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INVERSE SCATTERING FOR THE ONE DIMENSIONAL SCHRÖDINGER EQUATION WITH THE ENERGY DEPENDENT POTENTIAL AND DISCONTINUTY CONDITIONS

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Abstract. This work studies the direct and inverse scattering problems on the real axis for the one dimensional Schrödinger equation with the potential linearly dependent on the spectral parameter and with the discontinuity conditions at some point. Using the integral representations of the Jost solutions it is investigated the properties of the scattering data, obtained the main integral equations of the inverse scattering problem and the uniqueness theorem for recovering of the potential functions is proved.

1. Introduction

Consider the differential equation

$$-y'' + q(x)y + 2kp(x)y = k^2 y\ ,\ -\infty < x < +\infty,\ x \neq a \tag{1}$$

with discontinuity conditions at a point $a \in (-\infty, +\infty)$

$$y(a-0) = \alpha y(a+0),\ y'(a-0) = \alpha^{-1} y'(a+0), \tag{2}$$

where $1 \neq \alpha > 0$, k is a complex parameter, $q(x)$ and $p(x)$ are real-valued functions. Assume the following conditions are satisfied:

$$(a)\quad \int\limits_{-\infty}^{+\infty} (1+|x|)\,|q(x)|\,dx < +\infty; \int\limits_{-\infty}^{+\infty} |p(x)|\,dx < +\infty; \tag{3}$$

(b) $p(x)$ is bounded and continuous on $\mathbb{R}$.

This work deal with the inverse scattering problem for the equation (1) with the jump conditions (2).It is well known that one of the important method in the inverse problems theory is the method which was introduced by Marchenko [1, 2, 7]. He applied the transformation operators to the solution of the inverse problems for one dimensional Schrödinger operator on a finite interval and on the half line. Transformation operators were also used in the fundamental papers of Gelfand, Levitan [3] and Levitan, Gasymov [4], where they obtained necessary and sufficient conditions

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for recovering a Sturm-Liouville operator from its spectral characteristics. Later the idea of the Marchenko method was developed by Faddeev [5, 6] (see also [7]) who solved the inverse scattering problem on the real line.

The full-line inverse scattering problem for an energy dependent (or generalized) Schrödinger equation, as a generalization of the Marchenko method, was first investigated by Jaulent an Jean [10], and by Kaup [11] in connection with a nonlinear evolution equation (see also [12, 13]). The inverse scattering problem which was considered in [11], was also investigated in [14] by reduction this problem to the inverse scattering problem for the matrix-valued energy dependent Schrödinger equation. In the case when the potential functions are real valued differentiable functions belonging to the spaces of integrable functions together with derivatives the full-line inverse scattering problem (ISP) for (3) without discrete spectrum has been studied in [15]. This problem and the inverse scattering problem on the half line for the equation (1) recently has been investigated in [16] where the differentiability assumptions on the function $p(x)$ is not required. In [26] inverse scattering problem for the equation (1) with the jump conditions (2) is investigated in the class of integrable potentials.

The direct and inverse scattering problems, also some inverse problems of the spectral analysis for (1) in various statements were studied in details by many authors. We refer for further discussion to articles [8, 20, 21, 22, 23, 24, 25, 9, 17, 18, 19] and the references therein.

In this work using the integral representations of the Jost solutions of equation (1) with the jump conditions (2), in the class (3) of the coefficients , the main integral equation of the inverse scattering problem is derived and uniqueness theorem recovering the potential functions is proved.

2. Integral representations of the Jost solutions

Let $e^{\pm}(x,k)$ be solution of (1) satisfying the conditions (2) and the condition at infinity

$$\lim_{x\to\pm\infty} e^{\pm}(x,k)e^{\pm ikx} = 1. \tag{4}$$

The solution $e^{+}(x,k)$ and $e^{-}(x,k)$ will be called the right and the left Jost-type solutions of the problem (1), (2), (3). If $q(x)\equiv p(x)\equiv 0$ then the Jost-type solutions are

$$e_0^{\pm}(x,k) = \begin{cases} e^{ikx} & ,\ \pm x > \pm a \\ A^{+}e^{\pm ikx} \pm A^{-}e^{\pm ik(2a-x)} & ,\ \pm x < \pm a \end{cases}, \tag{5}$$

where $A^{\pm} = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right)$.The Jost solution $e^{\pm}(x,k)$ is equivalently defined as the solution of the integral equation

$$e^{\pm}(x,k) = e_0^{\pm}(x,k) \pm \int_x^{\pm\infty} S_0^{\pm}(x,t,k)\left[q(t)+2kp(t)\right]e^{\pm}(t,k)dt, \tag{6}$$

where

$$S_0^{\pm}(x,t,k) = \begin{cases} \frac{\pm\sin k(t-x)}{k}, \pm a < \pm x < \pm t & \text{or } \pm x < \pm t < \pm a \\ \frac{\pm A^{+}\sin k(t-x)}{k} + \frac{A^{-}\sin k(t-2a+x)}{k} & ,\ \pm x < \pm a < \pm t\ . \end{cases}$$

Let$\sigma^{\pm}(x)=\pm\int\limits_{x}^{\pm\infty}\left[(1+|t|)\left|q(t)\right|+2\left|p(t)\right|\right]dt.$

The following theorem is proved by the same way as in [26, 16].

Theorem 1. *If the conditions* (a) *are satisfied then*

$$e^{\pm}(x,k)=R_1^{\pm}(x)e^{\pm ikx}+R_2^{\pm}(x)e^{\pm ik(2a-x)}\pm\int\limits_{x}^{\pm\infty}K_t^{\pm}(x,t)e^{\pm ikt}dt,\ \ \operatorname{Im}k\geq 0,\ \ x\in\mathbb{R}, \tag{7}$$

where

$$R_1^{\pm}(x)=e^{i\omega_{\pm}(x)},\ \ R_2^{\pm}(x)=0,\ \ \pm x>\pm a, \tag{8}$$

$$R_1^{\pm}(x)=A^{+}e^{i\omega_{\pm}(x)},R_2^{\pm}\left(x\right)=\pm A^{-}e^{-i\omega_{\pm}(x)+2i\omega_{\pm}(a)},\ \ \pm x<\pm a, \tag{9}$$

$$\omega_{\pm}\left(x\right)=\pm\int\limits_{x}^{\pm\infty}p(s)ds$$

and the bounded kernels $K^{\pm}(x,t)$*, defined on* $x\leq t<\infty$ *and* $-\infty<t\leq x$ *respectively, are differentiable with respect to* t *almost everywhere and satisfy the inequalities*

$$\pm\int\limits_{x}^{\pm\infty}\left|K_t^{\pm}(x,t)\right|dt\leq Ce^{\sigma^{\pm}(x)}. \tag{10}$$

Moreover, the functions $K^{\pm}(x,t)$ *are continuous at* $t\neq 2a-x,\ x\neq a$ *and the following relations are satisfied:*

$$K_t^{\pm}(x,x)=\left(-\frac{i}{2}p(x)+\theta^{\pm}(x)\right)e^{i\omega_{\pm}(x)},\ \ \pm x>\pm a, \tag{11}$$

$$K_t^{\pm}(x,x)=A^{+}\left(-\frac{i}{2}p(x)+\theta^{\pm}(x)\right)e^{i\omega_{\pm}(x)},\ \ \pm x<\pm a, \tag{12}$$

$$\begin{gathered}K_t^{\pm}(x,2a-x\pm 0)-K_t^{\pm}(x,2a-x\mp 0)=\\ \pm A^{-}\left(-\frac{i}{2}p(x)+\theta^{\pm}(a)\mp\frac{1}{2}\int\limits_{x}^{a}(q\left(s\right)+p^2(s))ds\right)e^{-i\omega_{\pm}(x)+2i\omega_{\pm}(a)},\ \ \pm x<\pm a,\end{gathered} \tag{13}$$

where

$$\theta^{\pm}(x)=\left(\pm\frac{1}{2}\int\limits_{x}^{\pm\infty}\left(q\left(t\right)+p^2(t)dt\right)\right).$$

Note that from the relations $(11)-(13)$ we have

$$p(x)=\left\{\begin{array}{c}2A_t^{\pm}(x,x)\sin\omega_{\pm}\left(x\right)-2B_t^{\pm}(x,x)\cos\omega_{\pm}\left(x\right),\pm x>\pm a\\ \frac{2}{A^{+}}A_t^{\pm}(x,x)\sin\omega_{\pm}\left(x\right)-\frac{2}{A^{+}}B_t^{\pm}(x,x)\cos\omega_{\pm}\left(x\right),\pm x<\pm a\end{array}\right., \tag{14}$$

$$A_t^{\pm}(x,x)\cos\omega_{\pm}\left(x\right)+B_t^{\pm}(x,x)\sin\omega_{\pm}\left(x\right)=\left\{\begin{array}{c}\theta^{\pm}(x),\ \ \pm x>\pm a\\ A^{+}\theta^{\pm}(x),\ \ \pm x<\pm a\end{array}\right., \tag{15}$$

where $A_t^{\pm}(x,x)=\operatorname{Re}K_t^{\pm}(x,x)$ and $B_t^{\pm}(x,x)=\operatorname{Im}K_t^{\pm}(x,x)$.

3. Direct Scattering Problem

Since the functions $q(x)$, $p(x)$ and the number α are real, the functions $\overline{e^{+}(x,k)}$ and $\overline{e^{-}(x,k)}$ are also solutions of the problem $(1)-(2)$ for real k. Because of

$$e^{\pm}(x,k)=e^{\pm ikx}\left[1+o(1)\right],\quad e^{\pm}(x,k)'=e^{\pm ikx}\left[\pm ikR_1^{\pm}(x)+o(1)\right],\ x\to\pm\infty \tag{16}$$

which follows from the representation (7), the Wronskian

$$W\left[e^{\pm}(x,k),\overline{e^{\pm}(x,k)}\right]=:e^{\pm}(x,k)'\overline{e^{\pm}(x,k)}-e^{\pm}(x,k)\overline{e^{\pm}(x,k)}' \tag{17}$$

is equal to $\mp 2ki$ for all real k. Consequently, when $k\neq 0$, the pairs $e^{+}(x,k)$, $\overline{e^{+}(x,k)}$ and $e^{-}(x,k)$, $\overline{e^{-}(x,k)}$ form two fundamental systems of solutions. Hence for $k\in\mathbb{R}^{*}=\mathbb{R}-\{0\}$the representation

$$e^{-}(x,k)=b(k)e^{+}(x,k)+a(k)\overline{e^{+}(x,k)} \tag{18}$$

holds, where

$$a(k)=\frac{1}{2ki}W\left[e^{-}(x,k),e^{+}(x,k)\right],\quad k\in\mathbb{R}^{*} \tag{19}$$

$$b(k)=-\frac{1}{2ki}W\left[e^{-}(x,k),\overline{e^{+}(x,k)}\right],\quad k\in\mathbb{R}^{*}. \tag{20}$$

From the formulas (19), (20) we obtain

$$|a(k)|^2-|b(k)|^2=1,\quad k\in\mathbb{R}^{*} \tag{21}$$

(18) also implies

$$e^{+}(x,k)=-\overline{b(k)}e^{-}(x,k)+a(k)\overline{e^{-}(x,k)}. \tag{22}$$

Further, from (18) and (22) we have

$$\frac{1}{a(k)}e^{-}(x,k)=\frac{b(k)}{a(k)}e^{+}(x,k)+\overline{e^{+}(x,k)} \tag{23}$$

$$\frac{1}{a(k)}e^{+}(x,k)=-\frac{\overline{b(k)}}{a(k)}e^{-}(x,k)+\overline{e^{-}(x,k)} \tag{24}$$

We put

$$u^{\pm}(x,k)=\frac{e^{\mp}(x,k)}{a(k)},\quad r^{-}(k)=\frac{b(k)}{a(k)},\quad r^{+}(k)=-\frac{\overline{b(k)}}{a(k)},\quad t(k)=\frac{1}{a(k)} \tag{25}$$

Then (23) and (24) can be rewritten as

$$u^{\pm}(x,k)=r^{\pm}(k)e^{\pm}(x,k)+\overline{e^{\pm}(x,k)} \tag{26}$$

From (16) we obtain the asymptotic formulas

$$u^{\pm}(x,k)=r^{\pm}(k)e^{\pm ikx}+e^{\mp ikx}+o\left(1\right),\quad x\to\pm\infty$$

$$u^{\pm}(x,k)=t(k)e^{\mp ikx}+o(1),\quad x\to\mp\infty$$

The solution $u^{\pm}(x,k)$ are called the eigenfunctions of the left $(u^{-}(x,k))$ and the right $(u^{+}(x,k))$ scattering problems, the coefficients $r^{-}(k)$, $r^{+}(k)$ are called the left and right reflection coefficients, respectively, and $t(k)$ is called the transmission coefficient. Since $e^{+}(x,k)$ and $e^{-}(x,k)$ are analytic on the half plane $\operatorname{Im}k>0$, the function $a(k)$ is analytically continued to the half plane $\operatorname{Im}k>0$ by the same formula (19) . It can be proved that the function $a(k)$ may have only a finite number of zeros on the half plane $\operatorname{Im}k>0$ (see [15]). But here we suppose that $a(k)$ has not any zero on the half plane $\operatorname{Im} k > 0$. From (21) we also have that $a(k) \neq 0$ for $k \in \mathbb{R}^*$. Therefore we can assume that $a(k) \neq 0$ for all $k \neq 0$. Using Theorem 1 it is easy to prove the following lemma.

Lemma 2. *For $k \in \mathbb{R}^*$ the function $a(k)$, $b(k)$ defined by formulas (19), (20) have the following representations*

$$a(k) = \frac{1-ik}{2ik}\left(-2A^+e^{i\alpha_0}V + \int_{-\infty}^{0} G(t)e^{-ikt}dt\right), \tag{27}$$

$$b(k) = \frac{1-ik}{2ik}\left(2A^-e^{-2ika-ip_0}V + \int_{-\infty}^{\infty} L(t)Ve^{-ikt}dt\right), \tag{28}$$

where

$$\alpha_0 = \int_{-\infty}^{\infty} p(t)dt,$$

$$\int_{-\infty}^{0} |G(t)|\, dt < \infty \text{ and } \int_{-\infty}^{\infty} |L(t)|\, dt < \infty.$$

Proof. By equation (6) it follows that, for real $k \neq 0$,

$$e^-(x,k) = e^{-ikx}\left[A^+ - \int_{-\infty}^{a} \frac{A^+e^{ikt} + A^-e^{ik(2a-t)}}{2ik}\left[q(t)+2kp(t)\right]e^-(t,k)dt - \right.$$
$$\left. - \int_{a}^{+\infty} \frac{e^{ikt}}{2ik}\left[q(t)+2kp(t)\right]e^-(t,k)dt\right] + e^{ikx}\left[\int_{-\infty}^{a} \frac{A^+e^{-ikt} + A^-e^{-ik(2a-t)}}{2ik}\left[q(t)+2kp(t)\right]e^-(t,k)dt + \right.$$
$$\left. + \int_{a}^{+\infty} \frac{e^{-ikt}}{2ik}\left[q(t)+2kp(t)\right]e^-(t,k)dt\right] + o(1), x \to +\infty.$$

On the other hand by (22) , we have

$$e^-(x,k) = b(k)e^{ikx} + a(k)e^{-ikx} + o(1), x \to +\infty.$$

A comparison of corresponding terms shows that

$$a(k) = A^+ - \int_{-\infty}^{a} \frac{A^+e^{ikt} + A^-e^{ik(2a-t)}}{2ik}\left[q(t)+2kp(t)\right]e^-(t,k)dt -$$
$$- \int_{a}^{+\infty} \frac{e^{ikt}}{2ik}\left[q(t)+2kp(t)\right]e^-(t,k)dt, \tag{29}$$

$$b(k) = \int_{-\infty}^{a} \frac{A^+e^{-ikt} + A^-e^{-ik(2a-t)}}{2ik}\left[q(t)+2kp(t)\right]e^-(t,k)dt +$$
$$\int_{a}^{+\infty} \frac{e^{-ikt}}{2ik}\left[q(t)+2kp(t)\right]e^-(t,k)dt. \tag{30}$$

for real $k \neq 0$. Now, from the representation (7) of the solution $e^{-}(x,k)$ we get

$$a(k) = A^{+} \exp\left(\pm i \int\limits_{-\infty}^{+\infty} p(s)ds\right) - \frac{A^{+}}{2ik} W_A + \frac{1-ik}{2ik} \int\limits_{-\infty}^{0} G(s) e^{-iks} ds, \tag{31}$$

$$b(k) = -A^{-} e^{-2ika - ip_0} + \frac{A^{-} e^{-2ika}}{2ik} W_B + \frac{1-ik}{2ik} \int\limits_{-\infty}^{\infty} L(s) e^{-iks} ds, \tag{32}$$

where W_A, W_B are constants ,

$$p_0 = \int\limits_{-\infty}^{\infty} p(t) sgn(t-a) dt \tag{33}$$

Now the formulas (27) and (28) easily are derived from (31) and (32) (see [16]). □

From the previous lemma using the Wiener-Levy theorem we can prove the following lemma.

Lemma 3. *Let the conditions* (a) *are satisfied. Then the reflection coefficient* $r^{-}(k)$ *is expressed as*

$$r^{-}(k) - r_0^{-}(k) = \int\limits_{-\infty}^{\infty} F_0(s) e^{-iks} ds, \tag{34}$$

where

$$r_0^{-}(k) = -\frac{A^{-}}{A^{+}} e^{-2ika \mp i\gamma^{+}}, \ \gamma^{+} = \int\limits_{a}^{+\infty} p(s) ds \tag{35}$$

and $\int\limits_{-\infty}^{\infty} |F_0(s)|\, ds < \infty.$

4. Main integral equation and uniqueness theorem

In order to establish the main integral equations of the scattering problem we start from the formula (23) written in the form

$$\begin{aligned}\left(\frac{1}{a(k)} - \frac{1}{A^{+} e^{i\alpha_0}}\right) e^{-}(x,k) \quad &= \quad \left(r^{-}(k) - r_0^{-}(k)\right) e^{+}(x,k) + \overline{e^{+}(x,k)} + \\ &\quad + r_0^{-}(k)\, e^{+}(x,k) - \frac{1}{A^{+} e^{i\alpha_0}} e^{-}(x,k).\end{aligned} \tag{36}$$

Multiplying both sides of this equation by $\frac{1}{2\pi} e^{iky}$,where $y > x$ and integrating with respect to k from $-\infty$ to $+\infty$, we have

$$\frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \left(\frac{1}{a(k)} - \frac{e^{-i\alpha_0}}{A^{+}}\right) e^{-}(x,k) e^{iky} dk$$

$$= \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \left(r^{-}(k) - r_0^{-}(k)\right) e^{+}(x,k) e^{iky} dk +$$

$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\overline{e^{+}(x,k)}+r_{0}^{+}(k)\,e^{+}(x,k)-\frac{1}{A^{+}}e^{-i\alpha_{0}}e^{-}(x,k)\right)e^{iky}dk \tag{37}$$

Then using the representation (7) for the solution $e^{+}(x,k)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(r^{+}(k)-r_{0}^{+}(k)\right)e^{+}(x,k)e^{iky}dk=F^{+}(x,y)+\int_{x}^{+\infty}K_{t}^{+}(x,t)F_{0}^{+}(t+y)dt,$$

where

$$F^{+}(x,y)=\begin{cases}R_{1}^{+}(x)F_{0}(x+y) & ,\ x>a\\ R_{1}^{+}(x)F_{0}(x+y)+R_{2}^{+}(x)F_{0}(2a-x+y) & ,\ x<a\end{cases} \tag{38}$$

$$F_{0}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(r^{+}(k)-r_{0}^{+}(k)\right)e^{ikx}dk.$$

For the second integral in the right-hand side of (37) we have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\overline{e^{+}(x,k)}+r_{0}^{+}(k)\,e^{+}(x,k)-\frac{1}{A^{+}}e^{-i\alpha_{0}}e^{-}(x,k)\right]e^{iky}dk$$

$$=\overline{K_{y}^{+}(x,y)}-\frac{A^{-}}{A^{+}}e^{-2i\gamma^{+}}K_{y}^{+}(x,2a-y),$$

therefore equation (37) takes the form

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{a(k)}-\frac{e^{-i\alpha_{0}}}{A^{+}}\right)e^{-}(x,k)e^{iky}dy=F^{+}(x,y)+\int_{x}^{+\infty}K_{t}^{+}(x,t)F_{0}(t+y)dt$$

$$+\overline{K_{y}^{+}(x,y)}-\frac{A^{-}}{A^{+}}e^{-2i\gamma^{+}}K_{y}^{+}(x,2a-y),\quad (y>x) \tag{39}$$

Now consider the left-hand side of (39). Using the Lemma 1 and the Paley-Wiener theorem we can easily compute that (see [16])

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{a(k)}-\frac{e^{-i\alpha_{0}}}{A^{+}}\right)e^{-}(x,k)e^{iky}dy=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{a(k)}-\frac{e^{-i\alpha_{0}}}{A^{+}}\right)e^{-}(x,k)e^{ikx}e^{ik(y-x)}dk=0.$$

Then (39) shows that

$$F^{+}(x,y)+\overline{K_{y}^{+}(x,y)}-\frac{A^{-}}{A^{+}}e^{-2i\gamma^{+}}K_{y}^{+}(x,2a-y)+\int_{x}^{+\infty}K_{y}^{+}(x,t)F_{0}(t+y)dt=0,\quad (y\geq x)\,. \tag{40}$$

Integrating both sides of (40) , we get

$$\overline{K^{+}(x,t)}-\frac{A^{-}}{A^{+}}e^{-2i\gamma^{+}}K^{+}(x,2a-t)+\int_{x}^{+\infty}K_{r}^{+}(x,r)dr\int_{r+t}^{+\infty}F_{0}(s)ds+$$

$$+R_{1}^{+}(x)\int_{x+t}^{+\infty}F_{0}(s)ds+R_{2}^{+}(x))\int_{2a-x+t}^{+\infty}F_{0}(s)ds=0,t\geq x.$$

Performing integration by parts we have the following main integral equation:

$$K^{+}(x,t) \mp \frac{A^{-}}{A^{+}} e^{\mp 2i\gamma^{+}} K^{+}(x,2a-t) + \int\limits_{x}^{+\infty} K^{+}(x,r) F_0(r+t)dr +$$

$$+ h_1^{+}(x) \int\limits_{x+t}^{+\infty} F_0(s)ds + h_2^{+}(x) \int\limits_{2a-x+t}^{+\infty} F_0(s)ds = 0, \tag{41}$$

where

$$h_1^{+}(x) = R_1^{+}(x) - K^{+}(x,x),$$
$$h_2^{+}(x) = R_2^{+}(x) - K^{+}(x,2a-x+0) + K^{+}(x,2a-x-0).$$

Obviously,

$$h_1^{+}(x) + h_2^{+}(x) = e^{+}(x,0) \tag{42}$$

The uniqueness property and the solution of the inverse problem can be proved by the same arguments as in [26]. The main integral equations (41) can be rewritten in the form

$$e^{+}(x,0) \int\limits_{x+y}^{+\infty} F_0(s)ds + \overline{K^{+}(x,y)} +$$

$$+ \int\limits_{x}^{+\infty} K^{+}(x,t) F_0(t+y)dt = 0, \ x > a, \ y > x \ , \tag{42}$$

$$h_1^{+}(x) \int\limits_{x+y}^{+\infty} F_0(s)ds + h_2^{+}(x) \int\limits_{2a-x+y}^{+\infty} F_0(s)ds + \overline{K^{+}(x,y)} - \frac{A^{-}}{A^{+}} e^{-2i\gamma^{+}} K^{+}(x,2a-y) +$$

$$+ \int\limits_{x}^{+\infty} K^{+}(x,t) F_0(t+y)dt = 0, \ x < a, \ y < x < (2a-x)\,, \tag{43}$$

$$h_1^{+}(x) \int\limits_{x+y}^{+\infty} F_0(s)ds + h_2^{+}(x) \int\limits_{2a-x+y}^{+\infty} F_0(s)ds + \overline{K^{+}(x,y)} +$$

$$\int\limits_{x}^{+\infty} K^{+}(x,t) F_0(t+y)dt = 0, \ x < a, \ y > (2a-x) \tag{44}$$

Theorem 4. *If the conditions* (3) *are satisfied then equations* (42) - (44) *have the unique solutions* $K^{+}(x,.) \in L_1(x,\infty)$ *for each fixed* $x > -\infty$.

Proof: For each fixed $x > -\infty$ consider the operator (see [18])

$$(M_x^{+} f)y = \begin{cases} \overline{f(y)} & , \ x > a \\ \overline{f(y)} - \frac{A^{-}}{A^{+}} e^{-2i\gamma^{+}} f(2a-y) & , \ x < a \end{cases}$$

acting in the space $L_1(x,\infty)$ (and also $L_2(x,\infty)$). It is easy to show that the operator M_x^{+} is invertible. Using this operator the main equation (41) can be rewritten as

$$\overline{K^{+}(x,y)} + (M_x^{+})^{-1} F^{+}(x,y) + (M_x^{+})^{-1} \phi^{+} K^{+}(x,.)(y) = 0 \ , \ y > x \tag{45}$$

where the operator ϕ^+ is defined as

$$\phi^+ f(y) = \int_x^{+\infty} F_0(t+y) f(t) dt, \ y > x \tag{46}$$

for each fixed $x > -\infty$.

It is known that (see [7]) the operator ϕ^+ is a compact operator in the space $L_1(x,\infty)$ (also in $L_2(x,\infty)$). By the boundness of the operator M_x^{-1} we have that the operator $M_x^{-1}\phi^+$ is also a compact operator. Therefore, to prove the theorem, it is sufficient to show that the homogeneous equation

$$\overline{h_x(y)} - \frac{A^-}{A^+} e^{-2i\gamma^+} h_x(2a-y) + \int_x^{+\infty} h_x(t) F_0(t+y) dx = 0, \ y > x \tag{47}$$

has only the trivial solution $h_x(y) \in L_1(x,\infty)$. By conditions (3) the function $F_0^+(y)$ and the corresponding solution $h_x(y)$ are bounded in the half axis $x \le y < +\infty$. Therefore $h_x(.) \in L_2(x,\infty)$. Consequently, we have

$$0 = \int_x^{+\infty} |h_x(y)|^2 dy - \frac{A^-}{A^+} e^{-2i\gamma^+} \int_x^{+\infty} h_x(2a-y) h_x(y) dy + \int_x^{+\infty}\int_x^{+\infty} h_x(t) h_x(y) F_0(t+y) dt dy \tag{48}$$

Using the the Parsevall's identities we get

$$\int_x^{+\infty} |h_x(y)|^2 dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\widetilde{h}(\lambda)\right|^2 d\lambda,$$

$$-\frac{A^-}{A^+} e^{-2i\gamma^+} \int_x^{+\infty} h_x(y) h_x(2a-y) dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} r_0^-(\lambda) \widetilde{h}^2(\lambda) d\lambda,$$

where $\widetilde{h}(\lambda) = \int\limits_x^{+\infty} h_x(t) e^{-i\lambda t} dt$, we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\widetilde{h}(\lambda)\right|^2 d\lambda + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(r^-(\lambda) - r_0^-(\lambda)\right) \widetilde{h}^2(\lambda) d\lambda + \frac{1}{2\pi} \int_{-\infty}^{+\infty} r_0^-(\lambda) \widetilde{h}^2(\lambda) d\lambda = 0$$

i.e.

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\widetilde{h}(\lambda)\right|^2 d\lambda = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} r^-(\lambda) \widetilde{h}^2(\lambda) d\lambda.$$

Therefore

$$\int_{-\infty}^{+\infty} \left|\widetilde{h}(\lambda)\right|^2 d\lambda = \left| - \int_{-\infty}^{+\infty} r^-(\lambda) \widetilde{h}^2(\lambda) d\lambda \right| \le \int_{-\infty}^{+\infty} \left|r^-(\lambda)\right| \left|\widetilde{h}(\lambda)\right|^2 d\lambda,$$

that is

$$\int_{-\infty}^{+\infty} \left(1 - \left|r^-(\lambda)\right|\right) \left|\widetilde{h}(\lambda)\right|^2 d\lambda \le 0. \tag{49}$$

Since $|r^{-}(\lambda)|<1$ for $\lambda\neq 0$, (49) implies that $\widetilde{h}(\lambda)\equiv 0$. Consequently the equation (41) has a unique solution.

This theorem implies that the potential functions $q(x)$ and $p(x)$ from class (3) in problem $(1)-(2)$ without discrete spectrum are uniquely defined by the left reflection coefficient.

Theorem 5. *If conditions* (3) *are satisfied and there exist a pair real-valued matrix functions* $(q(x),p(x))$ *which has a given matrix* $r^{-}(k)$ *as its reflection coefficient, then* $(q(x),p(x))$ *is recovered from* $\widetilde{R}^{\pm}(k)$ *by* $(11)-(13)$ *where* $K^{+}(x,t)$ *is the solution of* (41) *with* $F_0(x)$ *defined by*

$$F_0(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(r^{+}(k)-r_0^{+}(k)\right)e^{ikx}dk \tag{50}$$

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