Electronic Journal of Mathematical Analysis and Applications
Vol. 8(1) Jan. 2020, pp. 327-335.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

ON THE ANALYTICAL AND NUMERICAL SOLUTIONS OF A MULTI-TERM NONLINEAR DIFFERENTIAL EQUATION WITH DEVIATED ARGUMENTS

E. A. A. ZIADA

Abstract. In this paper, we apply the two methods; Adomian decomposition method (ADM) and predictor-corrector method for solving a multi-term Deviated Nonlinear Differential Equation (DNDE). The existence and stability of a unique solution is proved. Convergence analysis of Adomian Decomposition Method (ADM) applied to these types of equations is discussed. Convergence analysis is reliable enough to estimate the maximum absolute truncated error of Adomian series solution.

1. Introduction

Deviated Nonlinear Differential Equations (DNDEs) arises in the context of nonlinear control, such as occurs in physiological systems and optical or neural network systems with delayed feedback [[1] -[9]]. In this paper, two methods are used to solve DNDEs. The first method is Adomian Decomposition Method (ADM) [[10] -[15]], this method has many advantages, it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization. The second method is the predictor-corrector method. The contribution of the work reported in this paper can be summarized in the following six points:

• Introducing the sufficient condition that guarantees the existence of a unique solution to our problem (see Theorem 1).

• Based on the above point and using Adomian polynomials formula suggested in [16], convergence of ADM is discussed (see Theorem 2).

• Using point two, the maximum absolute truncated error of the Adomian's series solution is estimated (see Theorem 3).

• Stability of the solution is discussed (see Theorem 4).

• Comparison between the results of the two methods (see the numerical examples).

• Preparation of an algorithm using MATHEMATICA package to solve the given numerical examples.

2010 *Mathematics Subject Classification.* 26D15, 26A51, 26A42.
Key words and phrases. Deviated nonlinear differential equation; Convergence analysis; Stability; Adomian method; Predictor-corrector method.
Submitted Aug. 18, 2019. Revised Dece. 5, 2019.

2. The Problem

Consider the following DNDE,

$$\begin{aligned} \frac{dx(t)}{dt} &= f(t, x(t), x(\phi_1(t)), \ldots, x(\phi_n(t))), \quad t \in [0,T] && (1)\\ x(0) &= x_0, \quad t \leq 0, && (2)\\ \phi_i(t) &\leq t, \qquad i = 1, 2, \cdots, n. \end{aligned}$$

where $x(t) \in C(J), J = [0,T]$ and f satisfies Lipschitz condition with Lipschitz constant k such as,

$$|f(t, x_1, x_2, \ldots, x_n) - f(t, y_1, y_2, \ldots, y_n)| \leq k \sum_{i=1}^{n} |x_i - y_i| \tag{3}$$

which implies that

$$\begin{aligned} &|f(t, x(t), x(\phi_1(t)), \ldots, x(\phi_n(t))) - f(t, y(t), y(\phi_1(t)), \ldots, y(\phi_n(t)))| \\ &\qquad\qquad\qquad \leq k \sum_{i=1}^{n} |x(\phi_i(t)) - y(\phi_i(t))| \end{aligned}$$

Operating with L^{-1} to both sides of equation (1), where $L^{-1}(.) = \int_0^t (.)\, dt$, we get

$$x(t) = x_0 + \int_0^t f(s, x(s), x(\phi_1(s)), \ldots, x(\phi_n(s)))\, ds, \tag{4}$$

3. The first method: ADM method

The solution algorithm of equation (4) using ADM is,

$$\begin{aligned} x_0(t) &= x_0, && (5)\\ x_m(t) &= \int_0^t A_{m-1}(s)\, ds. && (6) \end{aligned}$$

where A_m are Adomian polynomials of the nonlinear term $f(t, x(t), x(\phi_1(t)), \ldots, x(\phi_n(t)))$ which take the form,

$$A_m = \frac{1}{m!} \frac{d^m}{d\lambda^m} \left[f\left(t, \sum_{i=0}^{\infty} \lambda^i x_i(t), \sum_{i=0}^{\infty} \lambda^i x_i(\phi_1(t)), \ldots, \sum_{i=0}^{\infty} \lambda^i x_i(\phi_n(t))\right)\right]_{\lambda=0}$$

and the solution of problem (1)-(2) will be,

$$x(t) = \sum_{i=0}^{\infty} x_i(t) \tag{7}$$

4. Existence and uniqueness

Define the mapping $F : E \to E$ where E is the Banach space $(C(J), \|\cdot\|)$, the space of all continuous functions on J with the norm $\|x\| = \max_{t \in J} e^{-Nt} |x(t)|, N > 0$.

Theorem 1: *Let f satisfies the Lipschitz condition (3) then the DNDE has a unique solution $x \in E$.*

Proof: The mapping $F : E \to E$ is defined as,

$$Fx(t) = x_0 + \int_0^t f(s, x(s), x(\phi_1(s)), \ldots, x(\phi_n(s)))\, ds.$$

Let $x(t), y(t) \in E$, then

$$Fx(t) - Fy(t) = \int_0^t f(s, x(s), x(\phi_1(s)), \ldots, x(\phi_n(s))) - f(s, y(s), y(\phi_1(s)), \ldots, y(\phi_n(s)))\, ds$$

which implies that

$$\begin{aligned}
|Fx(t) - Fy(t)| &= \left| \int_0^t f(s, x(s), x(\phi_1(s)), \ldots, x(\phi_n(s))) \right. \\
&\qquad \left. - f(s, y(s), y(\phi_1(s)), \ldots, y(\phi_n(s)))\, ds \right| \\
e^{-Nt}|Fx(t) - Fy(t)| &\leq e^{-Nt} \int_0^t |f(s, x(s), x(\phi_1(s)), \ldots, x(\phi_n(s))) \\
&\qquad - f(s, y(s), y(\phi_1(s)), \ldots, y(\phi_n(s)))|\, ds \\
\max_{t \in J} e^{-Nt}|Fx(t) - Fy(t)| &\leq k \sum_{i=1}^n \max_{t \in J} \int_0^t e^{-N(t-\phi_i(s))} e^{-N\phi_i(s)} |x(\phi_i(s)) - y(\phi_i(s))|\, ds \\
\|Fx - Fy\| &\leq nk\|x - y\| \int_0^t e^{-N(t-\phi_i(s))} ds \\
&\leq nk\|x - y\| \int_0^t e^{-N(t-s)} ds \\
&\leq nk\left(\frac{1 - e^{-Nt}}{N}\right)\|x - y\| \\
&\leq \frac{nk}{N}\|x - y\|
\end{aligned}$$

Now choose N large enough such that $\beta = \frac{nk}{N} < 1$, then we get

$$\|Fx - Fy\| \leq \|x - y\|,$$

therefore the mapping F is contraction and there exists a unique solution $x \in C(J)$ of the problem (1)- (2) and this completes the proof. ■

Theorem 2: *The series solution (7) of the problem (1)- (2) using ADM converges if* $|x_1(t)| < c, c$ *is a positive constant.*

Proof: Define the sequence $\{S_p\}$ such that, $S_p = \sum_{i=0}^{p} x_i(t)$ is the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} x_i(t)$ since,

$$f(t, x(t), x(\phi_1(t)), \ldots, x(\phi_n(t))) = \sum_{i=0}^{\infty} A_i(t),$$

so we can write [[16]],

$$f(t, S_p(t), S_p(\phi_1(t)), \ldots, S_p(\phi_n(t))) = \sum_{i=0}^{p} A_i(t),$$

From equations (5) and (6) we have,

$$\sum_{i=0}^{\infty} x_i(t) = x_0 + \int_0^t \left(\sum_{i=0}^{\infty} A_{i-1}(s) \right) ds$$

Let S_p and S_q be two arbitrary partial sums with $p > q$, then we get,

$$S_p = \sum_{i=0}^{p} x_i(t) = x_0 + \int_0^t \left(\sum_{i=0}^{p} A_{i-1}(s) \right) ds$$

Now, we are going to prove that $\{S_p\}$ is a Cauchy sequence in this Banach space E.

$$\begin{aligned}
S_p(t) - S_q(t) &= \int_0^t \left(\sum_{i=0}^{p} A_{i-1}(s) \right) ds - \int_0^t \left(\sum_{i=0}^{q} A_{i-1}(s) \right) ds \\
&= \int_0^t \left[\sum_{i=q+1}^{p} A_{i-1}(s) \right] ds \\
&= \int_0^t \left[\sum_{i=q}^{p-1} A_i(s) \right] ds \\
&= \int_0^t [f(s, S_{p-1}(s), S_{p-1}(\phi_1(s)), \ldots, S_{p-1}(\phi_n(s))) \\
&\qquad - f(s, S_{q-1}(s), S_{q-1}(\phi_1(s)), \ldots, S_{q-1}(\phi_n(s)))] \, ds \\
|S_p(t) - S_q(t)| &= \left| \int_0^t [f(s, S_{p-1}(s), S_{p-1}(\phi_1(s)), \ldots, S_{p-1}(\phi_n(s))) \right. \\
&\qquad \left. - f(s, S_{q-1}(s), S_{q-1}(\phi_1(s)), \ldots, S_{q-1}(\phi_n(s)))] \, ds \right| \\
e^{-Nt} |S_p(t) - S_q(t)| &\leq e^{-Nt} \int_0^t |f(s, S_{p-1}(s), S_{p-1}(\phi_1(s)), \ldots, S_{p-1}(\phi_n(s))) \\
&\qquad - f(s, S_{q-1}(s), S_{q-1}(\phi_1(s)), \ldots, S_{q-1}(\phi_n(s)))| \, d\tau \\
\max_{t \in J} e^{-Nt} |S_p(t) - S_q(t)| &\leq k \sum_{i=1}^{n} \max_{t \in J} \int_0^t e^{-N(t-\phi_i(s))} e^{-N\phi_i(s)} |S_{p-1}(\phi_i(s)) - S_{q-1}(\phi_i(s))| \, d\tau \\
\|S_p - S_q\| &\leq \frac{nk}{N} \|S_{p-1} - S_{q-1}\| \\
&\leq \beta \|S_{p-1} - S_{q-1}\|
\end{aligned}$$

Let $p = q + 1$ then,

$$\|S_{q+1} - S_q\| \leq \beta \|S_q - S_{q-1}\| \leq \beta^2 \|S_{q-1} - S_{q-2}\| \leq \cdots \leq \beta^q \|S_1 - S_0\|$$

From the triangle inequality we have,

$$\begin{aligned}
\|S_p - S_q\| &\leq \|S_{q+1} - S_q\| + \|S_{q+2} - S_{q+1}\| + \cdots + \|S_p - S_{p-1}\| \\
&\leq \left[\beta^q + \beta^{q+1} + \cdots + \beta^{p-1}\right] \|S_1 - S_0\| \\
&\leq \beta^q \left[1 + \beta + \cdots + \beta^{p-q-1}\right] \|S_1 - S_0\| \\
&\leq \beta^m \left[\frac{1 - \beta^{p-q}}{1 - \beta} \right] \|x_1\|
\end{aligned}$$

Since, $0 < \beta < 1$, and $p > q$ then, $(1 - \beta^{p-q}) \leq 1$. Consequently,

$$\begin{aligned} \|S_p - S_q\| &\leq \frac{\beta^q}{1-\beta} \|x_1\| \\ &\leq \frac{\beta^q}{1-\beta} \max_{t \in J} e^{-Nt} |x_1(t)| \end{aligned}$$

but, $|x_1(t)| < c$ and as $q \to \infty$ then, $\|S_p - S_q\| \to 0$ and hence, $\{S_p\}$ is a Cauchy sequence in this Banach space E so, the series $\sum_{i=0}^{\infty} x_i(t)$ converges. ■

Theorem 3: *The maximum absolute truncation error of the series solution (7) to the problem (1)- (2) is estimated to be,*

$$\left\| x - \sum_{i=0}^{q} x_i \right\| \leq \frac{\beta^q}{1-\beta} \ \|x_1\|$$

Proof: From Theorem 2 we have,

$$\|S_p - S_q\| \leq \frac{\beta^q}{1-\beta} \max_{t \in J} e^{-Nt} |x_1(t)|$$

but, $S_p = \sum_{i=0}^{p} x_i(t)$ as $p \to \infty$ then, $S_p \to x(t)$ so,

$$\|x - S_q\| \leq \frac{\beta^q}{1-\beta} \ \|x_1\|$$

so, the maximum absolute truncation error in the interval J is,

$$\left\| x - \sum_{i=0}^{q} x_i \right\| \leq \frac{\beta^q}{1-\beta} \ \|x_1\|$$

and this completes the proof. ■

5. Stability of the solution

Theorem 4: *The solution of the problem (1)- (2) is uniformly stable.*

Proof: Let $x(t)$ be a solution of

$$x(t) = x_0 + \int_0^t f(s, x(s), x(\phi_1(s)), \ldots, x(\phi_n(s)))\, ds$$

and let $\widetilde{x}(t)$ be a solution of the above problem such that $\widetilde{x}(0) = \widetilde{x}_0$, then

$$\begin{aligned} x(t) - \widetilde{x}(t) &= x_0 - \widetilde{x}_0 \\ &\quad + \int_0^t [f(s, x(s), x(\phi_1(s)), \ldots, x(\phi_n(s))) - f(s, \widetilde{x}(s), \widetilde{x}(\phi_1(s)), \ldots, \widetilde{x}(\phi_n(s)))]\, ds \\ |x(t) - \widetilde{x}(t)| &\leq |x_0 - \widetilde{x}_0| \\ &\quad + \int_0^t |f(s, x(s), x(\phi_1(s)), \ldots, x(\phi_n(s))) - f(s, \widetilde{x}(s), \widetilde{x}(\phi_1(s)), \ldots, \widetilde{x}(\phi_n(s)))|\, ds \end{aligned}$$

$$\begin{aligned}
e^{-Nt}\left|x(t)-\widetilde{x}(t)\right| &\leq e^{-Nt}\left|x_0-\widetilde{x}_0\right| \\
&\quad +k\sum_{i=1}^{n}\int_0^t e^{-N(t-\phi_i(s))}e^{-N\phi_i(s)}\left|x(\phi_i(s))-\widetilde{x}(\phi_i(s))\right|ds
\end{aligned}$$

$$\begin{aligned}
\max_{t\in J} e^{-Nt}\left|x(t)-\widetilde{x}(t)\right| &\leq \max_{t\in J} e^{-Nt}\left|x_0-\widetilde{x}_0\right| \\
&\quad +k\sum_{i=1}^{n}\max_{t\in J}\int_0^t e^{-N(t-\phi_i(s))}e^{-N\phi_i(s)}\left|x(\phi_i(s))-\widetilde{x}(\phi_i(s))\right|ds \\
\left\|x-\widetilde{x}\right\| &\leq \left|x_0-\widetilde{x}_0\right| + nk\left\|x-\widetilde{x}\right\|\int_0^t e^{-N(t-s)}ds \\
&\leq \left|x_0-\widetilde{x}_0\right| + \frac{nk}{N}\left\|x-\widetilde{x}\right\| \\
\left(1-\frac{nk}{N}\right)\left\|x-\widetilde{x}\right\| &\leq \left|x_0-\widetilde{x}_0\right| \\
\left\|x-\widetilde{x}\right\| &\leq \left(1-\frac{nk}{N}\right)^{-1}\left|x_0-\widetilde{x}_0\right|
\end{aligned}$$

therefore, if $\left|x_0-\widetilde{x}_0\right| < \delta(\epsilon)$, then $\left\|x-\widetilde{x}\right\| < \epsilon$, which completes the proof. ■

6. The second method: Predictor-corrector method

Adams-type-predictor corrector method has been introduced in many references, see for example ([7]-[8]). In this section, we use Adams-type-predictor corrector method to solve the equations (1)-(2). The product trapezoidal quadrature formula is used $t_j.(j=0,1,.......k+1)$ taken with respect to the weight function $(t_{k+1}-.)^{\alpha-1}$ and the following approximation is applied:

$$\begin{aligned}
\int_{t_0}^{k+1} g\ (u)du &\approx \int_{t_0}^{k+1} g_{k+1}(u)du \\
&= \sum_{i=0}^{k+1}\tilde{a}_{j,k+1}g\ (t_{j)},
\end{aligned}$$

where,

$$\tilde{a}_{j,k+1}=\left\{\frac{h}{2}\left[\begin{array}{c} k^2-(k-1)(k+1), J=0 \\ (k-J+2)^2+(K-J)^2-2(K-J+1)^2, 1\leq J\leq K \\ 1 \end{array}\right]\right\},$$

and h is the step size, this yields the corrector-formula, i.e. the fractional variant of the one-step Adams Moulton method; the corrector formula is:

$$x_{k+1}=a(t_{k+1})+\left[\sum_{i=0}^{k}\tilde{a}_{j,k+1}f\ (t_j,x(t_j))+\tilde{a}_{k+1,k+1}f\ (t_{k+1},x^P(t_{k+1}))\right]$$

The remaining problem is the determination of the predictor formula that is need to calculate the X^P_{k+1}. The idea used to generalize the one step Adams–Bashforth method is the same as the one described above for the Adams-Moulton technique.

The integral on the right hand side of equation (4) is replaced by the following product rectangular rule:

$$\int_{t_0}^{k+1} g(u)du = \sum_{i=0}^{k} b_{j,k+1} g(t_j),$$

where,

$$b_{j,k+1} = h \ \ [(k+1-J)-(K-J)]$$

Thus, the Predictor x_{k+1}^{P} is determined by the fractional Adams-Bashforth method:

$$x_{k+1}^{P} = a(t_{k+1}) + \left[\sum_{i=0}^{k} b_{j,k+1} f(t_j, x(t_j))\right]$$

7. Numerical examples

Example 1 *Consider the following DNDE,*

$$\begin{aligned} \frac{dx\,(t)}{dt} &= \left(2t+\frac{t^2}{20}+t^4\right)-x^2\,(t)-\frac{1}{5}x\left(\frac{t}{2}\right), \quad t>0 \\ x(t) &= 0, \quad t\leq 0. \end{aligned} \tag{8}$$

which has the exact solution $\left(t^2\right)$. Applying ADM to equation (8), we have

$$x_0\,(t) = \int_0^t \left(2\tau+\frac{\tau^2}{20}+\tau^4\right) d\tau, \tag{9}$$

$$x_i\,(t) = \int_0^t \left(-A_{i-1}\,(\tau)-\frac{1}{5}x_{i-1}\left(\frac{\tau}{2}\right)\right) d\tau, \quad i\geq 1. \tag{10}$$

where A_i is Adomian polynomials of the nonlinear term $x^2\,(t)$. From equations (9)-(10), the solution of problem (8) is,$x(t) = \sum\limits_{i=0}^{m} x_i(t)$.

Table 1 shows the absolute error of ADM series solution $(m = 5)$ and $PECE$ solution.

Table1: Absolute Error

t	$\|x_{Exact} - x_{ADM}\|$	$\|x_{Exact} - x_{PECE}\|$
0.1	1.04321×10^{-20}	0.0000113735
0.2	1.18713×10^{-16}	0.0000429191
0.3	7.53583×10^{-14}	0.000115753
0.4	1.05908×10^{-11}	0.000258145
0.5	5.95277×10^{-10}	0.000506916
0.6	1.79337×10^{-8}	0.000929852
0.7	3.44187×10^{-7}	0.00160836
0.8	4.70735×10^{-6}	0.00270368
0.9	0.0000495637	0.0043627
1	0.000424683	0.0069078

Example 2 *Consider the following nonlinear DE,*

$$\begin{aligned} \frac{dx(t)}{dt} &= \left(3t^2+t^9+0.57884375t^3\right)-x^3(t)-\frac{1}{2}x(0.9t)-\frac{1}{4}x(0.95t), \quad t>0 \\ x(t) &= 0, \quad t\le 0. \end{aligned} \tag{11}$$

which has the exact solution $\left(t^3\right)$. Applying ADM to equation *(11)*, we have

$$x_0(t) = \int_0^t \left(3\tau^2+\tau^9+0.57884375\tau^3\right)d\tau, \tag{12}$$

$$x_i(t) = \int_0^t \left(-A_{i-1}(\tau)-\frac{1}{2}x_{i-1}(0.9\tau)-\frac{1}{4}x_{i-1}(0.95\tau)\right)d\tau, \quad i\ge 1. \tag{13}$$

where A_i is Adomian polynomials of the nonlinear term $x^3(t)$. From equations (12)- (13), the solution of problem (11) is,

$$x(t)=\sum_{i=0}^{m} x_i(t).$$

Table 2 shows the absolute error of ADM series solution ($m = 5$) and $PECE$ solution.

Table2: Absolute Error

t	$\lVert x_{Exact}-x_{ADM}\rVert$	$\lVert x_{Exact}-x_{PECE}\rVert$
0.1	7.94505×10^{-7}	0.0000148558
0.2	0.0000125885	0.0000609259
0.3	0.0000631091	0.000135945
0.4	0.000197459	0.000231983
0.5	0.000476659	0.000344733
0.6	0.000973614	0.000508431
0.7	0.00176027	0.000892094
0.8	0.00286937	0.0020606
0.9	0.00412554	0.00573262
1	0.00229944	0.0178213

References

[1] G. Chatzarakis, I.K. Purnaras and I. Stavroulakis, Oscillation tests for differential equations with deviating arguments. ,27, (2017), pp. 1-28.

[2] H. Akca, G.E. Chatzarakis and I. P. Stavrouiakis, An oscillation criterion for delay differential equations with several non-monotone arguments, Appl. Math. Lett., 59, (2016), pp. 101-108.

[3] E. Braverman, G.E. Chatzarakis and I. P. Stavrouiakis, Iterative oscillation tests for differential equations with several non-monotone arguments, Adv. Diference Equ. , (2016), pp. 18.

[4] G.E. Chatzarakis and I. P. Stavrouiakis, Differential equations with non-monotone arguments: Iterative oscillation results, J. Math. Comput., 6 (2016) No. 5, pp. 953-964.

[5] D. J. Evans, and K. R. Raslan, "The Adomian decomposition method for solving Delay differential equation", International Journal of Computer Mathematics, 82, (2005), pp. 49-54.

[6] D. Zwillinger, "Handbook of Differential Equations", Academic Press, (1997).

[7] B. Mensour, and A. Longtin, "Chaos control in multistable delay-differential equations and their singular limit maps", Pysical Review E, 58, (1998), pp. 410-422.

[8] J. M. Hefferan, and R. M. Corless, "Solving Some Delay Differential Equations with Computer Algebra", Applied Probability Trust, (2005), pp. 1-22.

[9] G. Adomian, "Solving Frontier Problems of Physics: The Decomposition Method", Kluwer, (1995).

[10] G. Adomian, "Stochastic System", Academic press, (1983).

[11] G. Adomian, "Nonlinear Stochastic Operator Equations", Academic press, San Diego, (1986).
[12] G. Adomian, "Nonlinear Stochastic Systems: Theory and Applications to Physics", Kluwer, (1989).
[13] K. Abbaoui, and Y. Cherruault, "Convergence of Adomian's method Applied to Differential Equations", Computers Math. Applic., 28, (1994), pp. 103-109.
[14] Y. Cherruault, G. Adomian, K. Abbaoui, and R. Rach, "Further remarks on convergence of decomposition method", International J. of Bio-Medical Computing., 38, (1995), pp. 89-93.
[15] I. L. El-kalla, "Error Analysis of Adomian Series Solution to a Class of Nonlinear Differential Equations", Applied Mathematics E-Notes, 7, (2007), pp. 214-221.
[16] I. L. El-kalla, "Convergence of the Adomian Method Applied to a Class of Nonlinear Integral Equations", Appl. Math. Letters, 21, (2008), pp. 372-376.

E. A. A. Ziada, Nile Higher Institute for Engineering & Technology, Mansoura, Egypt.
E-mail address: eng_emanziada@yahoo.com, dr.emanziada@gmail.com