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COMMON FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN PROBABILISTIC METRIC SPACES

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Abstract. In this paper, we define the concept of (owc)-property for two single-valued mappings and two multi-valued mappings in Probabilistic metric spaces and give some new common fixed point results for these mappings. Also, we give some examples to illustrate the main results in this paper.

1. Introduction

The concept of an abstract metric space, introduced by M. Frechet in 1906 [8], furnishes the common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of a " distance" appears. What matters is the possibility of associating a non-negative real number with each ordered pair of elements of a certain set, and that the numbers associated with pairs and triples of such elements satisfy certain conditions.

In 1942, K. Menger [14] was first who thought about distance distribution function in metric space and introduced the concept of probabilistic metric space. He replaced distance function $d(x, y)$, the distance between two point x, y by distance distribution function $\mathcal{F}_{x,y}(t)$ where the value of $\mathcal{F}_{x,y}(t)$ is interpreted as probability that the distance between x, y is less than t, $t > 0$. The history of probabilistic metric spaces is brief In the original paper, Menger gave postulates for the distribution functions $\mathcal{F}_{x,y}$ These included a generalized triangle inequality. In addition, he constructed a theory of betweeness and indicated possible fields of application. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. It has also important applications in nonlinear analysis [4].

In 1943, shortly after the appearance of Menger's paper, A. Wald published a paper [32] in which he criticized Menger's generalized triangle inequality and proposed an alternative one. On the basis of this new inequality, A. Wald constructed a theory of betweeness having certain advantages over Menger's theory [33].

In 1951, Menger continued his study of probabilistic metric spaces in a paper

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[15] devoted to a resume of the earlier work, the construction of several specific examples and further considerations of the possible applications of the theory. In this paper, K. Menger adopted Wald's version of the triangle inequality. PM-spaces have nice topological properties. Many different topological structures may be defined on a PM-space. The one that has received the most attention to date is the strong topology and it is the principal tool of this study. The convergence with respect to this topology is called strong convergence. Since the strong topology is first countable and Hausdorff, it can be completely specified in terms of the strong convergence of sequences.

One of the simplest and most useful results in the fixed point theory is the Banach-Caccioppoli contraction mapping principle. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Banach contraction principle has been generalized in different spaces by mathematicians over the years. In 1972, Sehgal and Bharucha-Reid [27] initiated the study of contraction mappings in PM-spaces. For other related fixed point results in Menger spaces and their applications, we refer to [4]. Many mathematicians weakened the notion of commutativity by introducing the notions of weak commutativity [28], compatibility [10] and weak compatibility [9] in metric spaces and proved a number of fixed point theorems using these notions. In 2008, Al-Thagafi and Shahzad [2] gave a definition which is proper generalization of nontrivial weakly compatible mappings which have coincidence points. Jungck and Rhoades [11] studied fixed point results for occasionally weakly compatible mappings. Many authors exploited these concepts (see for example, [1, 11, 18, 22, 23, 29, 30]) in framework of PM-spaces to obtain a number of common fixed point results.

In an interesting note, Doric et al. [7] have shown that in respect of single-valued mappings, the condition of occasionally weak compatibility reduces to weak compatibility in the presence of a unique point of coincidence (or a unique common fixed point) of the given pair of mappings. Thus, no generalization can be obtained by replacing weak compatibility with occasionally weak compatibility.

In 1976, Caristi [3] proved a fixed point theorem. Since the Caristis fixed point theorem does not require the continuity of the mappings, it has applications in many fields. In 1993, Zhang et al. [34] proved set-valued Caristis theorem in probabilistic metric spaces. Chuan [6] brought forward the concept of Caristi type hybrid fixed point in Menger spaces. In 2006, Chen and Chang [5] proved a common fixed point theorem for four single-valued and two set-valued mappings in a complete Menger spaces by using the notion of compatibility. Further, Pant et al. [19] proved common fixed point theorems for single-valued and set-valued mappings in Menger spaces using implicit relation. More recently, Pant et al. [20] improved the results of Chen and Chang [5] by using the notion of occasionally weak compatible mappings. Several interesting results for multi-valued mappings are also appeared in ([1, 7, 12, 17, 18, 21, 22, 23, 24, 25]).

In the present paper, we prove a common fixed point theorem for single-valued and set-valued occasionally weakly compatible mappings in Menger spaces. An example is furnished which demonstrates the validity of the hypotheses and degree of generality of our main result.

2. Preliminaries

The introduction of the general concept of statistical metric spaces is due to Karl Menger (1942), who dealt with probabilistic geometry. The new theory of fundamental probabilistic structures was developed later on by many authors. In this section, we start by recalling some basic concepts from Menger probabilistic metric spaces. For more details on such spaces, we refer to ([4]-[18]).

Definition 1[26] A mapping $\mathcal{F}:\mathbb{R}\longrightarrow\mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $sup_{x\in\mathbb{R}}\mathcal{F}(x)=1$ and $inf_{x\in\mathbb{R}}\mathcal{F}(x)=0$. We shall denote by D the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(x)=\begin{cases}0, & x\leq 0;\\ 1, & x>0.\end{cases}$$

Example 1 Let $G:\mathbb{R}\longrightarrow\mathbb{R}^+$ be a mapping defined by

$$G(x)=\begin{cases}0, & x\leq 0;\\ a, & 0<x\leq k;\\ b, & k<x\leq 3k;\\ 1, & 3k<x.\end{cases}$$

Where $0<a\leq b<1$ and k is any positive number. It is clear that G is non-decreasing and left continuous with $inf_{x\in\mathbb{R}}G(x)=0$ and $sup_{x\in\mathbb{R}}G(x)=1$, then G is called a distribution function.

Definition 2 [4] A probabilistic metric space (briefly, a PM-space) is an ordered pair $(X,\mathcal{F})$, where X is an abstract set and $\mathcal{F}$ is a mapping of $X\times X$ on to the set of all distributions function, i.e, $\mathcal{F}$ associates a distribution function $\mathcal{F}(p,q)$ with every pair (p,q) of points in X. we shall denote the distribution function $\mathcal{F}(p,q)$ by $\mathcal{F}_{p,q}$, whence the symbol $\mathcal{F}_{p,q}(x)$ will denote the value of $\mathcal{F}_{p,q}$ for the real argument x. the function $\mathcal{F}_{p,q}$ are assumed to satisfy the following condition:
(PM-1) $\mathcal{F}_{p,q}(x)=1$ for all $x>0$ if and only if $p=q$,
(PM-2) $\mathcal{F}_{p,q}(0)=0$,
(PM-3) $\mathcal{F}_{p,q}=\mathcal{F}_{p,q}$,
(PM-4) if $\mathcal{F}_{p,q}(x)=1$ and $\mathcal{F}_{p,q}(y)=1$, then $\mathcal{F}_{p,q}(x+y)=1$.
In view of Condition (PM-2), which evidently implies that $\mathcal{F}_{p,q}(0)=0$ for all $x\leq 0$, the Condition (PM-1) is equivalent to the statement: $p=q$ if and only if $\mathcal{F}_{p,q}=H$. Every metric space may be regarded as an PM-space of a special kind if we have only to set $\mathcal{F}_{p,q}(x)=H((x-d(p,q))$ for every pair of points (p,q) in the metric space. Furthermore, with the interpretation of $\mathcal{F}_{p,q}(x)$ as the probability that the distance from p to q is less than x, one sees that Conditions (PM-1),(PM-2),and (PM-3)are straightforward generalizations of the corresponding conditions (M-1), (M-2), (M-3). Condition (PM-4) is a ' minimal' generalization of the triangle inequality (M-4) which may be interpreted as follows : If it is certain that the distance of p and q is less than x, and likewise certain that the distance of q and r is less than y, then it is certain that the distance of p and r is less than $x+y$. Condition (PM-4) is always satisfied in metric spaces, where it reduces to the ordinary triangle inequality. However, in those PM-spaces in which the equality $\mathcal{F}_{p,q}(x)=1$ does not hold for any finite x, (PM-4) will be satisfied only vacuously.

Example 2 Let X be a set of all real numbers, and define:

$$\mathcal{F}_{p,q}(x) = \begin{cases} 0, & x \leq 0; \\ 1 - e^{-(\frac{x}{d(p,q)})}, & x > 0, \end{cases}$$

is a distribution function for all $p, q \in X$, where $d(p,q) = |p-q|$ for all $p, q \in X$. By verifying that $\mathcal{F}_{p,q}$ satisfy the axioms (PM-1) to (PM-4), then $(X, \mathcal{F}, \Delta)$ is probabilistic metric space.

Definition 3 [31] A mapping $\Delta : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a triangular norm (for short, a $t-$ norm) if the following conditions are satisfied:
$(\Delta - 1)$ $\Delta(a,1) = a$ and $\Delta(0,0) = 0$,
$(\Delta - 2)$ $\Delta(a,b) = \Delta(b,a)$,
$(\Delta - 3)$ $\Delta(a,c) = \Delta(b,d)$ for $a \geq b$, $c \geq d$,
$(\Delta - 4)$ $\Delta(a, \Delta(b,c)) = \Delta(\Delta(a,b), c)$.
Remark 1 From $(\Delta - 4)$, it is not difficult to find that

$$\Delta(\Delta(a,b), \Delta(c,d)) = \Delta(\Delta(\Delta(a,b),c),d) = \Delta(\Delta(\Delta(a,c),b),d) = \Delta(\Delta(a,c), \Delta(b,d)) = ...$$

Example 3 [4] The following are the three basic t-norms.
(1)The minimum $t-$norm: $\Delta_m(a,b) = min\{a,b\}$.
(2)The product $t-$norm: $\Delta_p(a,b) = a.b$.
(3)The Lukasiewicz $t-$norm: $\Delta_L(a,b) = max\{a+b-1, 0\}$.
In respect of above mentioned $t-$norms, we have the following ordering:

$$\Delta_L < \Delta_p < \Delta_m.$$

Definition 4[4] A Menger PM-space is a tripled $(X, \mathcal{F}, \Delta)$ where $(X, \mathcal{F})$ is a PM-space and Δ is a t-norm such that the inequality $\mathcal{F}_{p,r}(x+y) \geq \Delta(\mathcal{F}_{p,q}(x), \mathcal{F}_{q,r}(y))$ holds for all $p, q, r \in X$ and $x, y \geq 0$.
Example 4 Let X be a non empty set, and defined $\mathcal{F}_{p,q}$ by:

$$\mathcal{F}_{p,q}(x) = \begin{cases} G(x), & p \neq q; \\ H(x), & p = q, \end{cases}$$

where

$$G(x) = \begin{cases} 0, & x < 0; \\ x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$$

It is clear that $\mathcal{F}_{p,q}$ is a distribution function and satisfy the axioms (PM-1) to (PM-4), then $(X, \mathcal{F})$ is probabilistic metric space. For any triple of distinct points in X the inequality $\mathcal{F}_{p,r}(x+y) \geq \Delta(\mathcal{F}_{p,q}(x), \mathcal{F}_{q,r}(y))$ holds for all $p, q, r \in X$ and $x, y \geq 0$ under $\Delta_m(a,b) = min\{a,b\}$, since in all cases, we have $G(x+y) \geq min\{G(x), G(y)\}$ for all $x, y \geq 0$, then $(X, \mathcal{F}, \Delta)$ is Menger PM-space.
Lemma 1[4] If the points p, q, r are not all distinct, then the inequality $\mathcal{F}_{p,r}(x+y) \geq \Delta(\mathcal{F}_{p,q}(x), \mathcal{F}_{q,r}(y))$ holds for the triple p, q, r and all $x, y \geq 0$ under any choice of Δ satisfying $(\Delta - 1)$, $(\Delta - 2)$, $(\Delta - 3)$ and $(\Delta - 4)$.
Definition 4 let (X, F, Δ) be a Menger Probabilistic metric space and $B(X)$ be the family of all nonempty bounded subsets of X. For all $A, B \in B(X)$ and for every $t > 0$, We define the functions $\delta_{A,B}(t)$ by

$$\delta_{A,B}(t) = \inf\{\mathcal{F}_{a,b}(t) : a \in A \quad and \quad b \in B\}.$$

Remark 2 If A consists of a single point a, then we write $\delta_{A,B}(t) = \delta_{a,B}(t)$. If $A = \{a\}$ and $B = \{b\}$, then we write $\delta_{A,B}(t) = \delta_{a,b}(t)$. It follows immediately from the definition of δ that:

(1) $\delta_{A,B}(t) = \delta_{B,A}(t) \geq 0$,
(2) $\delta_{A,B}(t) = 1$ for all $t \geq 0$ if and only if $A = B = \{a\}$,

for all $A, B \in B(X)$.
Definition 5[4] Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space, $A, B \in CB(X)$ ($CB(X)$ be the family of all nonempty $\tau-$closed subsets of X) and $x \in X$. we defined
(1) The probabilistic distance between A and B is the function $\mathcal{F}_{A,B}$ defined by

$$\mathcal{F}_{A,B}(t) = \sup_{s<t} \Delta(\in \mathcal{F}_{x\in A} \sup_{y\in B} \mathcal{F}_{x,y}(s), \in \mathcal{F}_{y\in B} \sup_{x\in A} \mathcal{F}_{x,y}(s)), forall \quad t \in R.$$

(2) The probabilistic distance between x and A is the function $\mathcal{F}_{x,A}$ defined by

$$\mathcal{F}_{x,A}(t) = \sup_{y\in A} \mathcal{F}_{x,y}(t), forall \quad t \in R.$$

Forward, we denote by $Fix(T)$ the set of all fixed points of a multi-valued mapping T, that is,

$$Fix(T) = \{x \in X : x \in Tx\}.$$

Recall that $x \in X$ is called a coincidence point of $f : X \to X$ and $T : X \to B(X)$ if $fx \in Tx$.

Definition 6[16] Let (X, F, Δ) be a Menger PM-space. Two mappings $f, g : X \to X$ are said to be compatible (or asymptotically commuting) if

$$\lim_{n\to\infty} \mathcal{F}_{fgx_n,gfx_n}(t) = 1, forall \quad t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$$

for some $z \in X$.

Definition 7[16] Let (X, F, Δ) be a Menger PM-space. Two mappings $f : X \to X$ and $S : X \to CB(X)$ are said to be compatible if $fSx \in CB(X)$ for all $x \in X$ and

$$\lim_{n\to\infty} \mathcal{F}_{fSx_n,Sfx_n}(t) = 1, forall \quad t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = A$$

for some $A \in CB(X)$ and

$$\lim_{n\to\infty} fx_n = z \in A$$

for some $z \in X$.
Definition 8[9] Mappings $f : X \to X$ and $S : X \to B(X)$ are said to be weakly compatible if $fSx = Sfx$ whenever $fx \in Sx$.
It is easy to see that compatible mappings are weakly compatible, but the converse is not true.

Example 5 Let $X = [0, \infty)$ with usual metric. Define the mappings $f : X \to X$ and $S : X \to B(X)$ as: $f(x) = x^2$ for all $x \in X$ and

$$Sx = \begin{cases} \{x\}, & 0 \leq x \leq 1; \\ (0, 1), & 1 < x < \infty. \end{cases}$$

Then the mappings f and S are weakly compatible at their coincidence points.

Definition 9[11] Let (X, F, Δ) be a Menger PM-space. Two mappings $f, g : X \to X$ are said to be occasionally weakly compatible (shortly, $(owc)-$property) if there exists a point $u \in X$ such that $fu = gu$ and $fgu = gfu$.
Definition 10[1] Let (X, F, Δ) be a Menger PM-space. A single-valued mapping $f : X \to X$ and a multi-valued mapping $S : X \to CB(X)$ are said to occasionally weakly compatible (shortly, $(owc)-$property) if and only if there exists some point $x \in X$ such that $fx \in Sx$ and $fSx \subseteq Sfx$.

From the following example, it is clear that the notion of occasionally weakly compatible mappings is more general than weak compatibility.

Example 6 In the setting of Example 2, replace the mappings f and S by the following, besides retaining the rest:

$$fx = \begin{cases} 0, & 0 \leq x < 2; \\ x + 2, & 2 \leq x < \infty, \end{cases} \quad Sx = \begin{cases} x, & 0 \leq x < 2; \\ [2, x + 3)\,, & 2 \leq x < \infty. \end{cases}$$

Here, it can be easily verified that $x = 0, 2$ are the coincidence points of S and A, but f and S are not weakly compatible at $x = 2$ that is $Sf(2) = [2, 7] \neq fS(2) = [4, 7]$. Hence f and S are not compatible. However, the pair (f, S) is occasionally weakly compatible, since the pair (f, S) is weakly compatible at $x = 0$.

3. Main Results

In this section, we state and prove our main result.

Theorem 1 Let (X, F, Δ) be a Menger PM-space. Let $f, g : X \to X$ be a single-valued mappings and $S, T : X \to CB(X)$ be a multi-valued mappings satisfying the following conditions:
(1) the pairs (S, f) and (T, g) are the $(owc)-$property,
(2) for all $x, y \in X$, there exists k, where $0 < k < \mathcal{F}^p_{fx,gy}(t)$ such that

$$\delta^p_{Sx,Ty}(t) \geq \varphi\left(min\left\{\mathcal{F}^p_{fx,gy}(t), \frac{\mathcal{F}^p_{fx,Sx}(t)\mathcal{F}^p_{gy,Ty}(t)}{\mathcal{F}^p_{fx,gy}(t) - k}, \frac{\mathcal{F}^p_{fx,Ty}(t)\mathcal{F}^p_{gy,Sx}(t)}{\mathcal{F}^p_{fx,gy}(t) - k}\right\}\right),$$

where $p \geq 1$ and $\varphi : [0, 1] \to [0, 1]$ is a function such that $\varphi(1) = 1$, $\varphi(0) = 0$ and $\varphi(z) > z$ for all $0 < z < 1$. Then f, g, S and T have a unique common fixed point in X.
Proof. Since the pairs (S, f) and (T, g) satisfy the $(owc)-$property, there exist $u, v \in X$ such that

$$fu \in Su, \quad fSu \subset Sfu, \quad gv \in Tv, \quad gTv \subset Tgv,$$

which implies that $ffu \in Sfu$ and $ggv \in Tgv$.
Now, we prove that $fu = gv$. In fact, if $fu \neq gv$, then there exists a positive real number t such that $\mathcal{F}^p_{fu,gv}(t) < 1$. using the condition (2), we have

$$\begin{aligned}\delta^p_{Su,Tv}(t) &\geq \varphi\left(min\left\{\mathcal{F}^p_{fu,gv}(t), \frac{\mathcal{F}^p_{fu,Su}(t)\mathcal{F}^p_{gv,Tv}(t)}{\mathcal{F}^p_{fu,gv}(t)-k_1}, \frac{\mathcal{F}^p_{fu,Tv}(t)\mathcal{F}^p_{gv,Su}(t)}{\mathcal{F}^p_{fu,gv}(t)-k_1}\right\}\right)\\ &= \varphi\left(min\left\{\mathcal{F}^p_{fu,gv}(t), \frac{1}{1-k_1}, \frac{\mathcal{F}^p_{fu,Tv}(t)\mathcal{F}^p_{gv,Su}(t)}{\mathcal{F}^p_{fu,gv}(t)-k_1}\right\}\right)\\ &= \varphi\left(min\left\{\mathcal{F}^p_{fu,gv}(t), \frac{\mathcal{F}^p_{fu,Tv}(t)\mathcal{F}^p_{gv,Su}(t)}{\mathcal{F}^p_{fu,gv}(t)-k_1}\right\}\right),\end{aligned}$$

where $0 < k_1 < \mathcal{F}^p_{fu,gv}(t)$. Since $fu \in Su$ and $gv \in Tv$, we have

$$\frac{\mathcal{F}^p_{fu,Tv}(t)\mathcal{F}^p_{gv,Su}(t)}{\mathcal{F}^p_{fu,gv}(t)-k_1} \geq \frac{\mathcal{F}^p_{fu,gv}(t)\mathcal{F}^p_{gv,fu}(t)}{\mathcal{F}^p_{fu,gv}(t)-k_1} > \mathcal{F}^p_{fu,gv}(t)$$

and hence

$$\mathcal{F}^p_{fu,gv}(t) \geq \delta^p_{Su,Tv}(t) \geq \varphi\left(\mathcal{F}^p_{fu,gv}(t)\right) > \mathcal{F}^p_{fu,gv}(t)$$

which is a contradiction and so $fu = gv$.
Next, we prove that fu is a fixed point of f. Suppose that $ffu \neq fu$. Then there exists a positive real number t such that $\mathcal{F}^p_{ffu,fv}(t) < 1$. By using the condition (2), we have

$$\begin{aligned}\mathcal{F}^p_{ffu,fv}(t) &= \mathcal{F}^p_{ffu,gv}(t) \geq \delta^p_{Sfu,Tv}(t)\\ &\geq \varphi\left(min\left\{\mathcal{F}^p_{ffu,gv}(t), \frac{\mathcal{F}^p_{ffu,Sfu}(t)\mathcal{F}^p_{gv,Tv}(t)}{\mathcal{F}^p_{ffu,gv}(t)-k_2}, \frac{\mathcal{F}^p_{ffu,Tv}(t)\mathcal{F}^p_{gv,Sfu}(t)}{\mathcal{F}^p_{ffu,gv}(t)-k_2}\right\}\right),\end{aligned}$$

where $0 < k_2 < \mathcal{F}^p_{ffu,gv}(t)$. Since $ffu \in Sfu$ and $gv \in Tgv$, we have

$$\frac{\mathcal{F}^p_{ffu,Tv}(t)\mathcal{F}^p_{gv,Sfu}(t)}{\mathcal{F}^p_{ffu,gv}(t)-k_2} \geq \frac{\mathcal{F}^p_{ffu,Tv}(t)\mathcal{F}^p_{gv,ffu}(t)}{\mathcal{F}^p_{ffu,gv}(t)-k_2} \geq \mathcal{F}^p_{ffu,Tv}(t) \geq \mathcal{F}^p_{ffu,gv}(t)$$

and hence

$$\delta^p_{Sfu,Tv}(t) \geq \varphi\left(\mathcal{F}^p_{ffu,gv}(t)\right).$$

Thus it follows from the property of φ that

$$\mathcal{F}^p_{ffu,fv}(t) = \mathcal{F}^p_{ffu,gv}(t) \geq \delta^p_{Sfu,Tv}(t) \geq \varphi\left(\mathcal{F}^p_{ffu,gv}(t)\right) > \mathcal{F}^p_{ffu,gv}(t) = \mathcal{F}^p_{ffu,fv}(t)$$

which is a contradiction and so $ffu = fu$. Similarly, we can prove $fu = gfu = ffu$. Thus we have

$$fu = ffu \in Sfu$$

and

$$fu = gfu = ggv \in Tgv = Tfu.$$

Therefore, fu is a common fixed point of f, g, S and T. Moreover, by the condition (2), we have

$$\begin{aligned}\delta^p_{Sfu,Tfu}(t) &\geq \varphi\left(min\left\{\mathcal{F}^p_{ffu,gfu}(t), \frac{\mathcal{F}^p_{ffu,Sfu}(t)\mathcal{F}^p_{gfu,Tfu}(t)}{\mathcal{F}^p_{ffu,gfu}(t)-k_3}, \frac{\mathcal{F}^p_{ffu,Tfu}(t)\mathcal{F}^p_{gfu,Sfu}(t)}{\mathcal{F}^p_{ffu,gfu}(t)-k_3}\right\}\right)\\ &= \varphi\left(min\left\{1, \frac{1}{1-k_3}, \frac{1}{1-k_3}\right\}\right) = 1,\end{aligned}$$

where $0 < k_3 < \mathcal{F}^p_{ffu,gfu}(t)$. Therefore $Sfu = Tfu = \{fu\}$.
Next, assume that $w \neq z$ is another common fixed point of f, g, S and T,then there exists a positive real number t such that $\mathcal{F}^p_{z,w}(t) < 1$. From the condition (2), we have

$$\begin{aligned}\mathcal{F}^p_{z,w}(t) = \delta^p_{Sz,Tw}(t) &\geq \varphi\left(min\left\{\mathcal{F}^p_{fz,gw}(t), \frac{\mathcal{F}^p_{fz,Sz}(t)\mathcal{F}^p_{gw,Tw}(t)}{\mathcal{F}^p_{fz,gw}(t)-k_4}, \frac{\mathcal{F}^p_{fz,Tw}(t)\mathcal{F}^p_{gw,Sz}(t)}{\mathcal{F}^p_{fz,gw}(t)-k_4}\right\}\right)\\ &= \varphi\left(min\left\{\mathcal{F}^p_{z,w}(t), \frac{1}{\mathcal{F}^p_{z,w}(t)-k_4}, \frac{\mathcal{F}^p_{z,w}(t)\mathcal{F}^p_{w,z}(t)}{\mathcal{F}^p_{z,w}(t)-k_4}\right\}\right)\\ &= \varphi\left(\mathcal{F}^p_{z,w}(t)\right) > \mathcal{F}^p_{z,w}(t),\end{aligned}$$

where $0 < k_4 < \mathcal{F}^p_{fz,gw}(t)$ which is a contradiction. Thus the common fixed point z is unique. This completes the proof.

Example 7 Let $X = [0, \infty)$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$, define

$$\mathcal{F}_{p,q}(t) = \begin{cases} \frac{t}{t+|x-y|}, & t > 0;\\ 0, & t = 0,\end{cases}$$

for all $x, y \in X$. Clearly (X, F, Δ) be a Menger space, with t-norm Δ is defined by $\Delta(a, b) = min\{a, b\}$ for all $a, b?[0, 1]$. Define the mappings $f, g : X \to X$ and $S, T : X \to B(X)$ by

$$Sx = \begin{cases} \{x\}, & 0 \leq x < 1;\\ [1, x+2], & 1 \leq x < \infty,\end{cases} \quad Tx = \begin{cases} \{0\}, & 0 \leq x < 1;\\ [1, x+1], & 1 \leq x < \infty,\end{cases}$$

$$fx = \begin{cases} 0, & 0 \leq x < 1;\\ x+1, & 1 \leq x < \infty,\end{cases} \quad gx = \begin{cases} \frac{x}{2}, & 0 \leq x < 1;\\ 2x+3, & 1 \leq x < \infty.\end{cases}$$

Then the pairs (S, f) and (T, g) satisfy the $(owc)-$property because

$$f(0) \in S(0), \quad fS(0) \subseteq Sf(0), \quad g(0) \in T(0), \quad gT(0) \subseteq Tg(0).$$

Let $\varphi : [0, 1] \to [0, 1]$ be defined by $\varphi(z) = \sqrt{z}$ for all $0 \leq z \leq 1$. Then $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(z) > z$ for each $0 < z < 1$. Also the mappings f, g, S and T satisfy the condition (2) of Theorem 3 with $\varphi(z) = \sqrt{z}$, i.e., for all $x, y \in X$, there exists k, where $-1 < k < 0$ if $\mathcal{F}^p_{fx,gy}(t) = 0$ or $0 < k < \mathcal{F}^p_{fx,gy}(t)$ if $\mathcal{F}^p_{fx,gy}(t) \neq 0$ such that

$$\delta^p_{Sx,Ty}(t) \geq \varphi\left(min\left\{\mathcal{F}^p_{fx,gy}(t), \frac{\mathcal{F}^p_{fx,Sx}(t)\mathcal{F}^p_{gy,Ty}(t)}{\mathcal{F}^p_{fx,gy}(t)-k}, \frac{\mathcal{F}^p_{fx,Ty}(t)\mathcal{F}^p_{gy,Sx}(t)}{\mathcal{F}^p_{fx,gy}(t)-k}\right\}\right),$$

where $p \geq 1$. Then f, g, S and T have a unique common fixed point in X. Hence, 0 is the unique common fixed point of f, g, S and T.

If $p = 1$ in Theorem 3, then we have the following:

Corollary 1 Let (X, F, Δ) be a Menger PM-space. Let $f, g : X \to X$ be a single-valued mappings and $S, T : X \to CB(X)$ be a multi-valued mappings satisfying the following conditions:
(1) the pairs (S, f) and (T, g) are the $(owc)-$property,
(2) for all $x, y \in X$, there exists k, where $0 < k < \mathcal{F}^p_{fx,gy}(t)$ such that

$$\delta_{Sx,Ty}(t) \geq \varphi\left(min\left\{\mathcal{F}_{fx,gy}(t), \frac{\mathcal{F}_{fx,Sx}(t)\mathcal{F}_{gy,Ty}(t)}{\mathcal{F}_{fx,gy}(t) - k}, \frac{\mathcal{F}_{fx,Ty}(t)\mathcal{F}_{gy,Sx}(t)}{\mathcal{F}_{fx,gy}(t) - k}\right\}\right),$$

where $\varphi : [0, 1] \to [0, 1]$ is a function such that $\varphi(1) = 1$, $\varphi(0) = 0$ and $\varphi(z) > z$ for all $0 < z < 1$. Then f, g, S and T have a unique common fixed point in X.

If we take $S = T$ and $f = g$ in Theorem 3, then we have the following:

Corollary 2 Let (X, F, Δ) be a Menger PM-space. Let $f : X \to X$ be a single-valued mappings and $S : X \to CB(X)$ be a multi-valued mappings satisfying the following conditions:
(1) the pairs (S, f) satisfies the $(owc)-$property,
(2) for all $x, y \in X$, there exists k, where $0 < k < \mathcal{F}^p_{fx,gy}(t)$ such that

$$\delta^p_{Sx,Sy}(t) \geq \varphi\left(min\left\{\mathcal{F}^p_{fx,fy}(t), \frac{\mathcal{F}^p_{fx,Sx}(t)\mathcal{F}^p_{fy,Sy}(t)}{\mathcal{F}^p_{fx,gy}(t) - k}, \frac{\mathcal{F}^p_{fx,Sy}(t)\mathcal{F}^p_{fy,Sx}(t)}{\mathcal{F}^p_{fx,fy}(t) - k}\right\}\right),$$

where $p \geq 1$ and $\varphi : [0, 1] \to [0, 1]$ is a function such that $\varphi(1) = 1$, $\varphi(0) = 0$ and $\varphi(z) > z$ for all $0 < z < 1$. Then f, g, S and T have a unique common fixed point in X.

If S is a single-valued mapping in Corollary 3, then we have the following:

Corollary 3 Let (X, F, Δ) be a Menger PM-space and $f, S : X \to X$ be two single-valued mappings satisfying the following conditions:
(1) the pairs (S, f) satisfies the $(owc)-$property,
(2) for all $x, y \in X$, there exists k, where $0 < k < \mathcal{F}^p_{fx,gy}(t)$ such that

$$\delta^p_{Sx,Sy}(t) \geq \varphi\left(min\left\{\mathcal{F}^p_{fx,fy}(t), \frac{\mathcal{F}^p_{fx,Sx}(t)\mathcal{F}^p_{fy,Sy}(t)}{\mathcal{F}^p_{fx,gy}(t) - k}, \frac{\mathcal{F}^p_{fx,Sy}(t)\mathcal{F}^p_{fy,Sx}(t)}{\mathcal{F}^p_{fx,fy}(t) - k}\right\}\right),$$

where $p \geq 1$ and $\varphi : [0, 1] \to [0, 1]$ is a function such that $\varphi(1) = 1$, $\varphi(0) = 0$ and $\varphi(z) > z$ for all $0 < z < 1$. Then f, g, S and T have a unique common fixed point in X.

If both S and T are single-valued mappings in Theorem 3, then we have the following:

Corollary 4 Let (X, F, Δ) be a Menger PM-space and Let $f, g, S, T : X \to X$ be four mappings satisfying the following conditions:

(1) the pairs (S, f) and (T, g) are the $(owc)-$property,
(2) for all $x, y \in X$, there exists k, $0 < k < \mathcal{F}^p_{fx,gy}(t)$ such that

$$\delta^p_{Sx,Ty}(t) \geq \varphi\left(min\left\{\mathcal{F}^p_{fx,gy}(t), \frac{\mathcal{F}^p_{fx,Sx}(t)\mathcal{F}^p_{gy,Ty}(t)}{\mathcal{F}^p_{fx,gy}(t)-k}, \frac{\mathcal{F}^p_{fx,Ty}(t)\mathcal{F}^p_{gy,Sx}(t)}{\mathcal{F}^p_{fx,gy}(t)-k}\right\}\right),$$

where $p \geq 1$ and $\varphi : [0, 1] \to [0, 1]$ is a continuous monotone increasing function such that $\varphi(1) = 1$, $\varphi(0) = 0$ and $\varphi(z) > z$ for all $0 < z < 1$. Then f, g, S and T have a unique common fixed point in X.

If we take $\varphi = lz$ for some $l > 1$ in Corollary 3, then we have the following:

Corollary 5 Let (X, F, Δ) be a Menger PM-space and Let $f, g, S, T : X \to X$ be four mappings satisfying the following conditions:
(1) the pairs (S, f) and (T, g) are the $(owc)-$property,
(2) for all $x, y \in X$, there exists k, where $0 < k < \mathcal{F}^p_{fx,gy}(t)$ such that

$$\delta^p_{Sx,Ty}(t) \geq l min\left\{\mathcal{F}^p_{fx,gy}(t), \frac{\mathcal{F}^p_{fx,Sx}(t)\mathcal{F}^p_{gy,Ty}(t)}{\mathcal{F}^p_{fx,gy}(t)-k}, \frac{\mathcal{F}^p_{fx,Ty}(t)\mathcal{F}^p_{gy,Sx}(t)}{\mathcal{F}^p_{fx,gy}(t)-k}\right\}.$$

Then f, g, S and T have a unique common fixed point in X.

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