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UNBOUNDED PERTURBATION TO TIME-DEPENDENT SUBDIFFERENTIAL OPERATORS WITH DELAY

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Abstract. In this paper, we discuss the existence of solution for an evolution inclusion of the form $-\dot{x}(t) \in \partial\varphi(t, x(t)) + G(t, \tau(t)x)$ where $\partial\varphi(\cdot,\cdot)$ is a subdifferential of a proper convex lower semicontinous function φ and $G(\cdot,\cdot)$ is a set-valued unbounded perturbation with nonempty convex closed values with delay.

1. Introduction

Let $r > 0$ be a finite delay and $\mathcal{C}_0 = \mathcal{C}_H([-r, 0])$ be the Banach space of continuous functions defined on $[-r, 0]$ taking values in a Hilbert space H. In this paper, we establish the existence of solutions for an evolution inclusion of the form

$$(P_r)\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + G(t, \tau(t)x) \text{ a.e. } t \in [0, T], \\ x(s) = \psi(s) \quad \text{for all } s \in [-r, 0], \end{cases}$$

where $x : [-r, T] \to H$ is a continuous mapping such that its restriction to $[0, T]$ is absolutely continuous, $G : [0, T] \times \mathcal{C}_0 \rightharpoonup H$ is a scalarly upper semicontinuous set-valued mapping with convex closed values and for any $t \in [0, T]$, $\tau(t) : \mathcal{C}_H([-r, T]) \to \mathcal{C}_0$ is defined by $(\tau(t)x)(s) = x(t + s)$, $\forall\, s \in [-r, 0]$.
Such problems have been extensively studied without delay, see for instance [4, 5, 6, 7, 8, 13, 15, 16, 17], and with delay [18, 19]. In [7], we have established an existence result for the problem without delay when the system is subject to an unbounded perturbation.
A special case had great interest for a large number of researchers because of its various applications in elasto-plasticity, mechanics, economics, transportation, electrical circuits, dynamics, known as the *sweeping process*, that is when we take the subdifferential of the indicator function of a closed convex set. It is well known that this subdifferential is a maximal monotone operator governed by a normal cone, which makes possible to use of the projection property and to deduce an algorithm, called *catching up algorithm*. Several papers used this approach to generalize the pioneer work of Moreau [14], see for instance [1, 2, 3, 7, 10, 11, 20] and the references

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therein. It is well known that the delayed problem generalizes the problem without delay and the existence result can be obtained for the first problem by reducing it to the second one and using the known results for this case, see for instance [9, 12]. The main purpose of the present work is to show the existence of solution for (P_r). We proceed as follows: for each $n \in \mathbb{N}$, we consider a partition of the interval $[0, T]$, in each subinterval, making use of appropriate functions g_k^n (see the definition below), we replace the delayed perturbation G by an undelayed one $G_k^n(t,x) = G(t, \tau(t_{k+1}^n)g_k^n(\cdot, x))$ for which our result in [7] insures the existence of solution. In [18], a single valued perturbation satisfying a linear growth condition has been considered, here, we extend the result to a set-valued perturbation with non necessary bounded values. The paper is organized as follows. In Section 2, we give some preliminaries and recall some results which will be used in the paper. In Section 3, we establish the existence theorem for the considered problem (P_r).

2. Preliminaries

Through the paper, we will use the following notations and definitions:

• H is a separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$.

• For any r, $T \geq 0$, $\mathcal{C}_T = \mathcal{C}_H([-r, T])$ is the Banach space of all continuous mappings from $[-r, T]$ to the space H.

• Let $\varphi : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued lower semicontinuous function, which is proper in the sense that its effective domain $dom\varphi$ defined by

$$dom\ \varphi := \{x \in H \ : \ \varphi(x) < +\infty\}$$

is nonempty.

• The Fenchel conjugate of φ is defined by

$$\varphi^*(v) := \sup_{x \in H} [\langle v, x \rangle - \varphi(x)].$$

• The subdifferential $\partial\varphi(x)$ of φ at $x \in dom\ \varphi$ is defined by

$$\partial\varphi(x) = \{v \in H : \ \langle v, y - x \rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in dom\ \varphi\}$$

and its effective domain is $dom\ \partial\varphi = \{x \in \ H : \ \partial\varphi(x) \neq \emptyset\}$.

• If φ is a proper lower semicontinuous convex function, then its subdifferential operator $\partial\varphi$ is a maximal monotone operator. Any maximal monotone operator A satisfies the closure property, that is; if $x = \lim\limits_{n \longrightarrow \infty} x_n$ strongly in H and $y = \lim\limits_{n \longrightarrow \infty} y_n$ weakly in H, where $x_n \in domA$ and $y_n \in A(x_n)$, then, $x \in domA$ and $y \in A(x)$.

• The function φ is said to be inf-ball-compact if for every $\lambda > 0$, the set $\{x \in H : \ \varphi(x) \leq \lambda\}$ is ball-compact, i.e., its intersection with any closed ball in H is compact.

• For any subset C of H, $\overline{co}\, C$ stands for the closed convex hull of C.

• $\delta^*(\cdot, C)$ represents the support function of C, that is, $\delta^*(\xi, C) := \sup\limits_{x \in C} \langle \xi, x \rangle$, for all $\xi \in H$. Recall that for a closed convex subset C we have

$$d(x, C) = \sup_{x' \in \overline{\mathbf{B}}} [\langle x', x \rangle - \delta^*(x', C)].$$

• We denote by $Proj(\cdot, C)$ the metric projection mapping onto the closed set C, defined by

$$Proj(x, C) := \{v \in C : d(x, C) = \|v - x\|\}.$$

• A set-valued mapping $G : E \rightharpoonup H$ from a Hausdorff topological space E into H is said to be upper semicontinuous if, for any open subset $V \subset H$, the set $\{x \in E : G(x) \subset V\}$ is open in E.
• G is said to be scalarly upper semicontinuous or upper hemicontinuous if, for any $y \in H$, the real-valued function $x \mapsto \sigma(y, G(x))$ is upper semicontinuous.
Let us recall the following result due to [7].

Theorem 1 Let $\varphi : [0, T] \times H \to \mathbb{R}_+ \cup \{+\infty\}$ be such that:

($\mathcal{H}_1$) for each $t \in [0, T]$, the function $x \mapsto \varphi(t, x)$ is proper convex lower semicontinuous.

($\mathcal{H}_2$) there exist a ρ-Lipschitzean function $k : H \longrightarrow \mathbb{R}_+$ and an absolutely continuous function $a : [0, T] \longrightarrow \mathbb{R}$, with a non-negative derivative $\dot{a} \in L^2_{\mathbb{R}}([0, T])$, such that

$$\varphi^*(t, x) \leq \varphi^*(s, x) + k(x) \mid a(t) - a(s) \mid$$

for every $(t, s, x) \in [0, T] \times [0, T] \times H$.

($\mathcal{H}_3$) φ is inf-ball-compact for every $t \in [0, T]$.

Let $G : [0, T] \times H \rightharpoonup H$ be a set-valued mapping with nonempty closed convex values, satisfying the following assumptions:

($\mathcal{H}_4$) G is measurable on both variables and upper hemicontinuous with respect to the second variable.

($\mathcal{H}_5$) For some real $\alpha > 0$:

$$d(0, G(t, x)) \leq \alpha \quad \text{for all } t \in [0, T] \text{ and } x \in H.$$

Then, for any $x_0 \in dom\, \varphi(0, \cdot)$ the problem

$$(P) \begin{cases} -\dot{x}(t) \in \partial \varphi(t, x(t)) + G(t, x(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_0 \in dom\ \varphi(0, \cdot) \end{cases}$$

has at least one absolutely continuous solution, satisfying

$$\int_0^T \|\dot{x}(t)\|^2 dt \leq c,$$

where

$$c = 2c_0 \int_0^T \dot{a}^2(t)dt + \sigma\alpha^2 T + 2\Big(T + \varphi(0, x_0)\Big),$$

$$c_0 = \frac{1}{2}\big(k^2(0) + 3(\rho + 1)^2\big),$$

$$\sigma = k^2(0) + 3(\rho + 1)^2 + 4.$$

3. Existence result

We study here the existence of solution for the problem (P_r).

Theorem 2 Assume that $\varphi : [0,T] \times H \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ of Theorem 1. Let $G : [0,T] \times \mathcal{C}_0 \rightharpoonup H$ be a set-valued mapping with nonempty closed convex values, satisfying the following assumptions:

$(\mathcal{H}_4)$ G is measurable on both variables and upper hemicontinuous with respect to the second variable..

$(\mathcal{H}_5)$ For some real $\alpha > 0$, $d\big(0, G(t, \tau(t)x)\big) \leq \alpha$ for all $t \in [0,T]$ and $x \in \mathcal{C}_0$.

Then, for any $\psi \in \mathcal{C}_0$ with $\psi(0) \in dom\, \varphi(0,\cdot)$, the problem (P_r) has at least one absolutely continuous solution, satisfying

$$\int_0^T \|\dot{x}(t)\|^2 dt \leq c.$$

Proof. **Step 1:** We construct a sequence of continuous mappings $(x_n(\cdot))_n$ in $\mathcal{C}_H([-r,T])$. Define, for every $n \geq 1$, the partition of $[0,T]$: for each $0 \leq k \leq n$, $t_k^n = k\dfrac{T}{n}$ and $I_k^n =]t_k^n, t_{k+1}^n]$. Consider the function $g_0^n : [-r, t_1^n] \times H \longrightarrow H$ such that

$$g_0^n(t,x) = \begin{cases} \psi(t) & \text{if } t \in [-r, 0], \\ \psi(0) + \frac{n}{T} t (x - \psi(0)) & \text{if } t \in [0, t_1^n]. \end{cases}$$

Next, define the set-valued mapping $G_0^n : [0, t_1^n] \times H \rightharpoonup H$ by:

$$G_0^n(t,x) = G(t, \tau(t_1^n) g_0^n(\cdot, x)).$$

Now, let us prove that the function $x \mapsto \tau(t_1^n) g_0^n(\cdot, x)$ is continuous from H to $\mathcal{C}_0$. For each $x, y \in H$, we have

$$\begin{aligned} \| \tau(t_1^n) g_0^n(\cdot,x) - \tau(t_1^n) g_0^n(\cdot,y) \| &= \sup_{s\in[-r,0]} \| g_0^n(t_1^n + s, x) - g_0^n(t_1^n + s, y) \| \\ &= \sup_{s \in [-r + t_1^n, t_1^n]} \| g_0^n(s,x) - g_0^n(s,y) \| \\ &\leq \sup_{s\in[0,t_1^n]} \| g_0^n(s,x) - g_0^n(s,y) \| \leq \sup_{s\in[0,t_1^n]} \frac{n}{T} s \| x - y \| \leq \| x - y \| . \end{aligned}$$

Then, the function $x \mapsto \tau(t_1^n) g_0^n(\cdot,x)$ is continuous. Hence the set-valued mapping G_0^n is measurable on both variables, upper hemicontinuous with respect to the second variable and satisfies

$$d(0, G_0^n(t,x)) = d(0, G(t, \tau(t_1^n) g_0^n(\cdot, x))) < \alpha, \quad \forall\, (t,x) \in [0, t_1^n] \times H.$$

An application of Theorem gives an absolutely continuous solution $x_0^n : [0, t_1^n] \longrightarrow H$ such that

$$-\dot{x}_0^n(t) \in \partial \varphi(t, x_0^n(t)) + G(t, \tau(t_1^n) g_0^n(\cdot, x_0)),$$

let $z_0^n(t) = Proj\big(0, G(t, \tau(t_1^n) g_0^n(\cdot, x_0)\big)$ then

$$\| z_0^n(t) \| \leq \alpha,$$

$$-\dot{x}_0^n(t) \in \partial \varphi(t, x_0^n(t)) + z_0^n(t) \quad \text{for } a.e.\, t \in [0, t_1^n],$$

and

$$\int_0^{t_1^n} \| \dot{x}_0^n(t) \|^2\, dt \leq 2c_0 \int_0^{t_1^n} \dot{a}^2(t) dt + \sigma \int_0^{t_1^n} \alpha^2 dt + d_0,$$

with

$$d_0 = 2\Bigg((t_1^n - 0) + \varphi(0, x_0^n(0) - \varphi(t_1^n, x_0^n(t_1^n))\Bigg).$$

Now, define $g_1^n : [-r, t_2^n] \times H \to H$ by

$$g_1^n(t,x) = \begin{cases} \psi(t) & \text{if } t \in [-r, 0], \\ x_0^n(t) & \text{if } t \in [0, t_1^n], \\ x_0^n(t_1^n) + \frac{n}{T}(t - t_1^n)(x - x_0^n(t_1^n)) & \text{if } t \in [t_1^n, t_2^n]. \end{cases}$$

Let us show that the function $x \mapsto \tau(t_2^n) g_1^n(\cdot, x)$ from H to $\mathcal{C}_0$ is continuous. Indeed, for every $x, y \in H$ we have

$$\| \tau(t_2^n)g_1^n(\cdot,x) - \tau(t_2^n)g_1^n(\cdot,y) \| = \sup_{s\in[-r,0]} \| g_1^n(t_2^n + s, x) - g_1^n(t_2^n + s, y) \|$$

$$= \sup_{s\in[-r+t_2^n, t_2^n]} \| g_1^n(s,x) - g_1^n(s,y) \|$$

$$\leq \sup_{s\in[t_1^n,t_2^n]} \| g_1^n(s,x) - g_1^n(s,y) \| \leq \sup_{s\in[t_1^n,t_2^n]} \frac{n}{T}(s - t_1^n) \| x - y \| \leq \| x - y \| .$$

Then, the function $x \mapsto \tau(t_2^n) g_1^n(\cdot, x)$ is continuous. Let define a set-valued mapping $G_1^n : [t_1^n, t_2^n] \times H \rightharpoonup H$ by $G_1^n(t,x) = G(t, \tau(t_2^n) g_1^n(\cdot, x))$. Hence it is measurable on both variables, upper hemicontinuous with respect to the second variable and satisfies

$$d(0, G_1^n(t,x)) = d(0, G(t, \tau(t_2^n)g_1^n(\cdot,x))) < \alpha, \text{ for each } (t,x) \in [t_1^n, t_2^n] \times H.$$

An application of Theorem 1 gives an absolutely continuous solution $x_1^n : [t_1^n, t_2^n] \mapsto H$ such that

$$-\dot{x}_1^n(t) \in \partial\varphi(t, x_1^n(t)) + G(t, \tau(t_2^n)g_1^n(\cdot, x_1)),$$

let $z_1^n = Proj\big(0, G(t, \tau(t_1^n)g_1^n(\cdot, x_1)\big)$ then

$$-\dot{x}_1^n(t) \in \partial\varphi(t, x_1^n(t)) + z_1^n(t) \text{ for } a.e.\, t \in [t_1^n, t_2^n],$$

$$\| z_1^n(t) \| \leq \alpha,$$

$$\int_{t_1^n}^{t_2^n} \| \dot{x}_1^n(t) \|^2 \, dt \leq 2c_0 \int_{t_1^n}^{t_2^n} \dot{a}^2(t) dt + \sigma \int_{t_1^n}^{t_2^n} \alpha^2 dt + d_1,$$

with

$$d_1 = 2\Bigg((t_2^n - t_1^n) + \varphi(t_1^n, x_1^n(t_1^n) - \varphi(t_2^n, x_1^n(t_2^n))\Bigg).$$

Now suppose that $\big(x_{k-1}^n(\cdot), z_{k-1}^n(\cdot)_n\big)$ is defined on $[-r, t_k^n]$, $(1 \leq k \leq n)$ such that

$$\begin{cases} -\dot{x}_{k-1}^n(t) \in \partial\varphi(t, x_{k-1}^n(t)) + z_{k-1}^n(t) & \text{for a.e. } t \in [t_{k-1}^n, t_k^n], \\ z_{k-1}^n(t) = Proj\big(0, G(t, \tau(t_k^n)g_{k-1}^n(\cdot, x))\big), \\ \| z_{k-1}^n(t) \| \leq \alpha, \end{cases}$$

and

$$\int_{t_{k-1}^n}^{t_k^n} \| \dot{x}_{k-1}^n(t) \|^2 \, dt \leq 2c_0 \int_{t_{k-1}^n}^{t_k^n} \dot{a}^2(t) dt + \sigma \int_{t_{k-1}^n}^{t_k^n} \alpha^2 dt + d_{k-1},$$

with

$$d_{k-1} = 2\Bigg((t_k^n - t_{k-1}^n) + \varphi(t_{k-1}^n, x_{k-1}^n(t_{k-1}^n)) - \varphi(t_k^n, x_{k-1}^n(t_k^n))\Bigg),$$

where $g_{k-1}^n : [-r, t_k^n] \times H \to H$ is defined by

$$g_{k-1}^n(t,x) = \begin{cases} \psi(t) & \text{if } t \in [-r,0], \\ x_i^n(t) & \text{if } t \in [t_{i-1}^n, t_i^n] \quad (0 \le i \le k-1), \\ x_{k-2}^n(t_{k-1}^n) + \frac{n}{T}(t - t_{k-1}^n)(x - x_{k-2}^n(t_{k-1}^n)) & \text{if } t \in [t_{k-1}^n, t_k^n]. \end{cases}$$

Hence, we can define similarly $g_k^n : [-r, t_{k+1}^n] \times H \to H$ by

$$g_k^n(t,x) = \begin{cases} \psi(t) & \text{if } t \in [-r,0], \\ x_i^n(t) & \text{if } t \in [t_i^n, t_{i+1}^n] \quad (0 \le i \le k-1), \\ x_{k-1}^n(t_k^n) + \frac{n}{T}(t - t_k^n)(x - x_{k-1}^n(t_k^n)) & \text{if } t \in [t_k^n, t_{k+1}^n]. \end{cases}$$

So, we show that the mapping $x \mapsto \tau(t_{k+1}^n) g_k^n(\cdot, x)$ from H to $\mathcal{C}_0$ is continuous. Indeed, for every $x, y \in H$, we have

$$\| \tau(t_{k+1}^n) g_k^n(\cdot,x) - \tau(t_{k+1}^n) g_k^n(\cdot,y) \| = \sup_{s \in [-r,0]} \| g_k^n(t_{k+1}^n + s, x) - g_k^n(t_{k+1}^n + s, y) \|$$

$$= \sup_{s \in [-r + t_{k+1}^n, t_{k+1}^n]} \| g_k^n(s,x) - g_k^n(s,y) \|$$

$$\le \sup_{s \in [t_k^n, t_{k+1}^n]} \| g_k^n(s,x) - g_k^n(s,y) \| \le \sup_{s \in [t_k^n, t_{k+1}^n]} \frac{n}{T}(s - t_k^n) \| x - y \| \le \| x - y \| .$$

It follows that the set-valued mapping $G_k^n(t,x) = G(t, \tau(t_{k+1}^n) g_k^n(\cdot,x))$ is measurable on both variables, upper hemicontinuous with respect to the second variable and satisfies

$$d(0, G_k^n(t,x)) = d(0, G(t, \tau(t_{k+1}^n) g_k^n(\cdot,x))) < \alpha, \text{ for all } (t,x) \in [t_k^n, t_{k+1}^n] \times H.$$

An application of Theorem 1 gives an absolutely continuous solution $x_k^n : [t_k^n, t_{k+1}^n] \longrightarrow H$ and the function $z_k^n : [t_k^n, t_{k+1}^n] \to H$ such that

$$\begin{cases} -\dot{x}_k^n(t) \in \partial \varphi(t, x_k^n(t)) + z_k^n(t) & \text{for } a.e.\, t \in [t_k^n, t_{k+1}^n], \\ z_k^n(t) = Proj\big(0, G(t, \tau(t_{k+1}^n) g_k^n(\cdot, x_k^n)\big), \\ \| z_k^n(t) \| \le \alpha \end{cases}$$

and

$$\int_{t_k^n}^{t_{k+1}^n} \| \dot{x}_k^n(t) \|^2 \, dt \le 2c_0 \int_{t_k^n}^{t_{k+1}^n} \dot{a}^2(t) dt + \sigma \int_{t_k^n}^{t_{k+1}^n} \alpha^2 dt + d_k, \tag{1}$$

with

$$d_k = 2\Big((t_{k+1}^n - t_k^n) + \varphi(t_k^n, x_k^n(t_k^n) - \varphi(t_{k+1}^n, x_k^n(t_{k+1}^n)) \Big).$$

Now we define $x_n(\cdot) : [-r, T] \to H$ by

$$x_n(t) = \begin{cases} \psi(t) & \text{if } t \in [-r,0], \\ x_k^n(t) & \text{if } t \in [t_k^n, t_{k+1}^n] \quad (0 \le k \le n-1), \end{cases}$$

and $z_n : [0,T] \to H$ by $z_n(t) = z_k^n(t)$ for all $t \in]t_k^n, t_{k+1}^n]$, what allows to write

$$-\dot{x}_n(t) \in \partial \varphi(t, x_n(t)) + z_n(t), \quad \text{a.e. } t \in [0, T]$$

with

$$x_n(s) = \psi(s), \quad \text{for all } s \in [-r, 0],$$

and

$$\| z_n(t) \| \le \alpha.$$

So, for all $k \in \{0, \cdots, n-1\}$,

$$g_k^n(t,x) = \begin{cases} x_n(t) & \text{if } t \in [-r, t_k^n], \\ x_n(t_k^n) + \frac{n}{T}(t - t_k^n)(x - x_n(t_k^n)) & \text{if } t \in [t_k^n, t_{k+1}^n]. \end{cases}$$

Now, consider the functions $\delta_n, \theta_n : [0,T] \to [0,T]$ such that for any $k \in \{0, \cdots, n-1\}$, $\delta_n(t) = t_k^n \ \forall t \in [t_k^n, t_{k+1}^n[$, $\delta_n(T) = T$ and $\theta_n(t) = t_{k+1}^n \ \forall t \in]t_k^n, t_{k+1}^n]$, $\theta_n(0) = 0$. Then, for each $n \geq 1$, we have

$$z_n(t) \in G(t, \tau(\theta_n(t)) g^n_{\delta_n(t)\frac{n}{T}}(\cdot, x_n(t))), \tag{2}$$

Further, from (1),one has for any $k \in \{0, \cdots, n-1\}$

$$\sum_{k=0}^{n-1} \int_{t_k{}^n}^{t_{k+1}^n} \|\dot{x}_n(t)\|^2 dt \leq 2c_0 \sum_{k=0}^{n-1} \int_{t_k{}^n}^{t_{k+1}^n} \dot{a}^2(t) dt + \sigma\alpha^2 \sum_{k=0}^{n-1} \int_{t_k{}^n}^{t_{k+1}^n} dt + \sum_{k=0}^{n-1} d_k;$$

equivalently,

$$\begin{aligned} \int_0^T \|\dot{x}_n(t)\|^2 dt &\leq 2c_0 \int_0^T \dot{a}^2(t) dt + \sigma\alpha^2 \int_0^T dt + d_n \\ &\leq 2c_0 \int_0^T \dot{a}^2(t) dt + \sigma\alpha^2 T + d_n, \end{aligned}$$

with

$$d_n = 2\Big(T + \varphi(0, x_0) - \varphi(T, x_n(T))\Big),$$

because φ is non-negative, putting $d = 2(T + \varphi(0, x_0))$, we may write

$$\int_0^T \|\dot{x}_n(t)\|^2 dt \leq 2c_0 \int_0^T \dot{a}^2(t) dt + \sigma\alpha^2 T + d,$$

then, for all n

$$\int_0^T \|\dot{x}_n(t)\|^2 dt \leq D,$$

where

$$D = 2c_0 \int_0^T \dot{a}^2(t) dt + \sigma\alpha^2 T + d,$$

then

$$\sup_{n \in \mathbb{N}} \int_0^T \|\dot{x}_n(t)\|^2 dt \leq D. \tag{3}$$

Step 2: We will prove that $(x_n(\cdot))$ converges uniformly on $[-r, T]$ to a continuous mapping $x(\cdot) \in \mathcal{C}_T$.
Using the Cauchy-Schwartz inequality and (3), for all $s \in [0, T]$ we obtain

$$\| x_n(s) - x_n(0) \|^2 = \| x_n(s) - x_0 \|^2 \leq s \int_0^s \| \dot{x}_n(t) \|^2 dt \leq TD$$

and hence

$$\| x_n(s) \|^2 \leq 2 \| x_0 \|^2 + 2 \| x_n(s) - x_0 \|^2 \leq 2 \| x_0 \|^2 + 2TD.$$

Consequently, for each n, we get

$$\| x_n \|_\infty^2 \leq 2 \| x_0 \|^2 + 2TD.$$

Then

$$\| x_n(\cdot) \|_\infty \leq \Big(2 \| x_0 \|^2 + 2TD\Big)^{\frac{1}{2}}. \tag{4}$$

Therefore

$$\begin{aligned}\| x_n(t) - x_n(s) \| &= \| \int_s^t \dot{x}_n(\tau) d\tau \| \\ &\leq (t-s)^{\frac{1}{2}} (\int_s^t \| \dot{x}_n(\tau) \|^2 \, d\tau)^{\frac{1}{2}} \\ &\leq (t-s)^{\frac{1}{2}} (\int_0^T \| \dot{x}_n(\tau) \|^2 \, d\tau)^{\frac{1}{2}} \leq (t-s)^{\frac{1}{2}} D,\end{aligned}$$

so along with (4), the set $\{(x_n(\cdot))_n\}$ is bounded and equicontinuous in $\mathcal{C}_H([0,T])$, since φ is inf-ball-compact by assumption, the set $\{x_n(t);\ n \in \mathbb{N}\}$ is relatively compact in H, so by Ascoli's theorem, we can extract a subsequence of $(x_n(\cdot))_n$ that converges uniformly on $[0,T]$ to some map $w(\cdot) \in \mathcal{C}_H([0,T])$. From (3), $(\dot{x}_n)_n$ is bounded in $L^2_H([0,T])$, we may then extract a subsequence converging weakly in $L^2_H([0,T])$ to some map $v(\cdot)$. The equality

$$x_n(t) = x_n(0) + \int_0^t \dot{x}_n(s) ds \quad \text{for all } t \in [0,T],$$

then yields

$$w(t) = \psi(0) + \int_0^t v(s) ds \quad \text{for all } t \in [0,T].$$

For each $t \in [-r,T]$, we set

$$x(t) = \begin{cases} \psi(t) & \text{if } t \in [-r,0], \\ w(t) & \text{if } t \in [0,T]. \end{cases}$$

Then $(x_n(\cdot))_n$ converges to $x(\cdot) \in \mathcal{C}_H([-r,T])$.

Step 3: We show that $x(\cdot)$ is a solution of (P_r).

Since G is measurable on both variables and $\|z_n(t)\| \leq \alpha$ for all $n \in \mathbb{N}$ and $t \in [0,T]$, we may suppose that the sequence $(z_n(\cdot))_n$ converges weakly in $L^1_H([0,T])$ to a mapping $z(\cdot)$, with $\|z(t)\| \leq \alpha$ a.e. $t \in [0,T]$. Further, since $\big(\dot{x}_n(\cdot) + z_n(\cdot)\big)_n$ converges weakly in $L^1_H([0,T])$ to $(\dot{x}(\cdot) + z(\cdot))$ and $(x_n(\cdot))_n$ converges strongly in $L^1_H([0,T])$ to $x(\cdot)$ and since the operator $\partial\varphi(t,\cdot)$ satisfies the closure property as the subdifferential of a proper lower semicontinuous function one obtains

$$\dot{x}(t) + z(t) \in -\partial\varphi(t, x(t)).$$

Consequently, $-\dot{x}(t) \in \partial\varphi(t,x(t)) + z(t)$, a.e. $t \in [0,T]$. It remains to prove that $z(t) \in G(t, \tau(t)x)$ for almost every $t \in [0,T]$. First, we claim that $\tau(\theta_n(t)) g_{\delta_n(t)\frac{n}{T}}(\cdot, x_n(t))$ converges to $\tau(t)x$ in the space $\mathcal{C}_T([-r,0])$. Fix $t \in [0,T]$, for any $n \geq 1$, there exists an integer $0 \leq k \leq n-1$ such that $t \in [t^n_k, t^n_{k+1}]$. Then

$$\begin{aligned}\| \tau(\theta_n(t)) g^n_{\delta_n(t)\frac{n}{T}}(\cdot, x_n(t)) - \tau(t)x(\cdot) \|_{\mathcal{C}_0} &= \sup_{s\in[-r,0]} \| g^n_k(t^n_{k+1} + s, x_n(t)) - x(t+s) \| \\ &= \sup_{s \in [-r + t^n_{k+1}, t^n_{k+1}]} \| g^n_k(s, x_n(t)) - x(t + s - t^n_{k+1}) \| \\ &\leq \sup_{s \in [-r + t^n_k, t^n_{k+1}]} \| g^n_k(s, x_n(t)) - x(s) \| + \sup_{s \in [-r + t^n_k, t^n_{k+1}]} \| x(s) - x(t + s - t^n_{k+1}) \|\end{aligned}$$

$$\leq \sup_{s\in[-r,t_k^n]} \| x_n(s) - x(s) \| + \sup_{s\in[t_k^n,t_{k+1}^n]} \| g_k^n(s, x_n(t)) - x(s) \|$$
$$+ \sup_{s\in[-r+t_k^n,t_{k+1}^n]} \| x(s) - x(t + s - t_{k+1}^n) \| .$$

Clearly we have $\sup_{n\geq 1} \| x_n(s)-x(s) \|\leq\| x_n-x \|_{\mathcal{C}_T}$, because (x_n) converges uniformly to x, we get

$$\lim_{n\to+\infty} \sup_{s\in[-r,t_k^n]} \| x_n(s) - x(s) \|= 0.$$

As $x(\cdot)$ is uniformly continuous, we have

$$\lim_{n\to+\infty} \sup_{s\in[-r+t_{k+1}^n,t_{k+1}^n]} \| x(s) - x(t + s - t_{k+1}^n) \|= 0.$$

Indeed, let $\varepsilon > 0$, there is $\eta > 0$ such that $| s - (t + s - t_{k+1}^n) |\leq \eta$ for all $s \in [-r, 0]$ implies $\| x(s) - x(t + s - t_{k+1}^n) \|\leq \varepsilon$, hence

$$\sup \| x(s) - x(t + s - t_{k+1}^n) \|\leq \varepsilon.$$

Further on,

$$\sup_{s\in[t_k^n,t_{k+1}^n]} \| g_k^n(s, x_n(t)) - x(s) \|\leq \sup_{s\in[t_k^n,t_{k+1}^n]} \| x_n(t_k^n) - x(s) \| + \| x_n(t) - x_n(t_k^n) \|$$
$$\leq \sup_{s\in[t_k^n,t_{k+1}^n]} \| x_n(t_k^n) - x(s) \| + \| x_n(t) - x(t) \|$$
$$+ \| x(t) - x(\delta_n(t)) \| + \| x(\delta_n(t)) - x_n(\delta_n(t)) \| .$$

As $\lim_{n\to+\infty} \delta_n(t) = t$ and $x(\cdot)$ is continuous, we have $\lim_{n\to+\infty} \| x(t) - x(\delta_n(t)) \|= 0$, and

$$\lim_{n\to+\infty} \| x_n(t) - x(t) \|= 0,$$
$$\lim_{n\to+\infty} \sup_{s\in[t_k^n,t_{k+1}^n]} \| x_n(t_k^n) - x(s) \|= 0,$$

and

$$\lim_{n\to+\infty} \| x(\delta_n(t)) - x_n(\delta_n(t)) \|= 0.$$

So, we can conclude that

$$\lim_{n\to+\infty} \| \tau(\theta_n(t))g^n_{\delta_n(t)\frac{n}{T}}(\cdot, x_n(t)) - \tau(t)x(\cdot) \|_{\mathcal{C}_0}= 0. \tag{5}$$

As (z_n) converges weakly in $L^1_H([0, T])$ to z, using Mazur's lemma to (z_n) provides a sequence (ζ_n) such that

$$\zeta_n \in \bigcap_n \overline{co}\{z_q(t), \;\; q \geq n\}$$

and (ζ_n) converges strongly in $L^1_H([0, T])$ to z. We can extract from (ζ_n) a subsequence which converges a.e. to z. Then, there is a Lebesgue negligible set $\mathcal{N} \subset [0, T]$ such that for every $t \in [0, T] \setminus \mathcal{N}$

$$z(t) \in \bigcap_{n\geq 0} \overline{\{\zeta_m(t) : \; q \geq n\}} \subset \bigcap_{n\geq 0} \overline{co}\{z_m(t) : \; q \geq n\}. \tag{6}$$

Fix any $t \in [0, T] \setminus \mathcal{N}, n \geq n_0$ and $\xi \in H$, then the relation (6) gives

$$\langle \xi, z(t)\rangle \leq \limsup_{n\longrightarrow+\infty} \delta^*\Big(\xi, G\big(t, \tau(\delta_n(t))g^n_{\delta_n(t)\frac{n}{T}}(\cdot, x_n(t))\big)\Big).$$

By (5) and the upper hemicontinuity of $G(t,\cdot)$ we get for all $t \in [0,T]$

$$\langle \xi, z(t)\rangle \leq \delta^*\Big(\xi, G\big(t, \tau(t)x\big)\Big),$$

by the convexity of $G\big(t,\tau(t)x\big)$ we can write $d\big(z(t), G(t,\tau(t)x)\big) \leq 0$, then we get

$$z(t) \in G(t,\tau(t)x) \text{ a.e. } t \in [0,T]$$

because $G\big(t,\tau(t)x\big)$ is a closed set. The proof is therefore completed.

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