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INCLUSION PROPERTIES FOR CERTAIN $k-$UNIFORMLY SUBCLASSES OF $p-$VALENT FUNCTIONS ASSOCIATED WITH THE LIU-OWA OPERATOR

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Abstract. In this paper, we introduce several $k-$uniformly subclasses of $p-$valent functions defined by the Liu-Owa integral operator and investigate various inclusion relationships for these subclasses. Some interesting applications involving certain classes of integral operators are also considered.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n\,(p\in\mathbb{N}=\{1,2,3,...\}) \tag{1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f\prec g$ or $f(z)\prec g(z)$, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U})$, such that $f(z)=g(\omega(z))$ $(z\in\mathbb{U})$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$ (see [1] and [2]).

For $0\leq\gamma,\eta<p,k\geq0$ and $z\in\mathbb{U}$, we define $US_p^*(k;\gamma)$, $UC_p(k;\gamma)$, $UK_p(k;\gamma,\eta)$ and $UK_p^*(k;\gamma,\eta)$ the $k-$uniformly subclasses of $\mathcal{A}_p$ consisting of all analytic functions which are, respectively, $p-$valent starlike of order γ, $p-$valent convex of order γ, $p-$valent close-to-convex of order γ, and type η and $p-$valent quasi-convex of order γ, and type η as follows:

$$US_p^*(k;\gamma)=\left\{f\in\mathcal{A}_p:\Re\left(\frac{zf'(z)}{f(z)}-\gamma\right)>k\left|\frac{zf'(z)}{f(z)}-p\right|\right\}, \tag{2}$$

$$UC_p(k;\gamma)=\left\{f\in\mathcal{A}_p:\Re\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right)>k\left|1+\frac{zf''(z)}{f'(z)}-p\right|\right\}, \tag{3}$$

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$$UK_p(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in US_p^*(k;\eta),\Re\left(\frac{zf'(z)}{g(z)}-\gamma\right)>k\left|\frac{zf'(z)}{g(z)}-p\right|\right\},\tag{4}$$

$$UK_p^*(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in UC_p(k;\eta),\Re\left(\frac{\left(zf'(z)\right)'}{g'(z)}-\gamma\right)>k\left|\frac{\left(zf'(z)\right)'}{g'(z)}-p\right|\right\}.\tag{5}$$

These subclasses were introduced and studied by Al-Kharsani [3]. We note that
(i) $US_1^*(k;\gamma)=US^*(k;\gamma)$ and $UC_1(k;\gamma)=UC(k;\gamma)\,(0\leq\gamma<1)$ (see [4] and [5]);
(ii) $US_p^*(0;\gamma)=S_p^*(\gamma)\ \ (0\leq\gamma<p)$ (see [6] and [7]);
(iii) $UC_p(0;\gamma)=C_p(\gamma)\,(0\leq\gamma<p)$ (see [6]);
(iv) $UK_p(0;\gamma,\eta)=K_p(\gamma,\eta)\,(0\leq\gamma,\eta<p)$ (see [8]);
(v) $UK_p^*(0;\gamma,\eta)=K_p^*(\gamma,\eta)\,(0\leq\gamma,\eta<p)$ (see [9]).

Corresponding to a conic domain $\Omega_{p,k,\gamma}$ defined by

$$\Omega_{p,k,\gamma}=\left\{u+iv:u>k\sqrt{(u-p)^2+v^2}+\gamma\right\},\tag{6}$$

we define the function $q_{p,k,\gamma}(z)$ which maps $\mathbb{U}$ onto the conic domain $\Omega_{p,k,\gamma}$ such that $1\in\Omega_{p,k,\gamma}$ as the following:

$$q_{k,\gamma}(z)=\begin{cases}\dfrac{p+(p-2\gamma)z}{1-z} & (k=0),\\[2ex] \frac{p-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}\left(\cos^{-1}k\right)i\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{k^2p-\gamma}{1-k^2} & (0<k<1),\\[2ex] p+\dfrac{2(p-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & (k=1),\\[2ex] \frac{p-\gamma}{k^2-1}\sin\left\{\dfrac{\pi}{2\zeta(k)}\int_0^{\frac{u(z)}{\sqrt{k}}}\dfrac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right\}+\frac{k^2p-\gamma}{k^2-1} & (k>1),\end{cases}\tag{7}$$

where $u(z)=\frac{z-\sqrt{x}}{1-\sqrt{x}z},x\in(0,1)$ and $\zeta(k)$ is such that $k=\cosh\frac{\pi\zeta'(z)}{4\zeta(z)}$. By virtue of the properties of the conic domain $\Omega_{p,k,\gamma}$, we have

$$\Re\{q_{p,k,\gamma}(z)\}>\frac{kp+\gamma}{k+1}.\tag{8}$$

Making use of the principal of subordination between analytic functions and the definition of $q_{p,k,\gamma}(z)$, we may rewrite the subclasses $US_p^*(k;\gamma)$, $UC_p(k;\gamma)$, $UK_p(k;\gamma,\beta)$ and $UK_p^*(k;\gamma,\beta)$ as the following:

$$US_p^*(k;\gamma)=\left\{f\in\mathcal{A}_p:\frac{zf'(z)}{f(z)}\prec q_{p,k,\gamma}(z)\right\},\tag{9}$$

$$UC_p(k;\gamma)=\left\{f\in\mathcal{A}_p:1+\frac{zf''(z)}{f'(z)}\prec q_{p,k,\gamma}(z)\right\},\tag{10}$$

$$UK_p(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in US_p^*(k;\eta),\frac{zf'(z)}{g(z)}\prec q_{p,k,\gamma}(z)\right\},\tag{11}$$

$$UK_p^*(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in UC_p(k;\eta),\ \frac{\left(zf'(z)\right)'}{g'(z)}\prec q_{p,k,\gamma}(z)\right\}. \tag{12}$$

Motivated essentially by Jung et al. [10], Liu and Owa [11] introduced the integral operator $Q_{\beta,p}^{\alpha}:\mathcal{A}_p\to\mathcal{A}_p\ (\alpha\geq 0,\beta>-p,p\in\mathbb{N})$ as follows (see also [12]):

$$Q_{\beta,p}^{\alpha}f(z)=\begin{cases}\binom{p+\alpha+\beta-1}{p+\beta-1}\frac{\alpha}{z^{\beta}}\int_0^z\left(1-\frac{t}{z}\right)^{\alpha-1}t^{\beta-1}f(t)\,dt & (\alpha>0),\\ f(z) & (\alpha=0).\end{cases} \tag{13}$$

For $f\in\mathcal{A}_p$ given by (1), then from (13), we deduce that

$$Q_{\beta,p}^{\alpha}f(z)=z^p+\frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)}\sum_{n=p+1}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_nz^n\ \ (\alpha\geq 0;\beta>-p;p\in\mathbb{N}). \tag{14}$$

It is easily verified from the definition (14) that

$$z\left(Q_{\beta,p}^{\alpha+1}f(z)\right)'=(\alpha+\beta+p)\,Q_{\beta,p}^{\alpha}f(z)-(\alpha+\beta)\,Q_{\beta,p}^{\alpha+1}f(z). \tag{15}$$

We note that

$$Q_{c,p}^{1}f(z)=F_{c,p}(f)(z)=\frac{c+p}{z^c}\int t^{c-1}f(z)\,dt\ \ (c>-p), \tag{16}$$

where the operator $F_{c,p}$ is the generalized Bernardi–Libera–Livingston integral operator (see [13] and [14]). Also, we note that the one-parameter family of integral operator $Q_{\beta,1}^{\alpha}=Q_{\beta}^{\alpha}$ was defined by Jung et al. [10] and studied by Aouf [15] and Gao et al. [16].

Next, using the operator $Q_{\beta,p}^{\alpha}$, we introduce the following $k-$uniformly classes of $p-$valent functions for $\alpha\geq 0,\beta>-p,p\in\mathbb{N},k\geq 0$ and $0\leq\gamma,\eta<p$:

$$US_p^*(\alpha;k;\gamma)=\left\{f\in\mathcal{A}_p:Q_{\beta,p}^{\alpha}f(z)\in US_p^*(k;\gamma);z\in\mathbb{U}\right\}, \tag{17}$$

$$UC_p(\alpha;k;\gamma)=\left\{f\in\mathcal{A}_p:Q_{\beta,p}^{\alpha}f(z)\in UC_p(k;\gamma);z\in\mathbb{U}\right\}, \tag{18}$$

$$UK_p(\alpha;k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:Q_{\beta,p}^{\alpha}f(z)\in UK_p(k;\gamma,\eta);z\in\mathbb{U}\right\}, \tag{19}$$

$$UK_p^*(\alpha;k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:Q_{\beta,p}^{\alpha}f(z)\in UK_p^*(k;\gamma,\eta);z\in\mathbb{U}\right\}. \tag{20}$$

We also note that

$$f\in US_p^*(\alpha;k;\gamma)\Leftrightarrow\frac{zf'}{p}\in UC_p(\alpha;k;\gamma), \tag{21}$$

and

$$f\in UK_p(\alpha;k;\gamma,\eta)\Leftrightarrow\frac{zf'}{p}\in UK_p^*(\alpha;k;\gamma,\eta). \tag{22}$$

In this paper, we investigate several inclusion properties of the classes $US_p^*(\alpha;k;\gamma)$, $UC_p(\alpha;k;\gamma)$, $UK_p(\alpha;k;\gamma,\eta)$, and $UK_p^*(\alpha;k;\gamma,\eta)$ associated with the operator $Q_{\beta,p}^{\alpha}$. Some applications involving integral operators are also considered.

2. Inclusion properties involving the operator $Q^{\alpha}_{\beta,p}$

In order to prove the main results, we shall need The following lemmas.
Lemma 1 [17]. Let $h(z)$ be convex univalent in $\mathbb{U}$ with $\Re\{\eta h(z)+\gamma\}>0\,(\eta,\gamma\in\mathbb{C})$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=h(0)$, then

$$p(z)+\frac{zp'(z)}{\eta p(z)+\gamma}\prec h(z) \tag{23}$$

implies

$$p(z)\prec h(z). \tag{24}$$

Lemma 2 [1]. Let $h(z)$ be convex univalent in $\mathbb{U}$ and let w be analytic in $\mathbb{U}$ with $\Re\{w(z)\}\geq 0$. If $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=h(0)$, then

$$p(z)+w(z)\,zp'(z)\prec h(z) \tag{25}$$

implies

$$p(z)\prec h(z). \tag{26}$$

Theorem 1. Let $(\alpha+\beta)(k+1)+kp+\gamma>0$. Then,

$$US^{*}_{p}(\alpha;k;\gamma)\subset US^{*}_{p}(\alpha+1;k;\gamma). \tag{27}$$

Proof. Let $f\in US^{*}_{p}(\alpha;k;\gamma)$ and set

$$p(z)=\frac{z\left(Q^{\alpha+1}_{\beta,p}f(z)\right)'}{Q^{\alpha+1}_{\beta,p}f(z)}\quad(z\in\mathbb{U}), \tag{28}$$

where the function $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$. Using (15), (27) and (28), we have

$$\frac{z\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{Q^{\alpha}_{\beta,p}f(z)}=p(z)+\frac{zp'(z)}{p(z)+\alpha+\beta}\prec q_{p,k,\gamma}(z). \tag{29}$$

Since $(\alpha+\beta)(k+1)+kp+\gamma>0$, we see that

$$\Re\{q_{p,k,\gamma}(z)+\alpha+\beta\}>0\quad(z\in\mathbb{U}). \tag{30}$$

Applying Lemma 1 to (29), it follows that $p(z)\prec q_{p,k,\gamma}(z)$, that is, $f\in US^{*}_{p}(\alpha+1;k;\gamma)$. Therefore, we complete the proof of Theorem 1. □

Theorem 2. Let $(\alpha+\beta)(k+1)+kp+\gamma>0$. Then,

$$UC_{p}(\alpha;k;\gamma)\subset UC_{p}(\alpha+1;k;\gamma). \tag{31}$$

Proof. Applying (21) and Theorem 1, we observe that

$$\begin{aligned} f\in UC_{p}(\alpha;k;\gamma) &\iff \tfrac{zf'}{p}\in US^{*}_{p}(\alpha;k;\gamma)\\ &\Longrightarrow \tfrac{zf'}{p}\in US^{*}_{p}(\alpha+1;k;\gamma)\quad\text{(by Theorem 1)},\\ &\iff f\in UC_{p}(\alpha+1;k;\gamma),\end{aligned}$$

which evidently proves Theorem 2. □

Next, by using Lemma 2, we obtain the following inclusion relation for $UK_{p}(\alpha;k;\gamma,\eta)$.
Theorem 3. Let $(\alpha+\beta)(k+1)+kp+\eta>0$. Then,

$$UK_{p}(\alpha;k;\gamma,\eta)\subset UK_{p}(\alpha+1;k;\gamma,\eta). \tag{32}$$

Proof. Let $f\in UK_p(\alpha;k;\gamma,\eta)$. Then, from the definition of $UK_p(\alpha;k;\gamma,\eta)$, there exists a function $r(z)\in US_p^*(k;\eta)$ such that

$$\frac{z\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{r(z)}\prec q_{p,k,\gamma}(z). \tag{33}$$

Choose the function g such that $Q^{\alpha}_{\beta,p}g(z)=r(z)$. Then, $g\in US_p^*(\alpha;k;\eta)$ and

$$\frac{z\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{Q^{\alpha}_{\beta,p}g(z)}\prec q_{p,k,\gamma}(z). \tag{34}$$

Now let

$$p(z)=\frac{z\left(Q^{\alpha+1}_{\beta,p}f(z)\right)'}{Q^{\alpha+1}_{\beta,p}g(z)}\quad(z\in\mathbb{U}), \tag{35}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$. Since $g\in US_p^*(\alpha;k;\eta)$, by Theorem 1, we know that $g\in US_p^*(\alpha+1;k;\eta)$. Let

$$t(z)=\frac{z\left(Q^{\alpha+1}_{\beta,p}g(z)\right)'}{Q^{\alpha+1}_{\beta,p}g(z)}\quad(z\in\mathbb{U}), \tag{36}$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\}>\dfrac{kp+\eta}{k+1}$. Also, from (35), we note that

$$Q^{\alpha+1}_{\beta,p}zf'(z)=Q^{\alpha+1}_{\beta,p}g(z)\;p(z). \tag{37}$$

Differentiating both sides of (37) with respect to z, we obtain

$$\begin{aligned}\frac{z\left(Q^{\alpha+1}_{\beta,p}zf'(z)\right)'}{Q^{\alpha+1}_{\beta,p}g(z)} &= \frac{z\left(Q^{\alpha+1}_{\beta,p}g(z)\right)'}{Q^{\alpha+1}_{\beta,p}g(z)}p(z)+zp'(z)\\ &= t(z)p(z)+zp'(z).\end{aligned} \tag{38}$$

Now using the identity (15) and (36), we obtain

$$\begin{aligned}\frac{z\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{Q^{\alpha}_{\beta,p}g(z)} &= \frac{Q^{\alpha}_{\beta,p}zf'(z)}{Q^{\alpha}_{\beta,p}g(z)}=\frac{z\left(Q^{\alpha+1}_{\beta,p}zf'(z)\right)'+(\alpha+\beta)Q^{\alpha+1}_{\beta,p}zf'(z)}{z\left(Q^{\alpha+1}_{\beta,p}g(z)\right)'+(\alpha+\beta)Q^{\alpha+1}_{\beta,p}g(z)}\\ &= \frac{\dfrac{z\left(Q^{\alpha+1}_{\beta,p}zf'(z)\right)'}{Q^{\alpha+1}_{\beta,p}g(z)}+(\alpha+\beta)\dfrac{z\left(Q^{\alpha+1}_{\beta,p}f(z)\right)'}{Q^{\alpha+1}_{\beta,p}g(z)}}{\dfrac{z\left(Q^{\alpha+1}_{\beta,p}g(z)\right)'}{Q^{\alpha+1}_{\beta,p}g(z)}+\alpha+\beta}\\ &= \frac{t(z)p(z)+zp'(z)+(\alpha+\beta)p(z)}{t(z)+\alpha+\beta}\\ &= p(z)+\frac{zp'(z)}{t(z)+\alpha+\beta}.\end{aligned} \tag{39}$$

Since $(\alpha+\beta)(k+1)+kp+\gamma>0$ and $\Re\{t(z)\}>\dfrac{kp+\eta}{k+1}$, we see that

$$\Re\{t(z)+\alpha+\beta\}>0 \quad (z\in\mathbb{U}).$$

Hence, applying Lemma 2, we can show that $p(z)\prec q_{p,k,\gamma}(z)$ so that $f\in UK_p(\alpha;k;\gamma,\eta)$. Therefore, we complete the proof of Theorem 3. □

Theorem 4. Let $(\alpha+\beta)(k+1)+kp+\eta>0$. Then,

$$UK_p^*(\alpha;k;\gamma,\eta)\subset UK_p^*(\alpha+1;k;\gamma,\eta). \tag{2.18}$$

Proof. Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (21), we can also prove Theorem 4 by using Theorem 3 and the equivalence (22). □

3. Inclusion properties involving the integral operator $F_{c,p}$

In this section, we consider the generalized Libera integral operator $F_{c,p}$ defined by (16).

Theorem 5. Let $c>-p$ and $0\le\gamma<p$. If $f\in US_p^*(\alpha;k;\gamma)$, then $F_{c,p}(f)\in US_p^*(\alpha;k;\gamma)$.

Proof. Let $f\in US_p^*(\alpha;k;\gamma)$ and set

$$p(z)=\frac{z\left(Q_{\beta,p}^{\alpha}F_{c,p}(f)(z)\right)'}{Q_{\beta,p}^{\alpha}F_{c,p}(f)(z)} \quad (z\in\mathbb{U}), \tag{40}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$. From (16), we have

$$z\left(Q_{\beta,p}^{\alpha}F_{c,p}(f)(z)\right)'=(c+p)\,Q_{\beta,p}^{\alpha}f(z)-cQ_{\beta,p}^{\alpha}F_{c,p}(f)(z). \tag{41}$$

Then, by using (40) and (41), we obtain

$$(c+p)\frac{Q_{\beta,p}^{\alpha}f(z)}{Q_{\beta,p}^{\alpha}F_{c}(f)(z)}=p(z)+c. \tag{42}$$

Taking the logarithmic differentiation on both sides of (42) and multiplying by z, we have

$$\frac{z\left(Q_{\beta,p}^{\alpha}f(z)\right)'}{Q_{\beta,p}^{\alpha}f(z)}=p(z)+\frac{zp'(z)}{p(z)+c}\prec q_{k,\gamma}(z) \quad (z\in\mathbb{U}). \tag{43}$$

Hence, by virtue of Lemma 1, we conclude that $p(z)\prec q_{k,\gamma}(z)$ in $\mathbb{U}$, which implies that $F_{c,p}(f)\in US_p^*(\alpha;k;\gamma)$. □

Next, we derive an inclusion property involving $F_{c,p}(f)$, which is given by the following.

Theorem 6. Let $c>-p$ and $0\le\gamma<p$. If $f\in UC_p(\alpha;k;\gamma)$, then $F_{c,p}(f)\in UC_p(\alpha;k;\gamma)$.

Proof. By applying Theorem 5, it follows that

$$\begin{aligned} f\in UC_p(\alpha;k;\gamma) &\Longleftrightarrow \frac{zf'}{p}\in US_p^*(\alpha;k;\gamma) \\ &\Longrightarrow F_{c,p}\left(\frac{zf'}{p}\right)\in US_p^*(\alpha;k;\gamma) \quad (by\ Theorem\ 5) \\ &\Longleftrightarrow \frac{z\left(F_{c,p}(f)\right)'}{p}\in US_p^*(\alpha;k;\gamma) \\ &\Longleftrightarrow F_{c,p}(f)\in UC_p(\alpha;k;\gamma), \end{aligned} \tag{44}$$

which proves Theorem 6. □

Theorem 7. Let $c>-p$ and $0\le\gamma,\eta<p$. If $f\in UK_p(\alpha;k;\gamma,\eta)$, then $F_{c,p}(f)\in UK_p(\alpha;k;\gamma,\eta)$.

Proof. Let $f\in UK_p(\alpha;k;\gamma,\eta)$. Then, in view of the definition of the class $UK_p(\alpha;k;\gamma,\eta)$, there exists a function $g\in US_p^*(\alpha;k;\eta)$ such that

$$\frac{z\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{Q^{\alpha}_{\beta,p}g(z)}\prec q_{k,\gamma}(z). \tag{45}$$

Thus, we set

$$p(z)=\frac{z\left(Q^{\alpha}_{\beta,p}F_{c,p}(f)(z)\right)'}{Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)}\quad (z\in\mathbb{U}), \tag{46}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$. Since $g\in US_p^*(\alpha;k;\gamma)$, we see from Theorem 5 that $F_{c,p}(f)\in US_p^*(\alpha;k;\gamma)$. Let

$$t(z)=\frac{z\left(Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)\right)'}{Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)}\quad (z\in\mathbb{U}), \tag{47}$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\}>\dfrac{kp+\eta}{k+1}$. Also, from (46), we note that

$$Q^{\alpha}_{\beta,p}zF'_{c,p}(f)(z)=Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)\ .\ p(z). \tag{48}$$

Differentiating both sides of (48) with respect to z, we obtain

$$\begin{aligned} \frac{z\left(Q^{\alpha}_{\beta,p}zF'_{c,p}(f)(z)\right)'}{Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)} &= \frac{z\left(Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)\right)'}{Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)}p(z)+zp'(z) \\ &= t(z)\,p(z)+zp'(z). \end{aligned} \tag{49}$$

Now using the identity (41) and (49), we obtain

$$\begin{aligned}
\frac{z\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{Q^{\alpha}_{\beta,p}g(z)} &= \frac{z\left(Q^{\alpha}_{\beta,p}zF'_{c,p}(f)(z)\right)' + cQ^{\alpha}_{\beta,p}zF'_{c,p}(f)(z)}{z\left(Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)\right)' + cQ^{\alpha}_{\beta,p}F_{c,p}(g)(z)} \\
&= \frac{\dfrac{z\left(Q^{\alpha}_{\beta,p}zF'_{c,p}(f)(z)\right)'}{Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)} + c\dfrac{z\left(Q^{\alpha}_{\beta,p}F_{c,p}(f)(z)\right)'}{Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)}}{\dfrac{z\left(Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)\right)'}{Q^{\alpha}_{\beta,p}F_{c,p}(g)(z)} + c} \\
&= \frac{t(z)p(z) + zp'(z) + cp(z)}{t(z)+c} \\
&= p(z) + \frac{zp'(z)}{t(z)+c}. \qquad (50)
\end{aligned}$$

Since $c(k+1) + kp + \eta \geq 0$ and $\Re\{t(z)\} > \dfrac{kp+\eta}{k+1}$, we see that

$$\Re\{t(z)+c\} > 0 \quad (z \in \mathbb{U}). \qquad (51)$$

Hence, applying Lemma 2 to (50), we can show that $p(z) \prec q_{p,k,\gamma}(z)$ so that $f \in UK_p(\alpha; k; \gamma, \eta)$. □

Theorem 8. Let $c > -p$ and $0 \leq \gamma, \eta < p$. If $f \in UK^*_p(\alpha; k; \gamma, \eta)$, then $F_{c,p}(f)\, UK^*_p(\alpha; k; \gamma, \eta)$.

Proof. Just as we derived Theorem 6 as consequence of Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7. □

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