Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 151-164.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

GLOBAL STABILITY OF AN SIRSI EPIDEMIC MODEL WITH A DISTRIBUTED DELAYED AND GENERALIZED INCIDENCE FUNCTION

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ABSTRACT. In this paper we study the global dynamics of an SIRSI epidemic model with a distributed latent period and a general non-linear incidence function. By using suitable Lyapunov functionals and LaSalle's invariance principle, the global stability of a disease-free equilibrium and an endemic equilibrium is established for the SIRSI epidemic model with a distributed latent period.

1. Introduction

The latency period is the period during which an individual exposed to the disease can not pass it on to another individual. In the literature, several types of representation are presented, namely the introduction of an additional variable or the inclusion of a deviation in time: the discrete delay or the distributed delay (see [1, 2, 5, 6, 7, 8, 11, 12, 14, 16, 20, 23, 24, 25, 27, 28]).
Recently, considerable attention has been paid to model the relapse phenomenon, i.e. the return of signs and symptoms of a disease after a remission. Hence, the recovered individual can return to the infectious class (see [5, 6]).
For the biological explanations of the relapse phenomena, we cite two examples: Malaria and Tuberculosis.
On the other hand, the goal of research in epidemiology is to develop vaccines, treatments and intervention strategies for stopping the spread of infectious diseases and hence reducing the deaths:
In [26] Raul Nistal and al, proposed a discrete SEIADR epidemic model considering two extra subpopulation, and two types of vaccinations, one constant and another another proportional to the susceptible subpopulation and a traitement control applied to the infected subpopulation. The authors gave a characterization of the equilibrium point stability through the use of the next-generation matrix applied to the equilibrium point without disease. They gave, under positive conditions the characterization of the relations between the stability of the equilibrium point.

2010 *Mathematics Subject Classification.* 34D08, 34D20, 34D23.
Key words and phrases. Epidemic model, Distributed time delay, Lyapunov function, Global stability.
Submitted June 17, 2019. Revised Nov. 14, 2019.

In [32], Yuying He and al. consider an SIR epidemic model with time-varying pulse vaccination of the susceptible and time-varying pulse treatment of the infected population. The authors proved that the infection-free periodic solution is globally attractive if this threshold value is less than unity ($R_0 < 1$) and the system is permanent if this threshold value is larger than unity ($R_0 > 1$).
In [4] Amine Bernoussi and al introduced the term $e^{-\mu\tau}$ into the incidence function. This modification is based on the elimination of individuals exposed to the disease at time $t-\tau$ and who can not survive at time t, where τ is the duration of the latency period:

$$\begin{cases} \dfrac{dS}{dt} = A - \mu S - f(S, e^{-\mu\tau} I_\tau), \\ \dfrac{dI}{dt} = f(S, e^{-\mu\tau} I_\tau) - (\mu+\gamma)I + \delta R, \\ \dfrac{dR}{dt} = \gamma I - (\mu+\delta)R. \end{cases} \tag{1}$$

In the present paper, we study the model (1) with distributed time delay and Immunity loss

$$\begin{cases} \dfrac{dS}{dt} = A - \mu S - \displaystyle\int_0^h p(\tau) f(S, e^{-\mu\tau} I_\tau) I_\tau d\tau + \delta_1 R, \\ \dfrac{dI}{dt} = \displaystyle\int_0^h p(\tau) f(S, e^{-\mu\tau} I_\tau) I_\tau d\tau - (\mu+\gamma)I + \delta_2 R, \\ \dfrac{dR}{dt} = \gamma I - (\mu+\delta_1+\delta_2)R. \end{cases} \tag{2}$$

The initial condition for the above system is:

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta), \quad \theta \in [-h, 0]. \tag{3}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in \mathbb{C}$, such that $\varphi_i(\theta) = \varphi_i(0) \geq 0$, $(-h \leq \theta \leq 0$, $i = 1, 2, 3,)$ $\varphi_2(\theta) \geq 0$ $(-h \leq \theta \leq 0)$. $\mathbb{C}$ denotes the Banach space $\mathbb{C}([-h, 0], \mathbb{R}^3_{+0})$ of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}^3_{+0}$ with the supremum norm, where $\mathbb{R}_{+0} = \{x \in \mathbb{R} \mid x \geq 0\}$. From a biological meaning, we assume that $\varphi_i(0) > 0$ for $i = 1, 2, 3$.

Here $\psi_\tau = \psi(t-\tau)$ for any given function ψ, where A is the constant recruitment rate into the population, S represents the number of individuals who are susceptible to the disease, that is, who are not yet infected at time t, I represents the number of infected individuals who are infectious and are able to spread the disease by contact with susceptible individuals, I_τ is the number of infectious individuals at time $t-\tau$, R is the number of individuals who have been infected and temporarily recovered at time t, μ denotes birth and death rates, $f(S, I)I$ is the incidence function, i.e. the number of susceptible individuals infected through their contacts with the infectious individuals, the term $e^{-\mu\tau}$ is the probability of surviving from time $t-\tau$ to time t, τ is the duration of the latency period, γ is the rate at which infective individuals recover. Thus, the probabilities of remaining in the infective and recovered classes are assured to be exponentially distributed, δ_1, is the rate at which recovered individuals lose immunity and return to the susceptible class, δ_2 is a constant representing the rate at which an individual in the recovered class reverts to the infective class, and $\delta_2 > 0$ implies that the recovered individuals would lose the immunity, $\delta_2 = 0$ implies that the recovered individuals acquire permanent immunity, $p(\tau)$ is the latent period distribution, which denotes the fraction of individuals who become infective, satisfying $\int_0^h p(\tau)d\tau = 1$, and h is a maximum time taken by infected individual to become able to transmit the infection (maximum latent period).

In model (2) the function $f : \mathbb{R}^2_{+0} \to \mathbb{R}_{+0}$ is a locally Lipschitz continuous function satisfaying $f(0, I) = 0$ for $I \geq 0$ and the followings hold:

- **(H_1):** $f(S, I)$ is a strictly monotone increasing function of $S \geq 0$, for any fixed $I > 0$, and $f(S, I)$ is a monotone decreasing function of $I \geq 0$, for any fixed $S > 0$;
- **(H_2):** $\phi(S, I) = f(S, e^{-\mu\tau} I)I$ is a monotone increasing function of $I \geq 0$, for any fixed $S \geq 0$;
- **(H_3):** $\Psi(S, I_\tau) = \frac{\phi(S, I_\tau)}{I}$ is a monotone decreasing function of $I > 0$, for any fixed $S > 0$ for all $\tau \geq 0$.

The incidence function $f(S, I)I$ is considered by Hattaf and al [16] who generalizes several form of incidence: The first one is the saturated incidence $\frac{\beta SI}{d+S+I}$ [3], where β and d are the positive constants. The second one is the bilinear incidence βSI [15, 18, 29, 34, 35]. The third one is the saturated incidence $\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$ [3, 9, 10, 18, 19, 30, 31, 33], where α_1 and α_2 are the positive constants. The effect of saturation factor (refers to α_1 and α_2) stems from epidemic control and the protection measures. The fourth one is the standard incidence $\frac{\beta SI}{N}$ [13, 17].

The main results of this paper are as follows: The first one is the Lemma 6. The second one is the Lyapunov functional. The third one is the following:

Theorem 1.
Under the hypotheses (H_1), (H_2) and (H_3), the endemic equilibrium P^* of system (2) is globally asymptotically stable if $R_0 > 1$.

The organization of this article is as follows. In Section 2, we offer a basic result. In Section 3, we apply Lyapunov-LaSalle invariance principle to prove the global stability of the disease-free equilibrium and we apply Lyapunov- LaSalle invariance principle to prove the global stability of endemic equilibrium. In Section 4, we present some concluding remarks.

2. Preliminary

In this subsection, we prove the following basic result, which guarantees the existence and uniqueness of the solution $(S(t), I(t), R(t))$ for system (2) satisfying initial conditions (3).

Lemma 2. *The plane $S(t) + I(t) + R(t) = \frac{A}{\mu}$ is an invariant manifold of system (2), which is globally attractive in the first octant of $\mathbb{R}^3$, that is,*

$$\lim_{t\to+\infty}(S(t) + I(t) + R(t)) = \frac{A}{\mu}.$$

Proof. Let $N(t) = S(t) + I(t) + R(t)$. Then it follows from system (2) that

$$\frac{dN(t)}{dt} = A - \mu S(t) - \mu I(t) - \mu R(t)$$
$$= A - \mu N(t).$$

Hence, we obtain that $\lim_{t\to+\infty} N(t) = \frac{A}{\mu}$. This completes the proof.

Lemma 3. *The solution* (S, I, R) *of system* (2) *with initial condition* (3) *uniquely exists and is positive for all* $t \geq 0$.

Proof. We notice that the right hand side of system (2) is completely continous and locally Lipschitzian on $\mathbb{C}$. Then, it follows that the solution of system (2) exists and is unique on $[0, \alpha)$ for some $\alpha > 0$. It is easy to prove that $I(t) > 0$ for all $t \in [0, \alpha)$. Suppose on the contrary that there exists som $t_1 \in [0, \alpha)$ such that $I(t_1) = 0$ and $I(t) > 0$ for $t \in [0, t_1)$. by integrating from 0 to t_1 the second equation of system (2) we obtain

$$I(t_1) = I(0)e^{-(\mu+\gamma)t_1} + \int_0^{t_1} [\int_0^h \phi(S, I_\tau)d\tau + \delta_2 R]e^{-(\mu+\gamma)(t_1-\theta)}d\theta.$$

Solving $R(t)$ in the third equation of system (2), we have

$$R(t) = R(0)e^{-(\mu+\delta_1+\delta_2)t} + \int_0^t \gamma I(\sigma)e^{-(\mu+\delta_1+\delta_2)(t-\sigma)}d\sigma > 0, \ for\ all\ t \in [0, t_1).$$

We see that $I(t_1) > 0$. This contradicts $I(t_1) = 0$. From the third equation of system (2), we also have that $R(t) > 0$ for all $t \in [0, \alpha)$. Let us now show that $S(t) > 0$ for all $t \in [0, \alpha)$. Indeed, this follows from that $\frac{dS}{dt} = A + \delta_1 R(t) > 0$ for any $t \in [0, \alpha)$ when $S(t) = 0$.
Furthermore, for $t \in [0, \alpha)$, we obtain

$$\frac{dN(t)}{dt} = A - \mu N(t)$$

which implies that $(S(t), I(t), R(t))$ is uniformly bounded on $[0, \alpha)$. It follows that $(S(t), I(t), R(t))$ exists and is unique and positive for all $t \geq 0$, which completes the proof.

3. Global stability of the disease-free equilibrium and the endemic equilibrium

In this section, we discuss the global stability of the disease-free equilibrium P_0 and the endemic equilibrium P^* of system (2).
Firstly, we prove the existence and the uniqueness of the endemic equilibrium P^*.

Proposition 4. *System* (2) *always has a disease-free equilibrium* $P_0 = (\frac{A}{\mu}, 0, 0)$. *On the other hand, under the hypothesis* (H_1), *if*

$$R_0 := \frac{f(\frac{A}{\mu}, 0)}{\eta_1} > 1;$$

then system (2) *also admits a unique endemic equilibrium* $P^* = (S^*, I^*, R^*)$, *where* S^*, I^* *and* R^* *satisfying the following system:*

$$\begin{cases} A - \mu S - \int_0^h p(\tau) f(S, e^{-\mu\tau} I) I d\tau + \delta_1 R = 0, \\ \int_0^h p(\tau) f(S, e^{-\mu\tau} I) I d\tau - (\mu+\gamma) I + \delta_2 R = 0, \\ \gamma I - (\mu+\delta_1+\delta_2) R = 0. \end{cases} \tag{4}$$

With $\eta_1 = (\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}$.

Proof. At a fixed point (S, I, R) of system (2), the following equation hold

$$\begin{cases} A - \mu S - \int_0^h f(S, e^{-\mu\tau} I) I d\tau + \delta_1 R = 0, \\ \int_0^h f(S, e^{-\mu\tau} I) I d\tau - (\mu+\gamma) I + \delta_2 R = 0, \\ \gamma I - (\mu+\delta_1+\delta_2) R = 0. \end{cases} \tag{5}$$

Substituting the third equation into the seond equation and into the first equation of (5), we consider the following system

$$\begin{cases} A - \mu S - \int_0^h f(S, e^{-\mu\tau} I) I d\tau + \frac{\delta_1 \gamma I}{(\mu+\delta_1+\delta_2)} = 0, \\ \int_0^h f(S, e^{-\mu\tau} I) I d\tau - [(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}] I = 0, \\ R = \frac{\gamma I}{(\mu+\delta_1+\delta_2)}. \end{cases} \tag{6}$$

Substituting the second equation into first equation of (6), we consider the following system:

$$\begin{cases} A - \mu S - [(\mu+\gamma) - \frac{\delta_2\gamma}{(\mu+\delta_1+\delta_2)} - \frac{\delta_1\gamma}{(\mu+\delta_1+\delta_2)}] I = 0, \\ \int_0^h f(S, e^{-\mu\tau} I) I d\tau - [(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}] I = 0, \\ R = \frac{\gamma I}{(\mu+\delta_1+\delta_2)}. \end{cases} \tag{7}$$

From the second equation of (7), we get $I = 0$ or $\int_0^h f(S, e^{-\mu\tau} I) d\tau = [(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}]$ If $I = 0$, we obtain the disease-free equilibrium point $P_0 = (\frac{A}{\mu}, 0, 0)$. If $I \neq 0$, then using the (7), we get the following equation

$$\int_0^h f(S, \frac{(A-\mu S)e^{-\mu\tau}}{[(\mu+\gamma) - \frac{\gamma(\delta_1+\delta_2)}{(\mu+\delta_1+\delta_2)}]}) d\tau = [(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}].$$

We have $I = \frac{A-\mu S}{[(\mu+\gamma) - \frac{\gamma(\delta_1+\delta_2)}{(\mu+\delta_1+\delta_2)}]} \geq 0$ implies that $S \leq \frac{A}{\mu}$.
Hence, there is no positive equilibrium point if $S > \frac{A}{\mu}$.
Now, we consider the following fonction g_1 defined on the interval $[0, \frac{A}{\mu}]$

$$g_1(S) := \int_0^h f(S, \frac{(A-\mu S)e^{-\mu\tau}}{[(\mu+\gamma) - \frac{\gamma(\delta_1+\delta_2)}{(\mu+\delta_1+\delta_2)}]})d\tau - [(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}].$$

Since,

$$\begin{aligned} g_1(\frac{A}{\mu}) &= f(\frac{A}{\mu}, 0) - [(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}] \\ &= [(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}]\Big(\frac{f(\frac{A}{\mu},0)e^{-\mu\tau}}{[(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}]} - 1\Big) \\ &= [(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}](R_0 - 1) > 0 \end{aligned}$$

and

$$g_1(0) = -[(\mu+\gamma) - \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}] < 0.$$

Further

$$g_1^{'}(S) = \int_0^h (\frac{\partial f}{\partial S} - \frac{\mu e^{-\mu\tau}}{[(\mu+\gamma) - \frac{\gamma(\delta_1+\delta_2)}{(\mu+\delta_1+\delta_2)}]}\frac{\partial f}{\partial I})d\tau$$

by the hypothesis (H_1), we have $g_1^{'}(S) > 0$.
Hence, there existe a unique endemic equilibrium $P^* = (S^*, I^*, R^*)$ with $S^* \in]0, \frac{A}{\mu}[$ and $I^* > 0$, $R^* > 0$, satisfies the equations $I = \frac{A-\mu S}{[(\mu+\gamma) - \frac{\gamma(\delta_1+\delta_2)}{(\mu+\delta_1+\delta_2)}]}$ and $R = \frac{\gamma I}{(\mu+\delta_1+\delta_2)}$. Hence, we conclude the existence and uniquenss of the endemic equilibrium P^*.

Next we consider the global asymptotic stability of the disease-free equilibrium P_0 and the endemic equilibrium P^* of (2) by Lyapunov functionals, respectively.

Theorem 5. *If $R_{01} \leq 1$, then the disease-free equilibrium P_0 is globally asymptotically stable.*

Proof. Define the Lyapunov functional

$$\begin{aligned} V_0(t) &= \int_{\frac{A}{\mu}}^S (1 - \frac{f(\frac{A}{\mu},0)}{f(u,0)})du + I + \frac{\delta_2}{(\mu+\delta_1+\delta_2)}R \\ &\quad + \int_0^h p(\tau)\int_{t-\tau}^t \frac{f(\frac{A}{\mu},0)}{f(S(u+\tau),0)} f(S(u+\tau), e^{-\mu\tau}I(u))I(u)dud\tau. \end{aligned}$$

We will show that $\frac{dV_0(t)}{dt} \leq 0$ for all $t \geq 0$. We have

$$\begin{aligned}
\frac{dV_0(t)}{dt} &= (1 - \frac{f(\frac{A}{\mu},0)}{f(S,0)})\dot{S} + \int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau \\
&- (\mu+\gamma)I + \delta_2 R + \frac{\gamma\delta_2}{(\mu+\delta_1+\delta_2)}I - \delta_2 R \\
&+ \int_0^h p(\tau)\frac{f(\frac{A}{\mu},0)}{f(S(t+\tau),0)}f(S(t+\tau),e^{-\mu\tau}I)Id\tau - \int_0^h p(\tau)\frac{f(\frac{A}{\mu},0)}{f(S,0)}f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau \\
&= (1 - \frac{f(\frac{A}{\mu},0)}{f(S,0)})(A-\mu S) + \delta_1 R(1 - \frac{f(\frac{A}{\mu},0)}{f(S,0)}) \\
&+ \int_0^h p(\tau)\frac{f(\frac{A}{\mu},0)}{f(S(t+\tau),0)}f(S(t+\tau),e^{-\mu\tau}I)Id\tau - \eta_1 I \\
&= \mu(1 - \frac{f(\frac{A}{\mu},0)}{f(S,0)})(\frac{A}{\mu} - S) + \delta_1 R(1 - \frac{f(\frac{A}{\mu},0)}{f(S,0)}) \\
&+ \eta_1 I(\int_0^h p(\tau)\frac{f(S(t+\tau),e^{-\mu\tau}I)}{\eta_1}\frac{f(\frac{A}{\mu},0)}{f(S(t+\tau),0)}d\tau - 1)
\end{aligned}$$

Furthermore, it follows from the hypothesis (H_1) that

$$\begin{aligned}
\int_0^h p(\tau)\frac{f(S(t+\tau),e^{-\mu\tau}I)}{\eta_1}\frac{f(\frac{A}{\mu},0)}{f(S(t+\tau),0)}d\tau &\leq \int_0^h p(\tau)\frac{f(\frac{A}{\mu},0)}{f(S(t+\tau),0)}\frac{f(S(t+\tau),0)}{\eta_1}d\tau \\
&\leq \frac{f(\frac{A}{\mu},0)}{\eta_1} \\
&\leq R_0.
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{dV_0(t)}{dt} &\leq \mu(1 - \frac{f(\frac{A}{\mu},0)}{f(S,0)})(\frac{A}{\mu} - S) + \delta_1 R(1 - \frac{f(\frac{A}{\mu},0)}{f(S,0)}) \\
&+ \eta_1 I(R_0 - 1).
\end{aligned}$$

By the hypothesis (H_1), and non-negativity of the solution, we obtain that

$$(1 - \frac{f(\frac{A}{\mu},0)}{f(S,0)})(\frac{A}{\mu} - S) \leq 0.$$

and

$$\delta_1 R(1 - \frac{f(\frac{A}{\mu},0)}{f(S,0)}) \leq 0.$$

Where equality holds if and only if $S = \frac{A}{\mu}$.

Since $R_{01} \leq 1$, ensures that $\frac{dV_0(t)}{dt} \leq 0$ for all $t \geq 0$. Thus, the disease-free equilibrium P_0 is stable and $\frac{dV_0(t)}{dt} = 0$ holds if and only if $S = \frac{A}{\mu}$ and $I(R_{01} - 1) = 0$. Hence, we have $\lim_{t\to+\infty} S = \frac{A}{\mu}$. Which implies from Lemma 2, that $\lim_{t\to+\infty} I = 0$ and $\lim_{t\to+\infty} R = 0$ hold. By an extension of Lyapunov-LaSalle asymptotic stability theorem [21, 22], the disease-free equilibrium P_0 of system (2) is globally asymptotically stable. This completes the proof.

Next, we will study the global stability of the positive equilibrium P^*.
The following Lemma plays a key role to obtain Theorem 8.

Lemma 6. *Let (S, I, R) solution of system (2) with initial condition (3) and (S^*, I^*, R^*) the endemic equilibrium. Then we have the following proposition for all $t \geq T$ with T sufficiently large such that $S(T) + I(T) + R(T) = N(T) \cong \frac{A}{\mu}$:*

(i) *Suppose that $R(0) \leq R^*$, then we have the following:*
If $I \leq I^*$ then $R \leq R^*$ then $S \geq S^*$

(ii) *Suppose that $R(0) \geq R^*$, then we have the following:*
If $I \geq I^*$ then $R \geq R^*$ then $S \leq S^*$.

Proof. (i) By the third equation of system (2), we obtain

$$R(t) = R(0)e^{-(\mu+\delta_1+\delta_2)t} + \int_0^t \gamma I(\sigma)e^{-(\mu+\delta_1+\delta_2)(t-\sigma)}d\sigma$$

If $I \leq I^*$ then

$$\begin{aligned} R(t) &\leq R(0)e^{-(\mu+\delta_1+\delta_2)t} + \gamma I^* \int_0^t e^{-(\mu+\delta_1+\delta_2)(t-\sigma)}d\sigma \\ &= R(0)e^{-(\mu+\delta_1+\delta_2)t} + \frac{\gamma}{(\mu+\delta_1+\delta_2)}I^* - \frac{\gamma}{(\mu+\delta_1+\delta_2)}I^* e^{-(\mu+\delta_1+\delta_2)t} \end{aligned}$$

Using the relation $\frac{\gamma I^*}{(\mu+\delta_1+\delta_2)} = R^*$ and $R(0) \leq R^*$. We have $R(t) \leq R^*$.
Suppose that $I \leq I^*$ and $R \leq R^*$, we will prove that $S \geq S^*$ for all $t \geq T$ Suppose on the contrary that there exists som $t_1 > T$ such that $S(t_1) < S^*$. Then by the hypotheses we have

$$S(t_1) + I(t_1) + R(t_1) < S^* + I^* + R^*.$$

On the other hand, we have

$$S(t_1) + I(t_1) + R(t_1) = \frac{A}{\mu}$$

Contradiction with $S^* + I^* + R^* = \frac{A}{\mu}$.
(ii) By the third equation of system (2), we obtain

$$R(t) = R(0)e^{-(\mu+\delta_1+\delta_2)t} + \int_0^t \gamma I(\sigma)e^{-(\mu+\delta_1+\delta_2)(t-\sigma)}d\sigma$$

If $I \geq I^*$ then

$$\begin{aligned} R(t) &\geq R(0)e^{-(\mu+\delta_1+\delta_2)t} + \gamma I^* \int_0^t e^{-(\mu+\delta_1+\delta_2)(t-\sigma)}d\sigma \\ &= R(0)e^{-(\mu+\delta_1+\delta_2)t} + \frac{\gamma}{(\mu+\delta_1+\delta_2)}I^* - \frac{\gamma}{(\mu+\delta_1+\delta_2)}I^* e^{-(\mu+\delta_1+\delta_2)t} \end{aligned}$$

Using the relation $\frac{\gamma I^*}{(\mu+\delta_1+\delta_2)} = R^*$ and $R(0) \geq R^*$. We have $R(t) \geq R^*$
Suppose that $I \geq I^*$ and $R \geq R^*$, we will prove that $S \leq S^*$ for all $t \geq T$ Suppose on the contrary that there exists som $t_1 > T$ such that $S(t_1) > S^*$. Then by the hypotheses we have

$$S(t_1) + I(t_1) + R(t_1) > S^* + I^* + R^*.$$

On the other hand, we have

$$S^* + I^* + R^* = \frac{A}{\mu}$$

Contradiction with $S(t_1) + I(t_1) + R(t_1) = \frac{A}{\mu}$. This completes the proof.

Corollary 7. *Let (S, I, R) solution of system (2) and (S^*, I^*, R^*) the endemic equilibrium. Suppose the hypothesis (H_1) hold. Then we have for all $t \geq T$ with T sufficiently large such that $S(T) + I(T) + R(T) = N(T) \cong \frac{A}{\mu}$:*

$$\delta_1(R - R^*)(1 - \frac{f(S^*, I^*)}{f(S, I^*)}) \leq 0.$$

Proof. By the lemma 6 we have $(R - R^*)(S - S^*) \leq 0$.
Furthermore, it follows from the hypothesis (H_1) that

$$\delta_1(R - R^*)(1 - \frac{f(S^*, I^*)}{f(S, I^*)}) \leq 0.$$

This completes the proof.

Theorem 8. *If $R_{01} > 1$, then the endemic equilibrium P^* is globally asymptotically stable.*

Proof. We define the Lyapunov functional
$V(t) = V_1(t) + V_2(t)$, with

$$V_1(t) = \int_{S^*}^{S} (1 - \frac{\int_0^h p(\tau) f(S^*, e^{-\mu\tau} I^*) d\tau}{\int_0^h p(\tau) f(u, e^{-\mu\tau} I^*) d\tau}) du + (I - I^* - I^* \ln \frac{I}{I^*}),$$

$$V_2(t) = \frac{\delta_2}{(\mu + \delta_1 + \delta_2)} (R - R^* - R^* \ln \frac{R}{R^*}).$$

The function $f(S, I)$ is a strictly monotone increasing function of $S \geq 0$, for any fixed $I > 0$, and the function
$g(x) = x - 1 - \ln x$ is always positive for any $x > 0$, and $g(x) = 0$ if and only if $x = 1$, then we have $V(t) > 0$ for all $t > 0$ and $V(S^*, I^*, R^*) = 0$.
The time derivative of the functions $V_1(t)$ and $V_2(t)$ along the positive solution of system (2) is

$$\begin{aligned} \frac{dV_1(t)}{dt} &= \Big(1 - \frac{\int_0^h p(\tau) f(S^*, e^{-\mu\tau} I^*) d\tau}{\int_0^h p(\tau) f(S, e^{-\mu\tau} I^*) d\tau}\Big) \Big(A - \mu S - \int_0^h p(\tau) f(S, e^{-\mu\tau} I_\tau) I_\tau d\tau + \delta_1 R\Big) \\ &+ \big(1 - \frac{I^*}{I}\big) \Big(\int_0^h p(\tau) f(S, e^{-\mu\tau} I_\tau) I_\tau d\tau + \delta_2 R - (\mu + \gamma) I\Big) \end{aligned} \tag{8}$$

Using the relation $A = \mu S^* + \int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)I^* d\tau - \delta_1 R^*$, $(\mu+\gamma) = \frac{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)I^* d\tau + \delta_2 R^*}{I^*}$, and $(\mu + \delta_1 + \delta_2)R^* = \gamma I^*$. Simple calculations give that

$$
\begin{aligned}
\frac{dV_1(t)}{dt} &= \mu\Big(1 - \frac{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S, e^{-\mu\tau}I^*)d\tau}\Big)\Big(S^* - S\Big) \\
&+ \int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)I^* d\tau\Big(1 - \frac{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S, e^{-\mu\tau}I^*)d\tau}\Big) \\
&\times \Big(1 - \frac{\int_0^h p(\tau)f(S, e^{-\mu\tau}I_\tau)I_\tau d\tau}{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)I^* d\tau}\Big) + \delta_1(R - R^*)(1 - \frac{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S, e^{-\mu\tau}I^*)d\tau}) \\
&+ \Big(1 - \frac{I^*}{I}\Big)\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)I^* d\tau\Big(\frac{\int_0^h p(\tau)f(S, e^{-\mu\tau}I_\tau)I_\tau d\tau}{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)I^* d\tau} - \frac{I}{I^*}\Big) + \Big(1 - \frac{I^*}{I}\Big)\delta_2(R - \frac{IR^*}{I^*}) \\
&= \mu\Big(1 - \frac{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S, e^{-\mu\tau}I^*)d\tau}\Big)\Big(S^* - S\Big) \\
&+ \int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)I^* d\tau\{(2 - \frac{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S, e^{-\mu\tau}I^*)d\tau} - \frac{I^*}{I}\frac{\int_0^h p(\tau)f(S, e^{-\mu\tau}I_\tau)I_\tau d\tau}{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)I^* d\tau}) \\
&+ (\frac{\int_0^h p(\tau)f(S, e^{-\mu\tau}I_\tau)I_\tau d\tau}{\int_0^h p(\tau)f(S, e^{-\mu\tau}I^*)I^* d\tau} - \frac{I}{I^*})\} \\
&+ \delta_1(R - R^*)(1 - \frac{\int_0^h p(\tau)f(S^*, e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S, e^{-\mu\tau}I^*)d\tau}) + \delta_2(1 - \frac{I^*}{I})(R - \frac{IR^*}{I^*})
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dV_2(t)}{dt} &= \frac{\delta_2}{(\mu + \delta_1 + \delta_2)}(1 - \frac{R^*}{R})(\gamma I - (\mu + \delta_1 + \delta_2)R) \\
&= \frac{\delta_2}{(\mu + \delta_1 + \delta_2)}(\gamma I - (\mu + \delta_1 + \delta_2)R - \frac{\gamma I R^*}{R} + (\mu + \delta_1 + \delta_2)R^*) \\
&= \delta_2\frac{R^* I}{I^*} - \delta_2 R - \delta_2\frac{I(R^*)^2}{RI^*} + \delta_2 R^*
\end{aligned}
$$

Then we have

$$\begin{aligned}
\frac{dV(t)}{dt} &= \mu\Big(1-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau}\Big)\Big(S^*-S\Big)+\delta_1(R-R^*)(1-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau})\\
&+\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau\{3-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau}-\frac{I^*}{I}\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau}{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau}\\
&-\frac{I}{I^*}\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)I^*d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau}\}\\
&+\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau\{-1+\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)I^*d\tau}+\frac{I}{I^*}\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)I^*d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau}-\frac{I}{I^*}\}\\
&+\delta_2R^*(2-\frac{I^*R}{IR^*}-\frac{IR^*}{I^*R})\\
&=\mu\Big(1-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau}\Big)\Big(S^*-S\Big)+\delta_1(R-R^*)(1-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau})\\
&+\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau\{3-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau}-\frac{I^*}{I}\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau}{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau}\\
&-\frac{I}{I^*}\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)I^*d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau}\}\\
&+\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau\frac{I}{I^*}\Big(\frac{\int_0^h p(\tau)\phi(S,I^*)d\tau}{\int_0^h p(\tau)\phi(S,I_\tau)d\tau}-1\Big)\Big(1-\frac{\frac{\int_0^h p(\tau)\phi(S,I_\tau)d\tau}{I}}{\frac{\int_0^h p(\tau)\phi(S,I^*)d\tau}{I^*}}\Big)\\
&-\delta_2R^*\Big(\sqrt{\frac{I^*R}{IR^*}}-\sqrt{\frac{IR^*}{I^*R}}\Big)^2.
\end{aligned}$$

Cosequently, we obtain

$$\begin{aligned}
\frac{dV(t)}{dt} &= \mu\Big(1-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau}\Big)\Big(S^*-S\Big)+\delta_1(R-R^*)(1-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau})\\
&+\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau\{3-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau}-\frac{I^*}{I}\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau}{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau}\\
&-\frac{I}{I^*}\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)I^*d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau}\}\\
&+\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau\frac{I}{I^*}\Big(\frac{\int_0^h p(\tau)\phi(S,I^*)d\tau}{\int_0^h p(\tau)\phi(S,I_\tau)d\tau}-1\Big)\Big(1-\frac{\int_0^h p(\tau)\Psi(S,I_\tau)d\tau}{\int_0^h p(\tau)\Psi(S,I^*)d\tau}\Big)\\
&-\delta_2R^*\Big(\sqrt{\frac{I^*R}{IR^*}}-\sqrt{\frac{IR^*}{I^*R}}\Big)^2.
\end{aligned}$$

It follows from (H_1) and corollary 7 that

$$\mu\Big(1-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau}\Big)\Big(S^*-S\Big)\leq 0,$$

and

$$\delta_1(R-R^*)(1-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau})\leq 0.$$

and, according to the assumptions (H_2) and (H_3), we have

$$\Big(\frac{\int_0^h p(\tau)\phi(S,I^*)d\tau}{\int_0^h p(\tau)\phi(S,I_\tau)d\tau}-1\Big)\Big(1-\frac{\int_0^h p(\tau)\Psi(S,I_\tau)d\tau}{\int_0^h p(\tau)\Psi(S,I^*)d\tau}\Big)\leq 0.$$

Moreover, since the arithmetic mean is greater than or equal to the geometric mean, we obtain that

$$(3-\frac{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)d\tau}-\frac{I^*}{I}\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau}{\int_0^h p(\tau)f(S^*,e^{-\mu\tau}I^*)I^*d\tau}-\frac{I}{I^*}\frac{\int_0^h p(\tau)f(S,e^{-\mu\tau}I^*)I^*d\tau}{\int_0^h p(\tau)f(S,e^{-\mu\tau}I_\tau)I_\tau d\tau})\leq 0.$$

Therefore, $\frac{dV(t)}{dt}\leq 0$ for all $t\geq T$, where the equality holds only at the equilibrium point (S^*,I^*,R^*). Thus $\{P^*\}$ is the largest invariant set in $\{(S,I,R)\Big|\frac{dV(t)}{dt}=0\}$. Consequently, we obtain, by the Lyapunov-LaSalle asymptotic stability Theorem [21, 22], that P^* is globally asymptotically stable. This completes the proof.

4. Conclusion

In this paper, we presented a mathematical analysis for an SIRSI epidemiological model applied to the evolution of the spread of disease with relapse in a given population, with a distributed delay and incidence function rate of the form $f(S,I)I$. The incidence function rate used represents a variety of possible incidence functions that could be used in epidemic models as well as virus dynamics models. By constructing two suitable Lyapunov functionals, we found the sufficient conditions of the global stability for the endemic and disease-free equilibrium of the SIRSI epidemic model with distributed delay. The basic reproduction number R_0, remains a key parameter for the stability analysis of epidemiological models: if $R_0\leq 1$, then the disease free equilibrium is globally asymptotically stable and if $R_0>1$, then the unique endemic equilibrium is globally asymptotically stable.

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