n, m)-graph G,

$$M_1(\overline{G}) = M_1(G) + n(n-1)^2 - 4m(n-1).$$

2.2. **Theorem.** [1, 14] Let G be a (n, m)-graph. Then

$$M_1(G) + \overline{M_1}(G) = 2m(n-1).$$

2.3. **Theorem.** [1, 14] Let G be a simple graph. Then

$$\overline{M_1}(G) = \overline{M_1}(\overline{G}).$$

2.4. **Theorem.** [14] Let G be a graph of order n and size m. Then
i) $M_2(\overline{G}) = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \frac{2n-3}{2}M_1(G) - M_2(G)$,
ii)$\overline{M_2}(G) = 2m^2 - \frac{1}{2}M_1(G) - M_2(G)$,
iii)$\overline{M_2}(\overline{G}) = m(n-1)^2 - (n-1)M_1(G) + M_2(G)$.

2.5. **Theorem.** [2] Let G be an $(n, m)-$graph. Then
i) $M_2(S(G)) = 2M_1(G)$,
ii) $M_2(T_2(G)) = 4[M_1(G) + M_2(G)]$,
iii) $M_2(T_1(G)) = 2EM_1(G) + EM_2(G) + 2[M_1(G) + M_2(G)] + F(G) - 4m$,
iv) $M_2(T(G)) = 2M_1(G) + 8M_2(G) + 2EM_1(G) + EM_2(G) + 2F(G) - 4m$.

3. The Zagreb indices of F-tensor products of graphs

We start by stating the following proposition which will be needed to prove our main results:

3.1. **Proposition.** Let G_1 and G_2 be two graphs. Then $|V(G_1 \otimes_F G_2)| = n_2(n_1 + m_1)$ and

- **(i):** $|E(G_1 \otimes_S G_2)| = 4m_1m_2$
- **(ii):** $|E(G_1 \otimes_{T_2} G_2)| = 6m_1m_2$
- **(iii):** $|E(G_1 \otimes_{T_1} G_2)| = [M_1(G_1) + 2m_1]m_2$
- **(iv):** $|E(G_1 \otimes_T G_2)| = 4m_1m_2 + m_2M_1(G_1)$.

3.2. **Theorem.** Let G_1 and G_2 be the graphs. Then

$$M_1(G_1 \otimes_S G_2) = [M_1(G_1) + 4m_1]M_1(G_2).$$

Proof. By the definition of the first Zagreb index, we have

$$\begin{aligned} M_1(G_1 \otimes_S G_2) &= \sum_{(u,v)\in V(G_1\otimes_S G_2)} d^2_{G_1\otimes_S G_2}(u, v) \\ &= \sum_{u\in V(S(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} [d_{S(G_1)}(u)d_{G_2}(v)]^2 \\ &\quad + \sum_{z\in V(G_2)} \sum_{e\in V(S(G_1))\cap E(G_1)} [d_{S(G_1)}(e)d_{G_2}(z)]^2. \end{aligned}$$

Note that for $u \in V(S(G_1)) \cap V(G_1)$, $d_{S(G_1)}(u) = d_{G_1}(u)$ and for $e \in V(S(G_1)) \cap E(G_1)$, $d_{S(G_1)}(e) = 2$. Therefore

$$\begin{aligned} M_1(G_1 \otimes_S G_2) &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} [d_{G_1}(u)d_{G_2}(v)]^2 \\ &\quad + \sum_{z\in V(G_2)} \sum_{e\in E(G_1)} [2d_{G_2}(z)]^2 \\ &= M_1(G_1)M_1(G_2) + 4m_1 M_1(G_2) \end{aligned}$$

and hence we obtain

$$M_1(G_1 \otimes_S G_2) = [M_1(G_1) + 4m_1]M_1(G_2).$$

□

3.3. **Corollary.** Let G_1 and G_2 be any two graphs. Then

$$\begin{aligned} M_1(\overline{G_1 \otimes_S G_2}) &= [M_1(G_1) + 4m_1]M_1(G_2) \\ &+ [n_2(n_1+m_1) - 1]\{n_2(n_1+m_1)[n_2(n_1+m_1) - 1] - 16m_1m_2\}. \end{aligned}$$

Proof. From Theorem 2.1, we have

$$M_1(\overline{G_1 \otimes_S G_2}) = M_1(G_1 \otimes_S G_2) + n(n-1)^2 - 4m(n-1)$$

where n and m are number of vertices and edges of $G_1 \otimes_S G_2$. The result then follows by Theorem 3.2 and Proposition 3.1. □

3.4. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\overline{M_1}(G_1 \otimes_S G_2) = 8m_1m_2[n_2(n_1+m_1) - 1] - [M_1(G_1) + 4m_1]M_1(G_2).$$

Proof. From Theorem 2.2, we have

$$\overline{M_1}(G_1 \otimes_S G_2) = 2m(n-1) - M_1(G_1 \otimes_S G_2),$$

where n and m are number of vertices and edges of $G_1 \otimes_S G_2$. The required result follows by Theorem 3.2 and Proposition 3.1. □

3.5. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\overline{M}_1(\overline{G_1 \otimes_S G_2}) = 8m_1m_2[n_2(n_1+m_1) - 1] - [M_1(G_1) + 4m_1]M_1(G_2).$$

Proof. Apply Theorem 2.3 and Corollary 3.4. □

3.6. **Theorem.** Let G_1 and G_2 be two graphs. Then

$$M_1(G_1 \otimes_{T_2} G_2) = 4[M_1(G_1) + m_1]M_1(G_2).$$

Proof. By the definition of the first Zagreb index, we have

$$\begin{aligned} M_1(G_1 \otimes_{T_2} G_2) &= \sum_{(u,v)\in V(G_1 \otimes_{T_2} G_2)} d^2_{G_1 \otimes_{T_2} G_2}(u,v) \\ &= \sum_{u\in V(T_2(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} [d_{T_2(G_1)}(u)d_{G_2}(v)]^2 \\ &\quad + \sum_{z\in V(G_2)} \sum_{e\in V(T_2(G_1))\cap E(G_1)} [d_{T_2(G_1)}(e)d_{G_2}(z)]^2. \end{aligned}$$

Note that for $u \in V(T_2(G_1)) \cap V(G_1)$, $d_{T_2(G_1)}(u) = 2d_{G_1}(u)$ and for $e \in V(T_2(G_1)) \cap E(G_1)$, $d_{T_2(G_1)}(e) = 2$. Therefore

$$\begin{aligned} M_1(G_1 \otimes_{T_2} G_2) &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} 2^2[d_{G_1}(u)d_{G_2}(v)]^2 \\ &+ \sum_{z \in V(G_2)} \sum_{e \in E(G_1)} [2d_{G_2}(z)]^2 \\ &= 4M_1(G_1)M_1(G_2) + 4m_1M_1(G_2) \end{aligned}$$

and finally we have

$$M_1(G_1 \otimes_{T_2} G_2) = 4[M_1(G_1) + m_1]M_1(G_2).$$

□

3.7. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} M_1(\overline{G_1 \otimes_{T_2} G_2}) &= 4[M_1(G_1) + m_1]M_1(G_2) \\ &+ [n_2(n_1 + m_1) - 1]\{n_2(n_1 + m_1)[n_2(n_1 + m_1) - 1] - 24m_1m_2\}. \end{aligned}$$

Proof. From Theorem 2.1, we have

$$M_1(\overline{G_1 \otimes_{T_2} G_2}) = M_1(G_1 \otimes_{T_2} G_2) + n(n-1)^2 - 4m(n-1)$$

where n and m are number of vertices and edges of $G_1 \otimes_{T_2} G_2$. The required result then follows by Theorem 3.6 and Proposition 3.1. □

3.8. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\overline{M_1}(G_1 \otimes_{T_2} G_2) = 12m_1m_2[n_2(n_1 + m_1) - 1] - 4[M_1(G_1) + m_1]M_1(G_2).$$

Proof. From Theorem 2.2, we have

$$\overline{M_1}(G_1 \otimes_{T_2} G_2) = 2m(n-1) - M_1(G_1 \otimes_{T_2} G_2),$$

where n and m are number of vertices and edges of $G_1 \otimes_{T_2} G_2$. The result follows by Theorem 3.6 and Proposition 3.1. □

3.9. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\overline{M}_1(\overline{G_1 \otimes_{T_2} G_2}) = 12m_1m_2[n_2(n_1 + m_1) - 1] - 4[M_1(G_1) + m_1]M_1(G_2).$$

Proof. Apply Theorem 2.3 and Corollary 3.8. □

3.10. **Theorem.** Let G_1 and G_2 be two graphs. Then

$$M_1(G_1 \otimes_{T_1} G_2) = [M_1(G_1) + HM(G_1)]M_1(G_2).$$

Proof. By the definition of the first Zagreb index, we have

$$\begin{aligned} M_1(G_1 \otimes_{T_1} G_2) &= \sum_{(u,v) \in V(G_1 \otimes_{T_1} G_2)} d^2_{G_1 \otimes_{T_1} G_2}(u, v) \\ &= \sum_{u \in V(T_1(G_1)) \cap V(G_1)} \sum_{v \in V(G_2)} [d_{T_1(G_1)}(u)d_{G_2}(v)]^2 \\ &+ \sum_{z \in V(G_2)} \sum_{e \in V(T_1(G_1)) \cap E(G_1)} [d_{T_1(G_1)}(e)d_{G_2}(z)]^2. \end{aligned}$$

Note that for $u \in V(T_1(G_1)) \cap V(G_1)$, $d_{T_1(G_1)}(u) = d_{G_1}(u)$ and for $e \in V(T_1(G_1)) \cap E(G_1)$, $d_{T_1(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$ implying that

$$\begin{aligned} M_1(G_1 \otimes_{T_1} G_2) &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} [d_{G_1}(u) d_{G_2}(v)]^2 \\ &+ \sum_{z \in V(G_2)} \sum_{xy \in E(G_1)} [d_{G_1}(x) + d_{G_1}(y)]^2 d^2_{G_2}(z) \\ &= M_1(G_1) M_1(G_2) + HM(G_1) M_1(G_2) \end{aligned}$$

which finally gives

$$M_1(G_1 \otimes_{T_1} G_2) = [M_1(G_1) + HM(G_1)] M_1(G_2).$$

□

3.11. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} M_1(\overline{G_1 \otimes_{T_1} G_2}) &= [M_1(G_1) + HM(G_1)] M_1(G_2) \\ &+ [n_2(n_1 + m_1) - 1]\{n_2(n_1 + m_1)[n_2(n_1 + m_1) - 1] - 4m_2[M_1(G_1) + 2m_1]\}. \end{aligned}$$

Proof. From Theorem 2.1, we can write

$$M_1(\overline{G_1 \otimes_{T_1} G_2}) = M_1(G_1 \otimes_{T_1} G_2) + n(n-1)^2 - 4m(n-1)$$

where n and m are the number of vertices and edges of $G_1 \otimes_{T_1} G_2$. The result then follows from Theorem 3.10 and Proposition 3.1. □

3.12. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\overline{M_1}(G_1 \otimes_{T_1} G_2) = 2m_2[M_1(G_1) + 2m_1][n_2(n_1 + m_1) - 1] - [M_1(G_1) + HM(G_1)] M_1(G_2).$$

Proof. From Theorem 2.2, we have

$$\overline{M_1}(G_1 \otimes_{T_1} G_2) = 2m(n-1) - M_1(G_1 \otimes_{T_1} G_2),$$

where n and m are the number of vertices and edges of $G_1 \otimes_{T_1} G_2$. The result now follows by Theorem 3.10 and Proposition 3.1. □

3.13. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\overline{M}_1(\overline{G_1 \otimes_{T_1} G_2}) = 2m_2[M_1(G_1) + 2m_1][n_2(n_1 + m_1) - 1] - [M_1(G_1) + HM(G_1)] M_1(G_2).$$

Proof. Apply Theorem 2.3 and Corollary 3.12. □

3.14. **Theorem.** Let G_1 and G_2 be two graphs. Then

$$M_1(G_1 \otimes_T G_2) = [4M_1(G_1) + HM(G_1)] M_1(G_2).$$

Proof. By the definition of the first Zagreb index, we have

$$\begin{aligned} M_1(G_1 \otimes_T G_2) &= \sum_{(u,v) \in V(G_1 \otimes_T G_2)} d^2_{G_1 \otimes_T G_2}(u, v) \\ &= \sum_{u \in V(T_1(G_1)) \cap V(G_1)} \sum_{v \in V(G_2)} [d_{T(G_1)}(u) d_{G_2}(v)]^2 \\ &+ \sum_{z \in V(G_2)} \sum_{e \in V(T(G_1)) \cap E(G_1)} [d_{T(G_1)}(e) d_{G_2}(z)]^2. \end{aligned}$$

Note that for $u \in V(T(G_1)) \cap V(G_1)$, $d_{T(G_1)}(u) = 2d_{G_1}(u)$ and for $e \in V(T(G_1)) \cap E(G_1)$, $d_{T(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$ which implies that

$$\begin{aligned} M_1(G_1 \otimes_T G_2) &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} 2^2[d_{G_1}(u)d_{G_2}(v)]^2 \\ &\quad + \sum_{z\in V(G_2)} \sum_{xy\in E(G_1)} [d_{G_1}(x) + d_{G_1}(y)]^2 d^2_{G_2}(z) \\ &= 4M_1(G_1)M_1(G_2) + HM(G_1)M_1(G_2) \end{aligned}$$

which finally gives that

$$M_1(G_1 \otimes_T G_2) = [4M_1(G_1) + HM(G_1)]M_1(G_2).$$

□

3.15. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} M_1(\overline{G_1 \otimes_T G_2}) &= [4M_1(G_1) + HM(G_1)]M_1(G_2) \\ &+ [n_2(n_1 + m_1) - 1]\{n_2(n_1 + m_1)[n_2(n_1 + m_1) - 1] - 4[4m_1m_2 + m_2M_1(G_1)]\}. \end{aligned}$$

Proof. From Theorem 2.1, we have

$$M_1(\overline{G_1 \otimes_T G_2}) = M_1(G_1 \otimes_T G_2) + n(n-1)^2 - 4m(n-1)$$

where n and m are the number of vertices and edges of $G_1 \otimes_T G_2$. The required result then follows by Theorem 3.14 and Proposition 3.1. □

3.16. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} \overline{M_1}(G_1 \otimes_T G_2) &= 2[4m_1m_2 + m_2M_1(G_1)][n_2(n_1 + m_1) - 1] \\ &\quad -[4M_1(G_1) + HM(G_1)]M_1(G_2). \end{aligned}$$

Proof. From Theorem 2.2, we have

$$\overline{M_1}(G_1 \otimes_T G_2) = 2m(n-1) - M_1(G_1 \otimes_T G_2)$$

where n and m are the number of vertices and edges of $G_1 \otimes_T G_2$. The result follows now by Theorem 3.14 and Proposition 3.1. □

3.17. **Corollary.** Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} \overline{M}_1(\overline{G_1 \otimes_T G_2}) &= 2[4m_1m_2 + m_2M_1(G_1)][n_2(n_1 + m_1) - 1] \\ &\quad -[4M_1(G_1) + HM(G_1)]M_1(G_2). \end{aligned}$$

Proof. Apply Theorem 2.3 and Corollary 3.16. □

From Theorem 2.4, it is clear that if $M_1(G_1 \otimes_F G_2)$ and $M_2(G_1 \otimes_F G_2)$ are known, then also $M_2(\overline{G_1 \otimes_F G_2})$, $\overline{M_2}(G_1 \otimes_F G_2)$ and $\overline{M_2}(\overline{G_1 \otimes_F G_2})$ are known. As some expressions for $M_1(G_1 \otimes_F G_2)$ are known by Theorems 3.2, 3.6, 3.10 and 3.14, what really needs to be calculated are some expressions for $M_2(G_1 \otimes_F G_2)$. This we do now:

3.18. **Lemma.** Let G_1 and G_2 be two graphs. Then
$$M_2(G_1 \otimes_F G_2) = 2M_2(F(G_1))M_2(G_2).$$

Proof. By the definition of the second Zagreb index, we have
$$\begin{aligned}
M_2(G_1 \otimes_F G_2) &= \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_F G_2)} [d_{G_1+_F G_2}(u_1,v_1)d_{G_1+_F G_2}(u_2,v_2)] \\
&= 2\sum_{u_1u_2\in E(F(G_1))}\sum_{v_1v_2\in E(G_2)} [d_{F(G_1)}(u_1)d_{G_2}(v_1)][d_{F(G_1)}(u_2)d_{G_2}(v_2)] \\
&= 2\sum_{u_1u_2\in E(F(G_1))} [d_{F(G_1)}(u_1)d_{F(G_1)}(u_2)] \sum_{v_1v_2\in E(G_2)} [d_{G_2}(v_1)d_{G_2}(v_2)]
\end{aligned}$$

and hence we obtain
$$M_2(G_1 \otimes_F G_2) = 2M_2(F(G_1))M_2(G_2).$$
□

Applying Lemma 3.18 and Theorem 2.5, we reach the following theorem:

3.19. **Theorem.** Let G_1 and G_2 be two graphs. Then
i) $M_2(G_1 \otimes_S G_2) = 4M_2(G_2)M_1(G_1)$
ii) $M_2(G_1 \otimes_{T_2} G_2) = 8[M_1(G_1) + M_2(G_1)]M_2(G_2)$
iii) $M_2(G_1 \otimes_{T_1} G_2) = 2[2EM_1(G_1) + EM_2(G_1) + 2[M_1(G_1) + M_2(G_1)] + F(G_1) - 4m_1]M_2(G_2)$
iv) $M_2(G_1 \otimes_T G_2) = 2[2M_1(G_1) + 8M_2(G_1) + 2EM_1(G_1) + EM_2(G_1) + 2F(G_1) - 4m_1]M_2(G_2)$.

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