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STARLIKE AND CONVEXITY PROPERTIES FOR $p-$VALENT HYPERGEOMETRIC FUNCTIONS OF ORDER α AND TYPE β

A. O. MOSTAFA AND G. M. EL-HAWSH

ABSTRACT. Given the hypergeometric function $F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n$, we obtain conditions on a, b and c to guarante $z^p F(a,b,c;z)$ to be in various subclasses of $p-$valent starlike and $p-$valent convex functions. An operator related to the hypergeometric function is also examined.

1. Introduction

Denote by $\mathbb{S}(p)$ the class of $p-$valent analytic functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}, z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}). \tag{1}$$

A function $f(z) \in \mathbb{S}(p)$ is said to be in the class $\mathcal{S}_p^*(\alpha, \beta)$ of $p-$valently starlike functions of order α and type β (see [1], [2], [3] and also [4]) if :

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\alpha} \right| < \beta \quad (0 \le \alpha < p;\ 0 < \beta \le 1;\ z \in \mathbb{U}), \tag{2}$$

and is in the class $\mathcal{K}_p(\alpha, \beta)$ of $p-$valently convex functions of order α and type β (see [1], [2], [3] and also [4]) if :

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\alpha} \right| < \beta \ (0 \le \alpha < p; 0 < \beta \le 1;\ z \in \mathbb{U}). \tag{3}$$

From (2) and (3), we have

$$f(z) \in \mathcal{K}_p(\alpha, \beta) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha, \beta) \qquad (0 \le \alpha < p;\ 0 < \beta \le 1). \tag{4}$$

In particular:

(i) $\mathcal{S}_p^*(\alpha, 1) = \mathcal{S}_p^*(\alpha)$(see [1], [2], [5] and [6]);

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(ii) $\mathcal{K}_p(\alpha, 1) = \mathcal{K}_p(\alpha)$(see [1], [2] and [7]);
(iii) $\mathcal{S}_p^*(0, 1) = \mathcal{S}_p^*$ and $\mathcal{K}_p(0, 1) = \mathcal{K}_p$ (see [8]);
(iv) $\mathcal{S}_1^*(\alpha, 1) = \mathcal{S}^*(\alpha)$ and $\mathcal{K}_1(\alpha, 1) = \mathcal{K}(\alpha)$ (see [9]).

By $\mathbb{T}(p)$ we denote the subclass of $\mathbb{S}(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (a_{p+n} \geq 0;\ p \in \mathbb{N}). \tag{5}$$

By $\mathcal{T}_p^*(\alpha, \beta)$ and $\mathcal{C}_p(\alpha, \beta)$ we denote the classes obtained by taking interesctions, respectively, of the classes $\mathcal{S}_p^*(\alpha, \beta)$ and $\mathcal{K}_p(\alpha, \beta)$ with the class $\mathbb{T}(p)$

$$\begin{aligned} \mathcal{T}_p^*(\alpha, \beta) &= \mathcal{S}_p^*(\alpha, \beta) \cap \mathbb{T}(p), \\ \mathcal{C}_p(\alpha, \beta) &= \mathcal{K}_p(\alpha, \beta) \cap \mathbb{T}(p). \end{aligned}$$

We note that:
(i) $\mathcal{T}_p^*(\alpha, 1) = \mathcal{T}_p^*(\alpha)$ and $\mathcal{C}_p(\alpha, 1) = \mathcal{C}_p(\alpha)$ (see [1], [2], [10] and [11]);
(ii) $\mathcal{T}_1^*(\alpha, \beta) = \mathcal{T}^*(\alpha, \beta)$ and $\mathcal{C}_1(\alpha, \beta) = \mathcal{C}(\alpha, \beta)$ (see [12]);
(iii) $\mathcal{T}_1^*(\alpha, 1) = \mathcal{T}^*(\alpha)$ and $\mathcal{C}_1(\alpha, 1) = \mathcal{C}(\alpha)$ (see [13]).

For a, b and $c \in \mathbb{C}$ and $c \neq 0, -1, -2, ...$, the hypergeometric function is defined by

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in \mathbb{U}), \tag{6}$$

where $(\lambda)_n$ is defined by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & n = 0 \\ \lambda(\lambda + 1) ... (\lambda + n - 1) & n \in \mathbb{N} \end{cases}. \tag{7}$$

The series (6) represents an analytic function in $\mathbb{U}$ and has an analytic continuation throughout the finite complex plane except at most for the cut $[1, \infty)$. We note that $F(a, b, c; 1)$ converges for $\Re(a - b - c) > 0$ and is related to the Gamma function by

$$F(a, b, c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}. \tag{8}$$

Corresponding to the function $F(a, b, c; z)$ we define

$$h_p(a, b, c; z) = z^p F(a, b, c; z). \tag{9}$$

We observe that for a function $f(z)$ of the form (1), we have

$$h_p(a, b, c; z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p} (b)_{n-p}}{(c)_{n-p} (1)_{n-p}} z^n. \tag{10}$$

In [14] EL-Ashwah et al. gave necessary and sufficient conditions for $z^p F(a, b, c; z)$ to be in $\mathcal{T}_p^*(\alpha)$, $\mathcal{C}_p(\alpha)$, $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ and has also examined a linear operator acting on hypergeometric functions, (see also [15], [16], [17], [18], [19], [20], [21] and [22]).

In the present paper, we determine necessary and sufficient conditions for $h_p(a, b, c; z)$ to be in the classes $\mathcal{T}_p^*(\alpha, \beta)$, $\mathcal{C}_p(\alpha, \beta)$, $\mathcal{S}_p^*(\alpha, \beta)$ and $\mathcal{K}_p(\alpha, \beta)$. Furthermore, we consider an integral operator related to the hypergeometric function.

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \alpha < p$, $0 < \beta \leq 1$, $z \in \mathbb{U}$ and $p \in \mathbb{N}$.

The following lemmas will be required in our investigation.

Lemma 1 [3]. Let the function $f(z)$ defined by (1). Then

(i) A sufficient condition for $f(z) \in \mathbb{S}(p)$ to be in the class $\mathcal{S}_p^*(\alpha, \beta)$ is that

$$\sum_{n=p+1}^{\infty} \left[n - p + \beta\left(n + p - 2\alpha\right)\right] |a_n| \leq 2\beta\left(p - \alpha\right).$$

(ii) A sufficient condition for $f(z) \in \mathbb{S}(p)$ to be in the class $\mathcal{K}_p(\alpha, \beta)$ is that

$$\sum_{n=p+1}^{\infty} \frac{n}{p} \left[n - p + \beta\left(n + p - 2\alpha\right)\right] |a_n| \leq 2\beta\left(p - \alpha\right).$$

Lemma 2 [3]. Let the function $f(z)$ defined by (5). Then

(i) A sufficient condition for $f(z) \in \mathbb{T}(p)$ to be in the class $\mathcal{T}_p^*(\alpha, \beta)$ is that

$$\sum_{n=p+1}^{\infty} \left[n - p + \beta\left(n + p - 2\alpha\right)\right] a_n \leq 2\beta\left(p - \alpha\right).$$

(ii) A sufficient condition for $f(z) \in \mathbb{T}(p)$ to be in the class $\mathcal{C}_p(\alpha, \beta)$ is that

$$\sum_{n=p+1}^{\infty} \frac{n}{p} \left[n - p + \beta\left(n + p - 2\alpha\right)\right] a_n \leq 2\beta\left(p - \alpha\right).$$

Lemma 3 [23]. Let $f(z) \in \mathbb{T}(p)$ be defined by (5). Then $f(z)$ is $p-$valent in $\mathbb{U}$ if

$$\sum_{n=1}^{\infty} \left(p + n\right) a_{p+n} \leq p.$$

Theorem 1. If $a, b > 0$ and $c > a+b+1$, then a sufficient condition for $h_p(a, b, c; z)$ to be in $\mathcal{S}_p^*(\alpha, \beta)$, $0 \leq \alpha < p$ and $0 < \beta \leq 1$, is that

$$\frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\left[1 + \frac{ab\left(\beta+1\right)}{2\beta\left(p-\alpha\right)\left(c-a-b-1\right)}\right] \leq 2. \tag{11}$$

Condition (11) is necessary and sufficient for F_p defined by $F_p(a, b, c; z) = z^p\left(2 - F(a, b, c; z)\right)$ to be in $\mathcal{T}_p^*(\alpha, \beta)$.

Proof. Since $h_p(a, b, c; z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n$, according to Lemma 1 (i), we only need to show that

$$\sum_{n=p+1}^{\infty} \left[n - p + \beta\left(n + p - 2\alpha\right)\right] \frac{\left(a\right)_{n-p}\left(b\right)_{n-p}}{\left(c\right)_{n-p}\left(1\right)_{n-p}} \leq 2\beta\left(p - \alpha\right).$$

Now

$$\begin{aligned} &\sum_{n=p+1}^{\infty} \left[n - p + \beta\left(n + p - 2\alpha\right)\right] \frac{\left(a\right)_{n-p}\left(b\right)_{n-p}}{\left(c\right)_{n-p}\left(1\right)_{n-p}} \\ = \;& \left(\beta+1\right)\sum_{n=1}^{\infty} \frac{\left(a\right)_n\left(b\right)_n}{\left(c\right)_n\left(1\right)_{n-1}} + 2\beta\left(p-\alpha\right)\sum_{n=1}^{\infty} \frac{\left(a\right)_n\left(b\right)_n}{\left(c\right)_n\left(1\right)_n} \end{aligned} \tag{12}$$

Noting that $(\lambda)_n = \lambda\,(\lambda+1)_{n-1}$ and then applying (8), we get

$$\begin{aligned}
&\frac{ab}{c}\,(\beta+1)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}\,(b+1)_{n-1}}{(c+1)_{n-1}\,(1)_{n-1}}+2\beta\,(p-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,(1)_n}\\
=\;&\frac{ab}{c}\,(\beta+1)\,\frac{\Gamma\,(c+1)\,\Gamma\,(c-a-b-1)}{\Gamma\,(c-a)\,\Gamma\,(c-b)}+2\beta\,(p-\alpha)\left[\frac{\Gamma\,(c)\,\Gamma\,(c-a-b)}{\Gamma\,(c-a)\,\Gamma\,(c-b)}-1\right]\\
=\;&\frac{\Gamma\,(c)\,\Gamma\,(c-a-b)}{\Gamma\,(c-a)\,\Gamma\,(c-b)}\left[\frac{ab\,(\beta+1)}{(c-a-b-1)}+2\beta\,(p-\alpha)\right]-2\beta\,(p-\alpha)\,.
\end{aligned}$$

But this last expression is bounded above by $2\beta\,(p-\alpha)$ if and only if (11) holds.

Since $F_p(a,b,c;z) = z^p - \sum_{n=p+1}^{\infty}\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}}z^n$, the necessity of (11) for F_p to be in $\mathcal{T}_p^*(\alpha,\beta)$ follows from Lemma 2 (i). □

Remark 1. Condition (11) with $\alpha = 0$ and $\beta = 1$ is both necessary and sufficient for F_p to be in the class $\mathcal{T}_p^*$.

In the next theorem, we find constraints on a, b and c that lead to necessary and sufficient conditions for $h_p(a,b,c;z)$ to be in the class $\mathcal{T}_p^*(\alpha,\beta)$.

Theorem 2. If $a, b > -1, c > 0$ and $ab < 0$, then a necessary and sufficient condition for $h_p(a,b,c;z)$ to be in $\mathcal{T}_p^*(\alpha,\beta)$ is that $c \geq a+b+1-\frac{ab(\beta+1)}{2\beta(p-\alpha)}$. The condition $c \geq a+b+1-\frac{ab}{p}$ is necessary and sufficient for $h_p(a,b,c;z)$ to be in $\mathcal{T}_p^*$.

Proof. Since

$$\begin{aligned}
h_p(a,b,c;z) &= z^p+\sum_{n=p+1}^{\infty}\frac{(a)_{n-p}\,(b)_{n-p}}{(c)_{n-p}\,(1)_{n-p}}z^n\\
&= z^p+\frac{ab}{c}\sum_{n=p+1}^{\infty}\frac{(a+1)_{n-p-1}\,(b+1)_{n-p-1}}{(c+1)_{n-p-1}\,(1)_{n-p}}z^n\\
&= z^p-\left|\frac{ab}{c}\right|\sum_{n=p+1}^{\infty}\frac{(a+1)_{n-p-1}\,(b+1)_{n-p-1}}{(c+1)_{n-p-1}\,(1)_{n-p}}z^n, \qquad (13)
\end{aligned}$$

according to Lemma 2 (i), we must show that

$$\sum_{n=p+1}^{\infty}\left[n-p+\beta\,(n+p-2\alpha)\right]\frac{(a+1)_{n-p-1}\,(b+1)_{n-p-1}}{(c+1)_{n-p-1}\,(1)_{n-p}}\left|\frac{ab}{c}\right|\leq 2\beta\,(p-\alpha)\,. \qquad (14)$$

Note that the left side of (14) diverges if $c \leq a+b+1$. Now

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left[n+1+\beta\,(n+2p-2\alpha+1)\right]\frac{(a+1)_n\,(b+1)_n}{(c+1)_n\,(1)_{n+1}}\\
=\;&(\beta+1)\sum_{n=0}^{\infty}\frac{(a+1)_n\,(b+1)_n}{(c+1)_n\,(1)_n}+2\beta\,(p-\alpha)\left(\frac{c}{ab}\right)\sum_{n=1}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,(1)_n}\\
=\;&(\beta+1)\left[\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\right]+2\beta\,(p-\alpha)\left(\frac{c}{ab}\right)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]
\end{aligned}$$

Hence, (14) is equivalent to

$$\begin{aligned}&\frac{\Gamma\left(c+1\right)\Gamma\left(c-a-b-1\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\left[\left(\beta+1\right)+2\beta\left(p-\alpha\right)\left(\frac{c-a-b-1}{ab}\right)\right]\\ \leq\ &2\beta\left(p-\alpha\right)\left[\frac{c}{|ab|}+\frac{c}{ab}\right]=0. \qquad (15)\end{aligned}$$

Thus, (15) is valid if and only if

$$\left(\beta+1\right)+2\beta\left(p-\alpha\right)\left(\frac{c-a-b-1}{ab}\right)\leq 0,$$

or, equivalently,

$$c\geq a+b+1-\frac{ab\left(\beta+1\right)}{2\beta\left(p-\alpha\right)}.$$

Applying (i) of Lemma 2 , with $\alpha=0$ and $\beta=1$ the proof of Theorem 2 is completed. □

Our next theorems will parallel to Theorems 1 and 2 for the $p-$valent convex case.

Theorem 3. If $a,b>0$ and $c>a+b+2$, then a sufficient condition for $h_p(a,b,c;z)$ to be in $\mathcal{K}_p(\alpha,\beta)$, $0\leq\alpha<p$ and $0<\beta\leq 1$, is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{ab(3p\beta+p+\beta-2\alpha\beta+1)}{2p\beta(p-\alpha)(c-a-b-1)}+\frac{(a)_2(b)_2(\beta+1)}{2p\beta(p-\alpha)(c-a-b-2)_2}\right]\leq 2. \qquad (16)$$

Condition (16) is necessary and sufficient for $F_p(a,b,c;z)=z^p\left(2-F(a,b,c;z)\right)$ to be in $\mathcal{C}_p(\alpha,\beta)$.

Proof. In view of Lemma 1 (ii), we only need to show that

$$\sum_{n=p+1}^{\infty}n\left[n-p+\beta\left(n+p-2\alpha\right)\right]\frac{(a)_{n-p}\,(b)_{n-p}}{(c)_{n-p}\,(1)_{n-p}}\leq 2p\beta\left(p-\alpha\right).$$

Now

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(n+p+1)\left[n+1+\beta(n+2p-2\alpha+1)\right]\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
= &\sum_{n=0}^{\infty}(\beta+1)(n+1)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}+(3p\beta+p-2\alpha\beta)\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&+2p\beta(p-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
= &\sum_{n=0}^{\infty}(\beta+1)(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+(3p\beta+p-2\alpha\beta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} \\
&+2p\beta(p-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
= &\sum_{n=1}^{\infty}(\beta+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}}+(3p\beta+p-2\alpha\beta+\beta+1)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} \\
&+2p\beta(p-\alpha)\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}} \\
= &\sum_{n=0}^{\infty}(\beta+1)\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n}}+(3p\beta+p-2\alpha\beta+\beta+1)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} \\
&+2p\beta(p-\alpha)\left[\sum_{n=0}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}-1\right] \qquad (17)
\end{aligned}
$$

Since $(\lambda)_{n+k}=(\lambda)_k(\lambda+k)_n$, we may write (17) as

$$
\begin{aligned}
= &\frac{(a)_2(b)_2(\beta+1)}{(c)_2}\left[\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\right]+(3p\beta+p-2\alpha\beta+\beta+1) \\
&\times\left(\frac{ab}{c}\right)\left[\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\right]+2p\beta(p-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right].
\end{aligned}
$$

Upon simplification, we see that this last expression is bounded above by $2p\beta(p-\alpha)$ if and only if (16) holds. That (16) is also necessary for F_p to be in $\mathcal{C}_p(\alpha,\beta)$ follows from Lemma 2 (ii). □

Theorem 4. If $a,b>-1, c>a+b+2$ and $ab<0$, then a necessary and sufficient condition for $h_p(a,b,c;z)$ to be in $\mathcal{C}_p(\alpha,\beta)$ is that

$$
\begin{aligned}
&(a)_2(b)_2(\beta+1)+ab(3p\beta+p-2\alpha\beta+\beta+1)(c-a-b-2) \\
&+2p\beta(p-\alpha)(c-a-b-2)_2\geq 0. \qquad (18)
\end{aligned}
$$

Proof. Since $h_p(a,b,c;z)$ has the form (13), we see from Lemma 2 (ii) that our conclusion is equivalent to

$$
\sum_{n=p+1}^{\infty}n\left[n-p+\beta(n+p-2\alpha)\right]\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}\leq 2p\beta\left|\frac{c}{ab}\right|(p-\alpha). \qquad (19)
$$

Note that $c > a+b+2$ if the left side of (19) converges.Writing

$$\begin{aligned} & (n+p+1)\left[n+1+\beta\left(n+2p-2\alpha+1\right)\right] \\ = & (\beta+1)(n+1)^2+(3p\beta+p-2\alpha\beta)(n+1)+2p\beta(p-\alpha), \end{aligned}$$

we see that

$$\begin{aligned} & \sum_{n=p+1}^{\infty} n\left[n-p+\beta\left(n+p-2\alpha\right)\right]\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} \\ = & \sum_{n=0}^{\infty}(n+p+1)\left[n+1+\beta\left(n+2p-2\alpha+1\right)\right]\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ = & (\beta+1)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(3p\beta+p-2\alpha\beta)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\ & +2p\beta(p-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ = & \frac{(a+1)(b+1)(\beta+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} \\ & +(3p\beta+p-2\alpha\beta+\beta+1)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+2p\beta(p-\alpha)\left(\frac{c}{ab}\right)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\ = & \left[\frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\right] \\ & \times\left[\begin{array}{c}(a+1)(b+1)(\beta+1) \\ +(3p\beta+p-2\alpha\beta+\beta+1)(c-a-b-2)+\frac{2p\beta(p-\alpha)(c-a-b-2)_2}{ab}\end{array}\right] \\ & -2p\beta(p-\alpha)\left(\frac{c}{ab}\right). \end{aligned}$$

This last expression is bounded above by $2p\beta(p-\alpha)\left|\frac{c}{ab}\right|$ if and only if

$$\begin{aligned} & (a+1)(b+1)(\beta+1)+(3p\beta+p-2\alpha\beta+\beta+1)(c-a-b-2) \\ & +\frac{2p\beta(p-\alpha)(c-a-b-2)_2}{ab}\leq 0, \end{aligned}$$

which is equivalent to (18). □

Putting $p=\beta=1$ in Theorem 4, we obtain the following result.

Corollary 1. If $a,b>-1, c>a+b+2$ and $ab<0$, then a necessary and sufficient condition for $h(a,b,c;z)$ to be in $\mathcal{C}(\alpha)$ if and only if

$$(a)_2(b)_2+ab(3-\alpha)(c-a-b-2)+(1-\alpha)(c-a-b-2)_2\geq 0.$$

Remark 1. Corollary 1, corrects the result given by Silverman [13, Theorem 4].

3. A Family of Integral Operators

In this section, we obtain similar results in connection with a particular integral operator $G_p(a,b,c;z)$ acting on $F(a,b,c;z)$ as follows

$$G_p(a,b,c;z) = p\int_0^z t^{p-1}F(a,b,c;z)dt = z^p + \sum_{n=1}^{\infty}\left(\frac{p}{p+n}\right)\frac{(a)_n\,(b)_n}{(c)_n\,(1)_n}z^{n+p}. \quad (20)$$

We note that $\frac{zG_p^{'}}{p} = h_p$.

Now $G_p(a,b,c;z) \in \mathcal{K}_p(\alpha,\beta)$ if and only if

$$\frac{zG_p^{'}(a,b,c;z)}{p} = h_p(a,b,c;z) \in \mathcal{S}_p^*(\alpha,\beta).$$

This follows upon observing that $\frac{zG_p^{'}(a,b,c;z)}{p} = h_p(a,b,c;z)$, $\frac{zG_p^{''}(a,b,c;z)}{p} = h_p^{'}(a,b,c;z) - \frac{1}{p}G_p^{'}(a,b,c;z)$, and so

$$1 + \frac{zG_p^{''}(a,b,c;z)}{G_p(a,b,c;z)} = \frac{zh_p^{'}(a,b,c;z)}{h_p(a,b,c;z)}.$$

Thus any $p-$valent starlike about $h_p(a,b,c;z)$ leads to a $p-$valent convex about $G_p(a,b,c;z)$. Thus from Theorems 1 and 2, we have

Theorem 5. (i) If $a,b>0$ and $c>a+b+1$, then a sufficient condition for $G_p(a,b,c;z)$ defined by (20) to be in $\mathcal{K}_p(\alpha,\beta)$, $0\le\alpha<p$ and $0<\beta\le 1$, is that

$$\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\left[1+\frac{ab\,(\beta+1)}{2\beta\,(p-\alpha)\,(c-a-b-1)}\right]\le 2. \quad (21)$$

(ii) If $a,b>-1$, $c>a+b+2$ and $ab<0$, then a necessary and sufficient condition for $G_p(a,b,c;z)$ to be in $\mathcal{C}_p(\alpha,\beta)$ is that $c\ge a+b+1-\frac{ab(\beta+1)}{2\beta(p-\alpha)}$.

Remark 2.

(i) Putting $\beta=1$ in the above results, we obtain the results of El-Ashwah et al. [14];

(ii) Putting $\beta=1$ and $p=1$ in the above results, we obtain the results of Silverman [24];

(iii) Putting $p=1$ in the above results, we obtain the results of Mostafa [21].

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A. O. Mostafa
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
E-mail address: adelaeg254@yahoo.com

G. M. El-Hawsh
Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt
E-mail address: gma05@fayoum.edu.eg