Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 128-139
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

ON SOME IMAGE FORMULAS FOR GENERALIZED LOMMEL WRIGHT FUNCTION INVOLVING A GENERAL CLASS OF POLYNOMIALS

SANGEETA CHOUDHARY

Abstract. The objective of this paper is to apply generalized fractional integral operators given by Marichev-Saigo-Maeda, to the product of a general class of polynomial and generalized Lommel- Wright function. We also establish some new image formulas by implementing a new class of unified integrals to the above mentioned product. The results are expressed in terms of generalized Wright function. Some new results and a number of known results can be easily found as special cases of our main results.

1. INTRODUCTION & PRELIMINARIES

Special functions have great importance in pure and applied mathematics. The wide use of these functions has attracted many researchers to work on them in different directions. The integral formulas involving various special functions have gained importance due to the usefulness of these results in the evaluation of generalized integrals and generalized derivatives and the solution of differential and integral equation. In recent years, a remarkably large number of integral formulas involving a variety of special function have been developed by many authors [[1]-[7]]. In the present work, we aim at finding generalized integral formulas for the product of generalized Lommel-Wright function with a general class of polynomial, which are expressed in terms of the generalized (Wright's) hypergeometric functions. For our purpose, we begin by recalling some known functions, their definitions and earlier works.

2010 *Mathematics Subject Classification.* 26A33, 33C20, 44A20.
Key words and phrases. generalized fractional calculus operators; generalized Lommel-Wright function; generalized Wright hypergeometric function; a general class of polynomials; Lavoie-Trottier integral formula, Oberhettinger formula.
Submitted May 30, 2019. Revised Nov. 23, 2019.

The generalized Lommel-Wright function defined by de'Oteiza et al. [[8]] is represented as

$$\begin{aligned} J_{\omega,\upsilon}^{\mu,k}(z) &= \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(r+1)}{\Gamma(\upsilon+r+1)^k \Gamma(\omega+r\mu+\upsilon+1) r!} \left(\frac{z}{2}\right)^{\omega+2\upsilon+2r} \\ &= \left(\frac{z}{2}\right)^{\omega+2\upsilon} {}_1\Psi_{k+1} \left[(1,1); \underbrace{(\upsilon+1,1)}_{k-times}, (\omega+\upsilon+1,\mu); \frac{-z^2}{4} \right] \quad (1) \\ & z \in \mathrm{C}\backslash(-\infty,0],\ \mu>0,\ k\in \mathrm{N},\ \omega,\upsilon \in \mathrm{C} \end{aligned}$$

where ${}_p\Psi_q(z)$ is the generalized Wright hypergeometric function introduced by Wright [[9]] by the following series

$${}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q} \Gamma(b_j+\beta_j n)} \frac{z^n}{n!} \quad (2)$$

where $z, a_i, b_j \in \mathrm{C}$ and $\alpha_i, \beta_j \in \mathrm{R}-\{0\}\ (i=1,2,...,p; j=1,2,...q)$ and $\sum_{i=1}^{p}\alpha_i - \sum_{j=1}^{q}\beta_j \leq 1$.

Also we have the following relations of generalized Lommel-Wright functions with trigonometric functions and generalized Bessel function (see e.g. [10],p.353 , [11], p. 27, eq.(1.161)) and Struve function (see e.g., [11], p. 28, eq. (1.170)) :

$$J_{1/2,0}^{1,1}(z) = \sqrt{\left(\frac{2}{\pi z}\right)} \sin(z) \quad (3)$$

$$J_{-1/2,0}^{1,1}(z) = \sqrt{\left(\frac{2}{\pi z}\right)} \cos(z) \quad (4)$$

$$J_{\omega,\upsilon}^{\mu,1}(z) = J_{\omega,\upsilon}^{\mu}(z) \quad (5)$$

$$J_{\omega,1/2}^{1,1}(z) = H_{\omega}(z) \quad (6)$$

$$J_{\omega,0}^{1,1}(z) = J_{\omega}(z) \quad (7)$$

A lot of research work has been carried out to study various generalizations and special cases of Lommel-Wright function (see, for details, [12]-[14]).

The general class of polynomials introduced by Srivastava [[15], p. 1, eq.(1)] are represented in the following manner:

$$S_n^m(x) = \sum_{l=0}^{[n/m]} \frac{(-n)_{ml}}{l!} A_{n,l} x^l, \quad n=0,1,2,... \quad (8)$$

where n is an arbitrary positive integer and the coefficients $A_{n,l}$ $(n,l \geq 0)$ are arbitrary constants, real or complex.

The generalized fractional integral operators involving Appell's function or Horns function F_3 are introduced by Marichev [[16]] and later extended and studied by Saigo and Maeda [[17]] in the following form.

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathrm{C}$ and $x>0$ and $\mathrm{R}(\gamma)>0$, then

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt \quad (9)$$

and

$$\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^{\infty}(t-x)^{\gamma-1}t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\,dt \tag{10}$$

Many interesting applications of fractional integral operators in applicable mathematical analysis can be notably found in [[18], [19]].
Further, the image formulas for a power function, under operators (9) and (10) are given by (see [17], p. 394, eq. (4.18) and (4.19))

$$\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right](x)=\Gamma\left[\begin{array}{c}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho-\alpha'+\beta'\\ \rho+\beta',\rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta\end{array}\right]x^{\rho-\alpha-\alpha'+\gamma-1} \tag{11}$$

where $\mathrm{R}(\gamma)>0$ and $\mathrm{R}(\rho)>\max\{0,\mathrm{R}(\alpha+\alpha'+\beta-\gamma),\mathrm{R}(\alpha'-\beta')\}$
Also,

$$\left[I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right](x)=\Gamma\left[\begin{array}{c}1+\alpha+\alpha'-\gamma-\rho,1+\alpha+\beta'-\gamma-\rho,1-\beta-\rho\\ 1-\rho,1+\alpha+\alpha'+\beta'-\gamma-\rho,1+\alpha-\beta-\rho\end{array}\right]x^{\rho+\gamma-\alpha-\alpha'-1} \tag{12}$$

where $\mathrm{R}(\gamma)>0$ and $\mathrm{R}(\rho)<1+\min\{\mathrm{R}(-\beta),\mathrm{R}(\alpha+\beta'-\gamma),\mathrm{R}(\alpha+\alpha'-\gamma)\}$.
and $\Gamma\left[\begin{array}{c}a,b,c\\ d,e,f\end{array}\right]=\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(d)\,\Gamma(e)\,\Gamma(f)}$
We further recall the following known result of Lavoie and Trottier [[20]]:

$$\int_0^1 t^{\alpha-1}(1-t)^{2\beta-1}\left(1-\frac{t}{3}\right)^{2\alpha-1}\left(1-\frac{t}{4}\right)^{\beta-1}dt=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{13}$$

where $\mathrm{R}(\alpha)>0$ and $\mathrm{R}(\beta)>0$.
For our present investigation, we next recall following Oberhettinger's integral formula [[21]]:

$$\int_0^{\infty}x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta}dx=2\delta a^{-\delta}\left(\frac{a}{2}\right)^{\eta}\frac{\Gamma(2\eta)\,\Gamma(\delta-\eta)}{\Gamma(1+\delta+\eta)} \tag{14}$$

provided $0<\mathrm{R}(\eta)<\mathrm{R}(\delta)$
Integral formulas involving Lommel-Wright functions have been developed by many authors (see, e.g. [[22]-[25]]). In the present paper, we obtain some fractional integrals along with unified integral formulas for the product of a general class of polynomial and generalized Lommel-Wright function.

2. Image formulas for fractional integral operators

Here, we establish image formulas for the product of generalized Lommel-Wright function $J_{\omega,\upsilon}^{\mu,k}(\cdot)$ and a general class of polynomials $S_n^m(\cdot)$ involving Saigo-Maeda fractional integral operators given by equation (9) and (10).
Theorem 2.1: Let $\alpha,\alpha',\beta,\beta',\gamma,\upsilon\in\mathrm{C}$, $k\in\mathrm{N}$, $\mu>0$ and $x>0$ be such that $\mathrm{R}(\gamma)>0$ and $\mathrm{R}(\omega)>-1$;$\mathrm{R}(\rho+\omega\xi+2\upsilon\xi)>\max\{0,\mathrm{R}(\alpha+\alpha'+\beta-\gamma),\mathrm{R}(\alpha'-\beta')\}$, then the Left sided generalized fractional integration of generalized Lommel-Wright

function $J_{\omega,\upsilon}^{\mu,k}(\cdot)$ involving general class of polynomials $S_n^m(\cdot)$ is given by

$$\begin{aligned}&\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S_n^m\left(\sigma t^{\lambda}\right)J_{\omega,\upsilon}^{\mu,k}\left(\delta t^{\xi}\right)\right](x)\\&=x^{P-\alpha-\alpha'+\gamma-1}\left(\frac{\delta}{2}\right)^{\omega+2\upsilon}\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}\left(\sigma x^{\lambda}\right)^{l}\\&\times{}_4\Psi_{4+k}\left[\begin{array}{c|c}\begin{array}{c}(1,1)\,(P+\lambda l,2\xi)\,(P+\gamma-\alpha-\alpha'-\beta+\lambda l,2\xi)\\(P-\alpha'+\beta'+\lambda l,2\xi)\\(P+\gamma-\alpha-\alpha'+\lambda l,2\xi)\,(P+\gamma-\alpha'-\beta+\lambda l,2\xi)\\(P+\beta'+\lambda l,2\xi)\,(\omega+\upsilon+1,\mu)\underbrace{(\upsilon+1,1)}_{k-times}\end{array}&\frac{-\left(\delta x^{\xi}\right)^2}{4}\end{array}\right]\end{aligned}\tag{15}$$

where $P=\rho+\omega\xi+2\upsilon\xi$

Proof*:* On using definitions (8) and (1), writing the functions in series form, the LHS of equation (15) becomes

$$\begin{aligned}&\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S_n^m\left(\sigma t^{\lambda}\right)J_{\omega,\upsilon}^{\mu,k}\left(\delta t^{\xi}\right)\right](x)\\&=I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}\left(\sigma t^{\lambda}\right)^{l}\right.\\&\left.\times\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(r+1)}{\Gamma(\upsilon+r+1)^k\Gamma(\omega+r\mu+\upsilon+1)\,r!}\left(\frac{\delta t^{\xi}}{2}\right)^{\omega+2\upsilon+2r}t^{\rho-1}\right](x)\end{aligned}$$

Now, upon interchanging the order of integration and summation which is possible under given conditions, we obtain

$$\begin{aligned}&\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S_n^m\left(\sigma t^{\lambda}\right)J_{\omega,\upsilon}^{\mu,k}\left(\delta t^{\xi}\right)\right](x)\\&=\left[\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}(\sigma)^{l}\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(r+1)}{\Gamma(\upsilon+r+1)^k\Gamma(\omega+r\mu+\upsilon+1)\,r!}\right.\\&\left.\times\left(\frac{\delta}{2}\right)^{\omega+2\upsilon+2r}I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho+\omega\xi+2\upsilon\xi+2r\xi+\lambda l-1}\right](x)\end{aligned}$$

Further using the image formula (11), we get

$$\begin{aligned}&\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S_n^m\left(\sigma t^{\lambda}\right)J_{\omega,\upsilon}^{\mu,k}\left(\delta t^{\xi}\right)\right](x)=x^{P-\alpha-\alpha'+\gamma-1}\left(\frac{\delta}{2}\right)^{\omega+2\upsilon}\\&\times\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}\left(\sigma x^{\lambda}\right)^{l}\sum_{r=0}^{\infty}\frac{\Gamma(r+1)\,\Gamma(P+\lambda l+2r\xi)}{\Gamma(P+\gamma-\alpha-\alpha'+\lambda l+2r\xi)\,\Gamma(P+\gamma-\alpha'-\beta+\lambda l+2r\xi)}\\&\times\frac{\Gamma(P+\gamma-\alpha-\alpha'-\beta+\lambda l+2r\xi)\,\Gamma(P-\alpha'+\beta'+\lambda l+2r\xi)}{\Gamma(P+\beta'+\lambda l+2r\xi)\,\Gamma(\omega+\upsilon+1+\mu r)\,\Gamma(\upsilon+r+1)^k}\left(\frac{-\delta^2x^{2\xi}}{4}\right)^{r}\frac{1}{r!}\end{aligned}$$

Interpreting the RHS of above equation, in view of the definition (2), we arrive at result (15).

Theorem 2.2: Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon \in \mathrm{C}$, $k \in \mathrm{N}$, $\mu > 0$ and $x > 0$ be such that $\mathrm{R}(\gamma) > 0$ and $\mathrm{R}(\omega) > -1$; and $R(1-\eta-\rho-\omega\xi-2\upsilon\xi) < 1+\min\{R(-\beta), R(\alpha+\beta'-\gamma), R(\alpha+\alpha'-\gamma)\}$, then generalized fractional integration $I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of product of $J_{\omega,\upsilon}^{\mu,k}(\cdot)$ and $S_n^m(\cdot)$ is given by

$$\left[I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{-\rho-\eta} S_n^m\left(\sigma t^{\lambda}\right) J_{\omega,\upsilon}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x)$$
$$= x^{-P-\alpha-\alpha'+\gamma}\left(\frac{\delta}{2}\right)^{\omega+2\upsilon} \sum_{l=0}^{[n/m]} \frac{(-n)_{ml}}{l!} A_{n,l}\left(\sigma x^{\lambda}\right)^{l}$$
$$\times {}_4\Psi_{4+k}\left[\begin{array}{c|c} (1,1)\,(P+\alpha+\alpha'-\gamma-\lambda l,2\xi)\,(P+\alpha+\beta'-\gamma-\lambda l,2\xi) & \\ (P-\beta-\lambda l,2\xi) & \dfrac{-\left(\delta x^{-\xi}\right)^2}{4} \\ (P-\lambda l,2\xi)\,(P+\alpha+\alpha'+\beta'-\gamma-\lambda l,2\xi) & \\ (P+\alpha-\beta-\lambda l,2\xi)\,(\omega+\upsilon+1,\mu)\,\underbrace{(\upsilon+1,1)}_{k-times} & \end{array}\right] \tag{16}$$

where $P = \rho+\eta+\omega\xi+2\upsilon\xi$

Proof*:* On using equation (8) and (1) in the LHS of equation (16) and then interchanging order of integration and summation, we obtain

$$\left[I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{-\rho-\eta} S_n^m\left(\sigma t^{\lambda}\right) J_{\omega,\upsilon}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x)$$
$$= \left[\sum_{l=0}^{[n/m]} \frac{(-n)_{ml}}{l!} A_{n,l}(\sigma)^{l} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(r+1)}{\Gamma(\upsilon+r+1)^k \Gamma(\omega+r\mu+\upsilon+1)\, r!} \times \left(\frac{\delta}{2}\right)^{\omega+2\upsilon+2r} I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{-\rho-\eta-\omega\xi+2\upsilon\xi+2r\xi-\lambda l}\right](x)$$

which on using image formula (12), becomes

$$\left[I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{-\rho-\eta} S_n^m\left(\sigma t^{\lambda}\right) J_{\omega,\upsilon}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x)$$
$$= x^{-P-\alpha-\alpha'+\gamma}\left(\tfrac{\delta}{2}\right)^{\omega+2\upsilon} \sum_{l=0}^{[n/m]} \tfrac{(-n)_{ml}}{l!} A_{n,l}\left(\sigma x^{\lambda}\right)^{l}$$
$$\times \sum_{r=0}^{\infty} \frac{\Gamma(r+1)\Gamma\left(P+\alpha+\alpha'-\gamma-\lambda l+2r\xi\right)}{\Gamma(P+\alpha+\alpha'+\beta'-\gamma-\lambda l+2r\xi)\Gamma(P+\alpha-\beta-\lambda l+2r\xi)}$$
$$\times \frac{\Gamma\left(P+\gamma+\alpha+\beta'-\gamma-\lambda l+2r\xi\right)\Gamma(P-\beta-\lambda l+2r\xi)}{\Gamma(P-\lambda l+2r\xi)\Gamma(\omega+\upsilon+1+\mu r)\Gamma(\upsilon+r+1)^k}\left(\frac{-\delta^2 x^{-2\xi}}{4}\right)^r \frac{1}{r!}$$

Interpreting the RHS of above equation, in view of the definition (2), we arrive at result (16).

3. Generalized integral formulas

Theorem 3.1: Let $\alpha, \beta, \upsilon, \omega \in \mathrm{C}$, $k \in \mathrm{N}$, $\mu > 0$ and $x > 0$ be such that $\mathrm{R}(\alpha) > 0$ and $\mathrm{R}(\beta) > 0$ then for the product of generalized Lommel-Wright function $J_{\omega,\upsilon}^{\mu,k}(\cdot)$ and a general class of polynomials $S_n^m(\cdot)$ following integral formula holds true

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}$$
$$\times S_n^m\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right] J_{\omega,\upsilon}^{\mu,k}\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$$
$$=\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\left(\frac{y}{2}\right)^{\omega+2\upsilon}\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}y^l$$
$$\times {}_2\Psi_{2+k}\left[\begin{array}{c}(1,1)\,(\beta+l+\omega+2\upsilon,2)\\ (\alpha+\beta+l+\omega+2\upsilon,2)\,(\omega+\upsilon+1,\mu)\,\underbrace{(\upsilon+1,1)}_{k-times}\end{array}\middle|\,\frac{-y^2}{4}\right] \tag{17}$$

Proof: On using (8) and (1) in integrand of (17) and interchanging order of integration and summation which is valid under the conditions stated in the theorem, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{\beta-1}$$
$$\times S_n^m\left[y\left(1-\tfrac{x}{4}\right)(1-x)^2\right] J_{\omega,\upsilon}^{\mu,k}\left[y\left(1-\tfrac{x}{4}\right)(1-x)^2\right]dx$$
$$=\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}y^l\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(r+1)}{\Gamma(\upsilon+r+1)^k\Gamma(\omega+r\mu+\upsilon+1)r!}\left(\tfrac{y}{2}\right)^{\omega+2\upsilon+2r}$$
$$\times\int_0^1 x^{\alpha-1}(1-x)^{2(\beta+l+\omega+2\upsilon+2r)-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{\beta+l+\omega+2\upsilon+2r-1}dx$$

Now using result (13) in above equation, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{\beta-1}$$
$$\times S_n^m\left[y\left(1-\tfrac{x}{4}\right)(1-x)^2\right] J_{\omega,\upsilon}^{\mu,k}\left[y\left(1-\tfrac{x}{4}\right)(1-x)^2\right]dx$$
$$=\left(\tfrac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\left(\tfrac{y}{2}\right)^{\omega+2\upsilon}\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}y^l$$
$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r+1)\Gamma(\beta+l+\omega+2\upsilon+2r)}{\Gamma(\upsilon+r+1)^k\Gamma(\omega+r\mu+\upsilon+1)\Gamma(\alpha+\beta+l+\omega+2\upsilon+2r)}\left(\frac{-y^2}{4}\right)^r\frac{1}{r!}$$

Interpreting the RHS of above equation, in view of definition (2), we arrive at the result (17).

Theorem 3.2: Let $\alpha,\beta,\upsilon,\omega\in$ C, $k\in$ N, $\mu>0$ and $x>0$ be such that $\mathrm{R}(\alpha)>0$ and $\mathrm{R}(\beta)>0$ then for the product of generalized Lommel-Wright function $J_{\omega,\upsilon}^{\mu,k}(\cdot)$ involving $S_n^m(\cdot)$ following integral formula holds true

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{\beta-1}S_n^m\left[yx\left(1-\tfrac{x}{3}\right)^2\right]J_{\omega,\upsilon}^{\mu,k}\left[yx\left(1-\tfrac{x}{3}\right)^2\right]dx$$
$$=\left(\tfrac{2}{3}\right)^{2\alpha+\omega+2\upsilon}y^{\omega+2\upsilon}\Gamma(\beta)\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}\left(\tfrac{4}{9}y\right)^l$$
$$\times{}_2\Psi_{2+k}\left[\begin{array}{c}(1,1)\,(\alpha+l+\omega+2\upsilon,2)\\ (\alpha+\beta+l+\omega+2\upsilon,2)\,(\omega+\upsilon+1,\mu)\,\underbrace{(\upsilon+1,1)}_{k-times}\end{array}\middle|\,\frac{-4y^2}{81}\right] \tag{18}$$

Proof: On using (8) and (1) in the integrand of equation (18) and then interchanging the order of integration and summation, which is valid under the conditions stated in theorem, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} S_n^m\left[yx\left(1-\frac{x}{3}\right)^2\right] J_{\omega,\upsilon}^{\mu,k}\left[yx\left(1-\frac{x}{3}\right)^2\right] dx$$
$$=\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}y^l\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(r+1)}{\Gamma(\upsilon+r+1)^k\Gamma(\omega+r\mu+\upsilon+1)r!}\left(\frac{y}{2}\right)^{\omega+2\upsilon+2r}$$
$$\times\int_0^1 x^{\alpha+\omega+2\upsilon+l+2r-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2(\alpha+\omega+2\upsilon+l+2r)-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx$$

which on using image formula (13), becomes

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} S_n^m\left[yx\left(1-\frac{x}{3}\right)^2\right] J_{\omega,\upsilon}^{\mu,k}\left[yx\left(1-\frac{x}{3}\right)^2\right] dx$$
$$=\left(\frac{2}{3}\right)^{2(\alpha+\omega+2\upsilon)}\left(\frac{y}{2}\right)^{\omega+2\upsilon}\Gamma(\beta)\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}y^l\left(\frac{2}{3}\right)^{2l}$$
$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r+1)\Gamma(\alpha+l+\omega+2\upsilon+2r)}{\Gamma(\upsilon+r+1)^k\Gamma(\omega+r\mu+\upsilon+1)\Gamma(\alpha+\beta+l+\omega+2\upsilon+2r)}\left(\frac{-4y^2}{81}\right)^r\frac{1}{r!}$$

Interpreting the RHS of above equation, in view of definition (2), we arrive at the result (18).

Theorem 3.3 The following integral formula holds true for $\upsilon,\omega\in \mathrm{C}$, $k\in \mathrm{N}$, $\mu>0$ and $x>0$ with $0<\mathrm{R}(\eta)<\mathrm{R}(\delta)$:

$$\int_0^\infty x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta} S_n^m\left[\frac{y}{x+a+\sqrt{x^2+2ax}}\right] J_{\omega,\upsilon}^{\mu,k}\left[\frac{y}{x+a+\sqrt{x^2+2ax}}\right] dx$$
$$=\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}y^{l+\omega+2\upsilon}2^{1-\omega-2\upsilon-\eta}\Gamma(2\eta)a^{\eta-(\delta+l+\omega+2\upsilon)}$$
$${}_3\Psi_{3+k}\left[\left.\begin{matrix}(1,1)\,(\delta+l+\omega+2\upsilon-\eta,2)\,(\delta+l+\omega+2\upsilon+1,2)\\ \underbrace{(\upsilon+1,1)}_{k-times}(\omega+\upsilon+1,\mu)\,(1+\delta+l+\omega+2\upsilon+\eta,2)\,(\delta+l+\omega+2\upsilon,2)\end{matrix}\right|\frac{-y^2}{4a^2}\right] \tag{19}$$

Proof: On using (8) and (1), in the integrand of equation (19) and then interchanging the order of integration and summation, which is verified under the given conditions, we get

$$\int_0^\infty x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta} S_n^m\left[\frac{y}{x+a+\sqrt{x^2+2ax}}\right] J_{\omega,\upsilon}^{\mu,k}\left[\frac{y}{x+a+\sqrt{x^2+2ax}}\right] dx$$
$$=\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}\,y^l\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(r+1)}{\Gamma(\upsilon+r+1)^k\Gamma(\omega+r\mu+\upsilon+1)r!}\left(\frac{y}{2}\right)^{\omega+2\upsilon+2r}$$
$$\times\int_0^\infty x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta-l-\omega-2\upsilon-2r}dx$$

Further applying image formula (14), we arrive at

$$\int_0^\infty x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta} S_n^m\left[\frac{y}{x+a+\sqrt{x^2+2ax}}\right] J_{\omega,\upsilon}^{\mu,k}\left[\frac{y}{x+a+\sqrt{x^2+2ax}}\right] dx$$
$$=\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}y^{l+\omega+2\upsilon}2^{1-\omega-2\upsilon-\eta}\Gamma(2\eta)a^{\eta-(\delta+l+\omega+2\upsilon)}$$
$$\times\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(r+1)\Gamma(\delta+l+\omega+2\upsilon+2r-\eta)\Gamma(\delta+l+\omega+2\upsilon+2r+1)}{\Gamma(\upsilon+r+1)^k\Gamma(\omega+r\mu+\upsilon+1)\Gamma(1+\delta+l+\omega+2\upsilon+2r+\eta)\Gamma(\delta+l+\omega+2\upsilon+2r)r!}\left(\frac{y}{2a}\right)^{2r}$$

which in light of definition (2) gives the formula (19).

Theorem 3.4 The following integral formula holds true for $\upsilon, \omega \in \mathrm{C}$, $k \in \mathrm{N}$, $\mu > 0$ and $x > 0$ with $0 < \mathrm{R}(\eta) < \mathrm{R}(\delta)$:

$$\begin{aligned}
&\int_0^\infty x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta} S_n^m\left[\tfrac{xy}{x+a+\sqrt{x^2+2ax}}\right] J_{\omega,\upsilon}^{\mu,k}\left[\tfrac{xy}{x+a+\sqrt{x^2+2ax}}\right] dx \\
&= \sum_{l=0}^{[n/m]} \tfrac{(-n)_{ml}}{l!} A_{n,l} y^{l+\omega+2\upsilon} 2^{1-2\omega-4\upsilon-\eta-l}\, \Gamma(\delta-\eta)\, a^{\eta-\delta} \\
&{}_3\Psi_{3+k}\left[\begin{array}{c} (1,1)\,(2\omega+4\upsilon+2\eta+2l,4)\,(1+\delta+l+\omega+2\upsilon,2) \\ \underbrace{(\upsilon+1,1)}_{k-times}(\omega+\upsilon+1,\mu)\,(1+\delta+2l+2\omega+4\upsilon+\eta,4)\,(\delta+l+\omega+2\upsilon,2) \end{array}\middle|\ \tfrac{-y^2}{16}\right]
\end{aligned} \tag{20}$$

Proof: On using (8) and (1), in the integrand of equation (20) and then interchanging the order of integration and summation, which is verified under the given conditions, we get

$$\begin{aligned}
&\int_0^\infty x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta} S_n^m\left[\tfrac{xy}{x+a+\sqrt{x^2+2ax}}\right] J_{\omega,\upsilon}^{\mu,k}\left[\tfrac{xy}{x+a+\sqrt{x^2+2ax}}\right] dx \\
&= \sum_{l=0}^{[n/m]} \tfrac{(-n)_{ml}}{l!} A_{n,l} y^l \sum_{r=0}^\infty \tfrac{(-1)^r \Gamma(r+1)}{\Gamma(\upsilon+r+1)^k \Gamma(\omega+r\mu+\upsilon+1) r!} \left(\tfrac{y}{2}\right)^{\omega+2\upsilon+2r} \\
&\quad \times \int_0^\infty x^{\eta+l+\omega+2\upsilon+2r-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta-l-\omega-2\upsilon-2r} dx
\end{aligned}$$

Further applying image formula (14), we arrive at

$$\begin{aligned}
&\int_0^\infty x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta} S_n^m\left[\tfrac{y}{x+a+\sqrt{x^2+2ax}}\right] J_{\omega,\upsilon}^{\mu,k}\left[\tfrac{y}{x+a+\sqrt{x^2+2ax}}\right] dx \\
&= \sum_{l=0}^{[n/m]} \tfrac{(-n)_{ml}}{l!} A_{n,l} y^{l+\omega+2\upsilon} 2^{1-2\omega-4\upsilon-\eta-l}\, \Gamma(\delta-\eta)\, a^{\eta-\delta} \\
&\quad \times \sum_{r=0}^\infty \tfrac{(-1)^r \Gamma(r+1)\Gamma(\delta+l+\omega+2\upsilon+2r+1)\Gamma(2\omega+4\upsilon+2\eta+2l+4r)}{\Gamma(\upsilon+r+1)^k \Gamma(\omega+r\mu+\upsilon+1)\Gamma(1+\delta+2\omega+4\upsilon+\eta+2l+4r)\Gamma(\delta+l+\omega+2\upsilon+2r) r!} \left(\tfrac{y}{4}\right)^{2r}
\end{aligned}$$

which in light of definition (2) gives the formula (20).

4. Special cases

In this section, we derive some integral formulas involving trigonometric functions and generalized Bessel function and struve function.

Corollory 4.1: Setting $\mu = 1, k = 1, \omega = 1/2, \upsilon = 0$ in (17) and (18) and then by using (3), we obtain the following integral formulas:

$$\begin{aligned}
&\int_0^1 x^{\alpha-1}(1-x)^{2(\beta-1/2)-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{(\beta-1/2)-1} \\
&\quad \times S_n^m\left[y\left(1-\tfrac{x}{4}\right)(1-x)^2\right]\sin\left[y\left(1-\tfrac{x}{4}\right)(1-x)^2\right] dx \\
&= \sqrt{\pi}\left(\tfrac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\left(\tfrac{y}{2}\right)\sum_{l=0}^{[n/m]}\tfrac{(-n)_{ml}}{l!}A_{n,l}y^l \\
&\quad \times {}_1\Psi_2\left[\begin{array}{c}\left(\beta+l+1/2,2\right) \\ \left(\alpha+\beta+l+1/2,2\right)\left(3/2,1\right)\end{array}\middle|\ \tfrac{-y^2}{4}\right]
\end{aligned} \tag{21}$$

$$\begin{aligned}&\int_0^1 x^{\left(\alpha-1/2\right)-1}\left(1-x\right)^{2\beta-1}\left(1-\tfrac{x}{3}\right)^{2\left(\alpha-1/2\right)-1}\left(1-\tfrac{x}{4}\right)^{\beta-1}\\&\quad\times S_n^m\left[yx\left(1-\tfrac{x}{3}\right)^2\right]\sin\left[yx\left(1-\tfrac{x}{3}\right)^2\right]dx\\&=\sqrt{\pi}\left(\tfrac{2}{3}\right)^{2\alpha+1}\left(\tfrac{y}{2}\right)\Gamma\left(\beta\right)\sum_{l=0}^{[n/m]}\tfrac{(-n)_{ml}}{l!}A_{n,l}\left(\tfrac{4}{9}y\right)^l\\&\quad\times{}_1\Psi_2\left[\begin{array}{c}\left(\alpha+l+\frac{1}{2},2\right)\\\left(\alpha+\beta+l+\frac{1}{2},2\right)\left(\frac{3}{2},1\right)\end{array}\middle|\,\tfrac{-4y^2}{81}\right]\end{aligned}\tag{22}$$

Corollory 4.2 Taking $\mu=1, k=1, \omega=-1/2, \upsilon=0$ in (17) and (18) and then by using (4), we obtain

$$\begin{aligned}&\int_0^1 x^{\alpha-1}\left(1-x\right)^{2\left(\beta-1/2\right)-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{\left(\beta-1/2\right)-1}\\&\quad\times S_n^m\left[y\left(1-\tfrac{x}{4}\right)\left(1-x\right)^2\right]\cos\left[y\left(1-\tfrac{x}{4}\right)\left(1-x\right)^2\right]dx\\&=\sqrt{\pi}\left(\tfrac{2}{3}\right)^{2\alpha}\Gamma\left(\alpha\right)\sum_{l=0}^{[n/m]}\tfrac{(-n)_{ml}}{l!}A_{n,l}y^l\\&\quad\times{}_1\Psi_2\left[\begin{array}{c}\left(\beta+l-1/2,2\right)\\\left(\alpha+\beta+l-1/2,2\right)\left(1/2,1\right)\end{array}\middle|\,\tfrac{-y^2}{4}\right]\end{aligned}\tag{23}$$

$$\begin{aligned}&\int_0^1 x^{\left(\alpha-1/2\right)-1}\left(1-x\right)^{2\beta-1}\left(1-\tfrac{x}{3}\right)^{2\left(\alpha-1/2\right)-1}\left(1-\tfrac{x}{4}\right)^{\beta-1}\\&\quad\times S_n^m\left[yx\left(1-\tfrac{x}{3}\right)^2\right]\cos\left[yx\left(1-\tfrac{x}{3}\right)^2\right]dx\\&=\sqrt{\pi}\left(\tfrac{2}{3}\right)^{2\alpha-1}\Gamma\left(\beta\right)\sum_{l=0}^{[n/m]}\tfrac{(-n)_{ml}}{l!}A_{n,l}\left(\tfrac{4}{9}y\right)^l\\&\quad\times{}_1\Psi_2\left[\begin{array}{c}\left(\alpha+l-\frac{1}{2},2\right)\\\left(\alpha+\beta+l-\frac{1}{2},2\right)\left(\frac{1}{2},1\right)\end{array}\middle|\,\tfrac{-4y^2}{81}\right]\end{aligned}\tag{24}$$

Corollory 4.3 Taking $k=1$ in (17) and (18) and then by using (5), we obtain

$$\begin{aligned}&\int_0^1 x^{\alpha-1}\left(1-x\right)^{2\beta-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{\beta-1}\\&\quad\times S_n^m\left[y\left(1-\tfrac{x}{4}\right)\left(1-x\right)^2\right]J_{\omega,\upsilon}^{\mu}\left[y\left(1-\tfrac{x}{4}\right)\left(1-x\right)^2\right]dx\\&=\left(\tfrac{2}{3}\right)^{2\alpha}\Gamma\left(\alpha\right)\left(\tfrac{y}{2}\right)^{\omega+2\upsilon}\sum_{l=0}^{[n/m]}\tfrac{(-n)_{ml}}{l!}A_{n,l}y^l\\&\quad\times{}_2\Psi_3\left[\begin{array}{c}(1,1)\left(\beta+l+\omega+2\upsilon,2\right)\\\left(\alpha+\beta+l+\omega+2\upsilon,2\right)\left(\omega+\upsilon+1,\mu\right)\left(\upsilon+1,1\right)\end{array}\middle|\,\tfrac{-y^2}{4}\right]\end{aligned}\tag{25}$$

$$\begin{aligned}&\int_0^1 x^{\alpha-1}\left(1-x\right)^{2\beta-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{\beta-1}\\&\quad\times S_n^m\left[yx\left(1-\tfrac{x}{3}\right)^2\right]J_{\omega,\upsilon}^{\mu}\left[yx\left(1-\tfrac{x}{3}\right)^2\right]dx\\&=\left(\tfrac{2}{3}\right)^{2\alpha+\omega+2\upsilon}y^{\omega+2\upsilon}\Gamma\left(\beta\right)\sum_{l=0}^{[n/m]}\tfrac{(-n)_{ml}}{l!}A_{n,l}\left(\tfrac{4}{9}y\right)^l\\&\quad\times{}_2\Psi_3\left[\begin{array}{c}(1,1)\left(\alpha+l+\omega+2\upsilon,2\right)\\\left(\alpha+\beta+l+\omega+2\upsilon,2\right)\left(\omega+\upsilon+1,\mu\right)\left(\upsilon+1,1\right)\end{array}\middle|\,\tfrac{-4y^2}{81}\right]\end{aligned}\tag{26}$$

Corollory 4.4 Taking $k=1,\mu=1,\upsilon=1/2$ in (15), (16), (17), (18), (19) and (20) and then by using (6), we get

$$\begin{aligned}&\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S_n^m\left(\sigma t^{\lambda}\right)H_{\omega}\left(\delta t^{\xi}\right)\right](x)\\&=x^{P-\alpha-\alpha'+\gamma-1}\left(\frac{\delta}{2}\right)^{\omega+1}\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}\left(\sigma x^{\lambda}\right)^{l}\\&\times{}_4\Psi_5\left[\begin{matrix}(1,1)\,(P+\lambda l,2\xi)\,(P+\gamma-\alpha-\alpha'-\beta+\lambda l,2\xi)\\(P-\alpha'+\beta'+\lambda l,2\xi)\\(P+\gamma-\alpha-\alpha'+\lambda l,2\xi)\,(P+\gamma-\alpha'-\beta+\lambda l,2\xi)\\(P+\beta'+\lambda l,2\xi)\left(\omega+\frac{3}{2},1\right)\left(\frac{3}{2},1\right)\end{matrix}\middle|\frac{-\left(\delta x^{\xi}\right)^{2}}{4}\right]\end{aligned}\tag{27}$$

where $P=\rho+\omega\xi+\xi$

$$\begin{aligned}&\left[I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{-\rho-\eta}S_n^m\left(\sigma t^{\lambda}\right)H_{\omega}\left(\delta t^{-\xi}\right)\right](x)\\&=x^{-P-\alpha-\alpha'+\gamma}\left(\frac{\delta}{2}\right)^{\omega+1}\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}\left(\sigma x^{\lambda}\right)^{l}\\&\times{}_4\Psi_5\left[\begin{matrix}(1,1)\,(P+\alpha+\alpha'-\gamma-\lambda l,2\xi)\,(P+\alpha+\beta'-\gamma-\lambda l,2\xi)\\(P-\beta-\lambda l,2\xi)\\(P-\lambda l,2\xi)\,(P+\alpha+\alpha'+\beta'-\gamma-\lambda l,2\xi)\\(P+\alpha-\beta-\lambda l,2\xi)\left(\omega+\frac{3}{2},1\right)\left(\frac{3}{2},1\right)\end{matrix}\middle|\frac{-\left(\delta x^{-\xi}\right)^{2}}{4}\right]\end{aligned}\tag{28}$$

where $P=\rho+\eta+\omega\xi+\xi$

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{\beta-1}\\&\quad\times S_n^m\left[y\left(1-\tfrac{x}{4}\right)(1-x)^2\right]\mathrm{H}_{\omega}\left[y\left(1-\tfrac{x}{4}\right)(1-x)^2\right]dx\\&=\left(\tfrac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\left(\tfrac{y}{2}\right)^{\omega+1}\sum_{l=0}^{[n/m]}\tfrac{(-n)_{ml}}{l!}A_{n,l}y^{l}\\&\quad\times{}_2\Psi_3\left[\begin{matrix}(1,1)\,(\beta+l+\omega+1,2)\\(\alpha+\beta+l+\omega+1,2)\left(\omega+\frac{3}{2},1\right)\left(\frac{3}{2},1\right)\end{matrix}\middle|\tfrac{-y^2}{4}\right]\end{aligned}\tag{29}$$

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\tfrac{x}{3}\right)^{2\alpha-1}\left(1-\tfrac{x}{4}\right)^{\beta-1}\\&\quad\times S_n^m\left[yx\left(1-\tfrac{x}{3}\right)^2\right]\mathrm{H}_{\omega}\left[yx\left(1-\tfrac{x}{3}\right)^2\right]dx\\&=\left(\tfrac{2}{3}\right)^{2\alpha+\omega+1}y^{\omega+1}\Gamma(\beta)\sum_{l=0}^{[n/m]}\tfrac{(-n)_{ml}}{l!}A_{n,l}\left(\tfrac{4}{9}y\right)^{l}\\&\quad\times{}_2\Psi_3\left[\begin{matrix}(1,1)\,(\alpha+l+\omega+1,2)\\(\alpha+\beta+l+\omega+1,2)\left(\omega+\frac{3}{2},1\right)\left(\frac{3}{2},1\right)\end{matrix}\middle|\tfrac{-4y^2}{81}\right]\end{aligned}\tag{30}$$

$$\begin{aligned}&\int_0^{\infty} x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta}\\&\quad\times S_n^m\left[\tfrac{y}{x+a+\sqrt{x^2+2ax}}\right]\mathrm{H}_{\omega}\left[\tfrac{y}{x+a+\sqrt{x^2+2ax}}\right]dx\\&=\sum_{l=0}^{[n/m]}\tfrac{(-n)_{ml}}{l!}A_{n,l}y^{l+\omega+1}2^{-\omega-\eta}\,\Gamma(2\eta)\,a^{\eta-(\delta+l+\omega+1)}\\&\times{}_3\Psi_4\left[\begin{matrix}(1,1)\,(\delta+l+\omega+1-\eta,2)\,(\delta+l+\omega+2,2)\\\left(\frac{3}{2},1\right)\left(\omega+\frac{3}{2},1\right)(\delta+l+\omega+2+\eta,2)\,(\delta+l+\omega+1,2)\end{matrix}\middle|\tfrac{-y^2}{4a^2}\right]\end{aligned}\tag{31}$$

$$\int_0^\infty x^{\eta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\delta}$$
$$\times S_n^m\left[\frac{xy}{x+a+\sqrt{x^2+2ax}}\right]\mathrm{H}_\omega\left[\frac{xy}{x+a+\sqrt{x^2+2ax}}\right]dx$$
$$=\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}y^{l+\omega+1}2^{-2\omega-1-\eta-l}\,\Gamma(\delta-\eta)\,a^{\eta-\delta}$$
$$\times{}_3\Psi_4\left[\begin{array}{c}(1,1)\,(2\omega+2+2\eta+2l,4)\,(\delta+l+\omega+2,2)\\ \left(\frac{3}{2},1\right)\left(\omega+\frac{3}{2},1\right)(\delta+2l+2\omega+3+\eta,4)\,(\delta+l+\omega+1,2)\end{array}\middle|\frac{-y^2}{16}\right] \tag{32}$$

5. Further remarks and observations

We conclude our present study by remarking that it is not difficult to obtain several analogues and variations of the derived formulas exhibited here by (15), (16), (17), (18), (19) and (20) involving the generalized Lommel–Wright function $J_{\omega,\upsilon}^{\mu;k}(\cdot)$itself and its other variants.

For suitable choices of the parameters μ, $\omega$and υ, each of our integral formulasy (15), (16), (17), (18), (19) and (20) (with $k=1$) give some known as well as new results for the generalized Bessel function $J_{\omega,\upsilon}^{\mu}(z)$, the Struve function $\mathrm{H}_\omega(z)$ and the classical Bessel function $J_\omega(z)$, which are related to the generalized Lommel–Wright function $J_{\omega,\upsilon}^{\mu;k}(z)$ by means of (5), (6) and (7).

Particularly, if we take $l=0$in equations (15), (16), (17) and (18), we obtain known results derived by Agrawal et al.[[23]] and Haq et al [[22]].

References

References

[1] Baleanu, D., Kumar, D., Purohit, S. D. (2016). Generalized fractional integrals of product of two H-functions and a general class of polynomials, International Journal of Computer Mathematics, 93, 1320–1329.

[2] Kumar, D., Purohit, S. D., Choi, J. (2016). Generalized fractional integrals involving product of multivariable H-function and a general class of polynomials, The Journal of Nonlinear Science and its Applications, 9, 8–21.

[3] Purohit, S. D., Kalla, S. L., Suthar, D. L. (2011). Fractional integral operators and the multiindex Mittag-Leffler functions, Scientia Series A: Mathematical Sciences (N.S.), 21, 87–96.

[4] Choi, J., Agarwal, P., Mathur, S., Purohit, S.D. (2014). Certain new integral formulas involving the generalized Bessel functions, Bull. Korean Math. Soc, 51, No. 4, 995-1003.

[5] Jain, S., Agarwal, P. (2015). A new class of integral relation involving general class of Polynomials and I- functionmsa, Walailak J. Sci. & Tech.

[6] Menaria, N., Purohit, S. D., Parmar, R. K. (2016). On a new class of integrals involving generalized Mittag-leffler function, Surveys in Mathematics and its Applications, 11, 1-9.

[7] Suthar, D.L. , Reddy, Gv, Tsegaye, T. (2017). Unified Integrals Formulas Involving Product of Srivastava's Polynomials and Generalized Bessel-Maitland Function, International Journal of Scientific Research, 6(2), 708-710.

[8] Oteiza MBM, de Kalla, S., Conde, S. (1986). Un estudio sobre la funcition Lommel-Maitland, Rev Técnica Facult Ingenieria Univers Zulia, 9, 33–40.

[9] Wright, E. M. (1935). The asymptotic expansion of the generalized hypergeometric functions, Journal of the London Mathematical Society, 10, 286–293.

[10] Kiryakova, V. (2006). On two Saigo's fractional integral operators in the class of univalent functions, Fract Calc Appl Anal. , 9, 159–76.

[11] Mathai, A.M., Saxena, R.K., Haubold, H.J. (2010). The H-Function Theory and Applications, New York, NY: Springer-Verlag.

[12] Bhowmick, K.N. (1962). Some relation between generalized Struve function and hypergeometric function, Vijnana Parishad Anusandhan Patrika, 5, 93-99.

[13] Konovska, J.P. (2007). Theorems on the convergence of series in generalized Lommel-Wright functions, Fractional Calculus and Applied Analysis, 10(1).
[14] Singh, R.P. (1988). Some integral representation of generalized Struve's functions, Math.Ed (Siwan), 22(3), 91-94.
[15] Srivastava, H.M. (1972). A contour integral involving Fox's H function, Indian J. Math., 14, 1-6.
[16] Marichev, O. I. (1974). Volterra equation of Mellin convolution type with a Horn function in the kernel, Izvestiya Akademii Nauk BSSR. Seriya Fiziko-Matematicheskikh Nauk, 1, 128–129, (Russian)
[17] Saigo, M., Maeda, N. (1996). More generalization of fractional calculus Transform Methods and Special Functions, Varna, Bulgaria, 386–400.
[18] Kim, Y.C., Lee, K.S., Srivastava, H.M. (1996). Some applications of fractional integral operators and Ruscheweyh derivatives, Journal of Mathematical Analysis and Applications, vol.197, no. 2, pp.505–517.
[19] Kiryakova, V. (1997). All the special functions are fractional differintegrals of elementary functions, Journal of Physics A: Mathematical and General, vol.30, no.14, 5085–5103.
[20] Lavoie, J.L., Trottier, G. (1969). On the sum of certain Appell's series, Ganita, (1) 20, 31-32.
[21] Oberhettinger, F. (1974). Tables of Mellin Transforms, Springer, New York.
[22] Haq, S., Nisar, K.S., Khan, A.H. Some New Results Associated with the Generalized Lommel-Wright Function, *Preprints* 2018, 2018020155 (doi: 10.20944/preprints201802.0155.v1).
[23] Agarwal R., Jain S., Agarwal R.P., Baleanu D. (2018). A Remark on the Fractional Integral Operators and the Image Formulas of Generalized Lommel–Wright Function. Front. Phys. 6:79. doi: 10.3389/fphy.2018.00079
[24] Nisar K. S., Khan, Ah., Haq, S., Beta type integral formula associated with the generalized Lommel-Wright function, DOI: 10.5899/2018/cna-00373.
[25] Prieto, A. I., Romero, S. S., Srivastava, H. M. (2007). Some fractional-calculus results involving the generalized Lommel-Wright and related functions, Appl. Math. Lett., 20(1), 17–22.

Sangeeta Choudhary
Department of Mathematics ,
Swami Keshvanand Institute of Technology, Management & Gramothan,
Jaipur-302017, India
Email address: sangeeta11rc@gmail.com