Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 109-114
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

ON THE STABILITY OF A SYSTEM OF DIFFERENCE EQUATIONS

FARIDA BELHANNACHE

Abstract. This paper is devoted to discuss the stability of positive solutions of a system of two nonlinear difference equations. Our discussion is based on the method of linearization.

1. Introduction

Many natural phenomena are described by difference equations see, for example, [2, 13, 20, 22] and the references cited therein. Nonlinear difference equations are important in practical classes of difference equations. In most of the cases, it is difficult to find exact solution of a nonlinear difference equation. Recently, many researchers have investigated the behavior of the solutions of rational difference equations and systems.
In [22] the authors discussed the behavior of the solutions of the system

$$x_{n+1} = \frac{x_n + x_{n-1}}{A + y_n y_{n-1}}, \; y_{n+1} = \frac{y_n + y_{n-1}}{B + x_n x_{n-1}}, \; n = 0, 1, ...,$$

Belhannache et al. [3] investigated the global behavior of the rational third-order difference equation

$$x_{n+1} = \frac{A + Bx_{n-1}}{C + Dx_n^p x_{n-2}^q}, \; n = 0, 1, ...$$

Inspired and motivated by the above mentioned papers, our aim in this paper is to investigate the asymptotic behavior of

$$x_{n+1} = \frac{x_n + x_{n-1}}{A + y_n^p y_{n-1}^q}, \; y_{n+1} = \frac{y_n + y_{n-1}}{B + x_n^p x_{n-1}^q}, \; n = 0, 1, ..., \tag{1}$$

where the initial conditions x_{-1}, x_0, y_{-1}, y_0 are non-negative real numbers, the parameters A, B are positive real numbers and p, q, r are fixed positive integers.

1991 *Mathematics Subject Classification.* 39A10, 40A05.

Key words and phrases. System of rational difference equations; Stability; Global behavior; Oscillatory.

Submitted Oct. 22, 2019.

2. Preliminaries

In this section, we present some preliminaries that we will use in the sequal, for more details, we refer to [5], [12], [15], [17] and [19].
Let I, J be two intervals of real numbers and

$$f : I^2 \times J^2 \to I, \ g : I^2 \times J^2 \to J,$$

be two continuously differentiable functions. Then for every set of initial conditions $(x_i,\, y_i) \in I \times J$, $i = -1, 0$, the following system

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), \\ y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}), \end{cases} \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{(x_n,\ y_n)\}_{n=-1}^{+\infty}$.

Definition 2.1. *A point $(\overline{x}, \overline{y}) \in I \times J$ is called an equilibrium point of system (2) if*

$$\overline{x} = f(\overline{x}, \overline{x}, \overline{y}, \overline{y}),$$

and

$$\overline{y} = g(\overline{x}, \overline{x}, \overline{y}, \overline{y}).$$

We can rewrit the system (2) in the following vector form,

$$X_{n+1} = F(X_n), \ n = 0, 1, ..., \tag{3}$$

where

$$\begin{array}{llll} F: & I^2 \times J^2 & \longrightarrow & I^2 \times J^2 \\ & \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} & \longmapsto & F\left(\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} f(u_1, v_1, u_2, v_2) \\ u_1 \\ g(u_1, v_1, u_2, v_2) \\ u_2 \end{pmatrix} \end{array}$$

and

$$X_n = (x_n,\, x_{n-1},\, y_n,\, y_{n-1})^T .$$

Definition 2.2. *A vector $\overline{X} \in I^2 \times J^2$ is called an equilibrium point of system (3) if*

$$\overline{X} = F(\overline{X}).$$

Remark 2.3. *Clearly, $(\overline{x}, \overline{y})$ is an equilibrium point of system (2) if and only if $\overline{X} = (\overline{x}, \overline{x}, \overline{y}, \overline{y})$ is an equilibrium point of system (3).*

Definition 2.4. *Let $\overline{X}$ be an equilibrium point of system (3),*

$$X_0 = (x_0, x_{-1}, y_0, y_{-1}),$$

and $\| . \|$ any norm.

- **(i):** *The equilibrium point $\overline{X}$ is called stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\| X_0 - \overline{X} \| < \delta$ implies*

  $$\| X_n - \overline{X} \| < \epsilon \text{ for } n \geq 0.$$

  Otherwise the equilibrium $\overline{X}$ is called unstable.
- **(ii):** *The equilibrium point $\overline{X}$ is called asymptotically stable if it is stable and there exists $\gamma > 0$ such that $\left\|X_0 - \overline{X}\right\| < \gamma$ implies*

  $$\lim_{n \to +\infty} \| X_n - \overline{X} \| = 0.$$

(iii): *The equilibrium point* $\overline{X}$ *is called globally asymptotically stable relative to* $I^2 \times J^2$ *if it is asymptotically stable and, if for every* $X_0 \in I^2 \times J^2$*, we have*

$$\lim_{n\to+\infty} \parallel X_n - \overline{X} \parallel = 0.$$

The linearized system, associated to system (3), about the equilibrium point

$$\overline{X} = (\overline{x}, \overline{x}, \overline{y}, \overline{y}),$$

is given by

$$X_{n+1} = CX_n,\ n = 0, 1, ..., \tag{4}$$

where C is the Jacobian matrix of the map F at the equilibrium point $\overline{X}$.

Theorem 2.5 ([12])**.** *Let* $\overline{X}$ *be an equilibrium point of system (4).*

(i): *If all eigenvalues of the Jacobian matrix* C *lie inside the open unit disk then* $\overline{X}$ *is asymptotically stable.*

(ii): *If at least one of eigenvalues of the Jacobian matrix* C *has absolute value greater than one, then* $\overline{X}$ *is unstable.*

Definition 2.6. *Let* $\{(x_n, y_n)\}_{n=-1}^{+\infty}$ *be a solution of system (2) and* $(\overline{x}, \overline{y})$ *an equilibrium point of the same system.*

(i): *A function* x_n *(resp.* y_n*) oscillates about* $\overline{x}$ *(resp.* $\overline{y}$*) if for every* $\tau \in N$ *there exist* s, $m \in N$, $s \geq \tau$, $m \geq \tau$ *such that* $(x_s - \overline{x})(x_m - \overline{x}) \leq 0$ *(resp.* $(y_s - \overline{y})(y_m - \overline{y}) \leq 0$*).*

(ii): *We say that* $\{(x_n, y_n)\}_{n=-1}^{+\infty}$ *oscillates about* $(\overline{x}, \overline{y})$ *if* x_n *oscillates about* $\overline{x}$ *or* y_n *oscillates about* $\overline{y}$*.*

(iii): $\{(x_n, y_n)\}_{n=-1}^{+\infty}$ *is called non-oscillatory if both* x_n *and* y_n *are not oscillatory.*

3. Main Results

In this section we will present and prove our main results. First, we note that $(\overline{x}_1, \overline{y}_1) = (0, 0)$ is always an equilibrium point of system (1) and when $A < 2$ and $B < 2$, system (1) also possesses the unique positive equilibrium $(\overline{x}_2, \overline{y}_2) = (\sqrt[p+q]{2-B},\ \sqrt[p+q]{2-A})$.

Theorem 3.1. *Assume that* $A > 2$ *and* $B > 2$*. Then the equilibrium point* $(\overline{x}_1, \overline{y}_1) = (0, 0)$ *of system (1) is globally asymptotically stable.*

Proof. The linearized system of (1) about the equilibrium point $(\overline{x}_1, \overline{y}_1) = (0, 0)$ is

$$X_{n+1} = C\ X_n, \tag{5}$$

where $X_n = (x_n, x_{n-1}, y_n, y_{n-1})$ and

$$C = \begin{pmatrix} \frac{1}{A} & \frac{1}{A} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{B} & \frac{1}{B} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of system (5) is

$$P(\lambda) = P_1(\lambda)P_2(\lambda) = 0, \tag{6}$$

where

$$P_1(\lambda) = \lambda^2 - \frac{1}{A}\lambda - \frac{1}{A}$$

and

$$P_2(\lambda) = \lambda^2 - \frac{1}{B}\lambda - \frac{1}{B}.$$

It is clear that

$$P_1(0) = -\frac{1}{A} < 0, \ \ P_1(-1) = 1 > 0 \quad \text{and} \quad P_1(1) = 1 - \frac{2}{A} > 0.$$

Hence all solutions of the equation $P_1(\lambda) = 0$ lie inside the unit disk.
Similarly we obtain that all the solutions of the equation $P_2(\lambda) = 0$ lie inside the unit disk. Then all the solutions of the characteristic equation (6) lie inside the unit disk. So the unique equilibrium $(0, 0)$ is asymptotically stable.
Now we shall prove that $\underset{n\to+\infty}{lim} x_n = 0$. From system (1) we get

$$x_{n+1} \leq \frac{1}{A}x_n + \frac{1}{A}x_{n-1}.$$

Let $\{z_n\}_{n=-1}^{+\infty}$ be the solution of the following linear difference equation

$$z_{n+1} = \frac{1}{A}z_n + \frac{1}{A}z_{n-1} \tag{7}$$

such that $z_0 = x_0$ and $z_{-1} = x_{-1}$. Then,

$$x_n \leq z_n, \ \forall n \geq 0.$$

It is clear that $A > 2$ implies $\lim_{n\to+\infty} z_n = 0$. Hence

$$\underset{n\to+\infty}{lim} x_n = 0.$$

Similarly we can prove that $\underset{n\to\infty}{lim} y_n = 0$ and so $(0, 0)$ is a global attractor.
The global asymptotically stability of $(0, 0)$ is obtained by combining the global attractivity and the asymptotic stability of $(0, 0)$ when $A > 2$ and $B > 2$. ∎

Theorem 3.2. *If $A < 2$ and $B < 2$. Then the equilibrium points $(\overline{x}_1, \overline{y}_1) = (0, 0)$ and $(\overline{x}_2, \overline{y}_2) = (\sqrt[p+q]{2 - B}, \sqrt[p+q]{2 - A})$ of system (1) are unstable.*

Proof. **(i):** It is clear that if $A < 2$ and $B < 2$ we have

$$\begin{aligned} \lim_{\lambda\to+\infty} P_1(\lambda) &= +\infty, \ P_1(1) = 1 - \frac{2}{A} < 0, \ \lim_{\lambda\to+\infty} P_2(\lambda) = +\infty, \\ P_2(1) &= 1 - \frac{2}{B} < 0. \end{aligned}$$

Hence the characteristic equation (6) has at least a root with absolute value greater than one. Therefore, the equilibrium point $(\overline{x}_1, \overline{y}_1) = (0, 0)$ is unstable.

(ii): The linearized system of system (1) about the equilibrium point $(\overline{x}_2, \overline{y}_2) = (\sqrt[p+q]{2 - B}, \sqrt[p+q]{2 - A})$ is

$$X_{n+1} = C\ X_n, \tag{8}$$

where

$$C = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -p\alpha & -q\alpha \\ 1 & 0 & 0 & 0 \\ -p\mu & -q\mu & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$\alpha = \frac{1}{2}(2-A)^{\frac{p+q-1}{p+q}}(2-B)^{\frac{1}{p+q}}$ and $\mu = \frac{1}{2}(2-B)^{\frac{p+q-1}{p+q}}(2-A)^{\frac{1}{p+q}}$. The characteristic equation of system (8) is

$$P_3(\lambda) = \lambda^4 - \lambda^3 - (\frac{3}{4} + \alpha\mu p^2)\lambda^2 + (\frac{1}{2} - 2\alpha\mu pq)\lambda + \frac{1}{4} - \alpha\mu q^2 = 0.$$

Then

$$P_3(1) = -\alpha\mu p^2 - 2\alpha\mu pq - \alpha\mu q^2 < 0$$

and $\lim_{\lambda\to+\infty} P_3(\lambda) = +\infty$. Hence P_3 has a root λ_1 in $(1, +\infty)$, which completes the proof.

■

In the following result, we are concerned with the oscillation of positive solutions of system (1) about the equilibrium point $(\overline{x}_2, \overline{y}_2)$.

Theorem 3.3. *Assume that* $A < 2$, $B < 2$. *Let* $\{(x_n, y_n)\}_{n=-1}^{+\infty}$ *be a solution of system (1) such that*

(i): x_{-1}, $x_0 \geq \overline{x}_2$, y_{-1}, $y_0 < \overline{y}_2$ *or*
(ii): x_{-1}, $x_0 < \overline{x}_2$, y_{-1}, $y_0 \geq \overline{y}_2$.

Then $\{(x_n, y_n)\}_{n=-1}^{+\infty}$ *non-oscillates about the equilibrium point* $(\overline{x}_2, \overline{y}_2)$.

Proof. Assume that case (i) holds, the case (ii) is similar and will be omitted. From (1), we have

$$\begin{aligned} x_1 &= \frac{x_0 + x_{-1}}{1 + y_0^p y_{-1}^q} > \frac{2\overline{x}_2}{1 + \overline{y}_2^{p+q}} = \overline{x}_2, \\ y_1 &= \frac{y_0 + y_{-1}}{1 + x_0^p x_{-1}^q} < \frac{2\overline{y}_2}{1 + \overline{x}_2^{p+q}} = \overline{y}_2. \end{aligned}$$

Then the result follows by induction. ■

For confirming the results of this paper, we give the following numerical example.

Example 3.4. *Figure* (1) *shows the behavior of the solutions of system (1) with the initial conditions* $x_{-1} = 2.73$, $x_0 = 0.47$, $y_{-1} = 0.15$, $y_0 = 1.18$ *and the parameters* $A = 3.2$, $B = 4.35$, $p = 1$, $q = 3$.

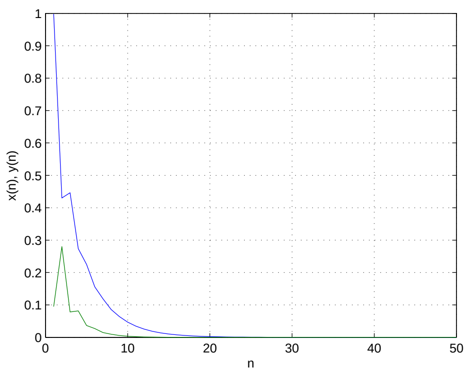

Figure 1.

References

[1] H. Bao, *On a System of Second-Order Nonlinear Difference Equations*, J. Appl. Math. Phys. **3** (2015), 903-910.

[2] J.R. Beddington, C.A. Free, J.H. Lawton, *Dynamic complexity in predator prey models framed in difference equations*, Nature. **255** (1975), 58-60.

[3] F. Belhannache, N. Touafek, R. Abo-Zeid, *Dynamics of a third-order rational difference equation*, Bull. Math. Soc. Sci. Math. Roum. vol. **59** (107) (1) (2016), 13-22.

[4] Q. Din, *Asymptotic Behavior of a second-order Fuzzy Rational Difference Equation*, Jour. Disc. Math. vol. 2015, Article ID 524931, 7 pages (2015).

[5] S. Elaydi, *An Introduction to Difference Equations*, third edition, Undergraduate Texts in Mathematics, Springer, New York, 1999.

[6] E. M. Elsayed, *Solutions of Rational Difference System of Order Two*, Math. Comp. Mode. **55** (2012), 378-384.

[7] E. M. Elsayed, *Behavior and expression of the solutions of some rational difference equations*, J. Comp. Anal. Appl. **15** (1) (2013), 73-81.

[8] E. M. Elsayed, *Solution for systems of difference equations of rational form of order two*, Comp. App. Math. **33** (3) (2014), 751-765.

[9] E. M. Elsayed, *On the solutions and periodic nature of some systems of difference equations*, Int. J. Bioma. **7** (6) (2014), 1450067.

[10] E. M. Elsayed and A. M. Ahmed, *Dynamics of a three-dimensional systems of rational difference equations*, Math. Meth. Appl. Sci. **39** (5) (2016), 1026-1038.

[11] E. M. Elsayed and T. F. Ibrahim, *Periodicity and solutions for some systems of nonlinear rational difference equations*, Hac. J. Math. Stat. **44** (6) (2015), 1361-1390.

[12] E. A. Grove, G. Ladas, *Periodicities in nonlinear difference equations*, Adv. Disc. Math. Appl. Volume 4, Chapman and Hall, CRS Press, 2005.

[13] E.A. Grove, G. Ladas, N.R. Prokup, R. Levis, *On the global behavior of solutions of a biological model*, Commun. Appl. Nonl. Anal. **7** (2) (2000), 33-46.

[14] A. Q. Khan, Q. Din, M. N. Qureshi, and T. F. Ibrahim, *Global behavior of an anti-competitive system of fourth-order rational difference equations*, Comp. Eco. Sof. **4** (1) (2014), 35-46.

[15] V. L. Kocic, G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[16] M.R.S. Kulenovic and O. Merino, *Discrete dynamical systems and difference equations with mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.

[17] M. R. S. Kulenovic, G. Ladas, *Dynamics of second order rational difference equations with open problems and conjectures*, Chapman and Hall, CRC Press, 2001.

[18] A. S. Kurbanli, *On the behavior of solutions of the system of rational difference equations:* $x_{n+1} = x_{n-1}/x_{n-1}y_n - 1,\ y_{n+1} = y_{n-1}/y_{n-1}x_n - 1,\ z_{n+1} = z_{n-1}/z_{n-1}y_n - 1$, Dis. Dyn. Nat. Soc. Volume 2011, Article ID 932362, 12 pages.

[19] V. Lakshmikantham, D. Trigiante, *Theory of difference equations: Numerical methods and applications*, MarcelDekker, Inc. New York-Basel, Second edition, 2002.

[20] S. Stević, *On a Discrete Epidemic Model*, Dis. Dyn. Nat. Soc. **10** (2007), Article ID 87519.

[21] D.C. Zhang, B. Shi, *Oscillation and global asymptotic stability in a discrete epidemic model*, J. Math. Anal. Appl. **278** (2003), 194-202.

[22] Q. Zhang, W. Zhang, *On the System of Nonlinear Rational Difference Equations*, Int. J. Math. Comp. Phy. Elec. Comp. Eng. **8** (4) (2014), 702-705.

Farida Belhannache

LMPA Laboratory, Mathematics Department, Mohammed Seddik Ben Yahia University, Jijel 18000 Algeria

E-mail address: fbelhannache@yahoo.fr